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A New Unicity Theorem and Erdös' Problem for Polarized Semi-Abelian Varieties

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1 Introduction

The subject which we are going to deal with has a quite classical background in the complex function theory. Cf. Corvaja-Noguchi [4] for the details of this talk.

(a) Nevanlinna's unicity theorem. We begin with the famous five points theorem of R. Nevnalinna.

Theorem 1.1. (Unicity Theorem) *Let* $f, g : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ *be two non-constant meromorphic functions. If there are five distinct points* $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq 5$ *such that* $\mathrm{Supp}\ f^*a_i = \mathrm{Supp}\ g^*a_i$ $(1 \leq i \leq 5)$, *then* $f \equiv g$.

This follows from Nevanlinna's Second Fundamental Theorem, also called Second Main Theorem (Acta 1925, "Second Théorème fondamental" due to [6]; abbreviated "SFT"):

Theorem 1.2. (SFT) *Let* $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$ *be a meromorphic function, and* $a_i \in \mathbf{P}^1(\mathbf{C}), 1 \leq i \leq q$, *be distinct* q *points. Then*

$$(q-2)T_f(r) \leq \sum_{i=1}^{q} N(r, \mathrm{Supp}\ f^*a_i) + \text{small-term}.$$

Here $T_f(r)$ denotes the order function (energy integral) of $f : \mathbf{C} \to \mathbf{P}^1(\mathbf{C})$, and $N(r, *)$ denotes the counting function for a point distribution in the disk of radius r with center at the origin (cf. §3 for notation).

Proof of Theorem 1.1. By Nevanlinna's SFT 1.2 we have

$$(5-2=3)T_{f(\text{ or } g)}(r) \leq \sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^*(\text{ or } g^*)a_i) + \text{small-term}.$$

Suppose $f \not\equiv g$. Then the assumption implies that

$$\sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^*a_i) \le N(r, (f-g)_0) \le T_{f-g}(r) + O(1)$$

$$\le T_f(r) + T_g(r) + O(1) \le \frac{2}{3} \sum_{i=1}^{5} N(r, \mathrm{Supp}\ f^*a_i) + \text{small-term}.$$

Thus, $1 \le \frac{2}{3}$; a contradiction. □

Remark. The number 5 in the above unicity theorem is optimal for the following trivial reason: Set $f(z) = e^z$, $g(z) = e^{-z}$; $a_1 = 0, a_2 = \infty, a_3 = 1, a_4 = -1$. Then $f^*a_i = g^*a_i, 1 \le i \le 4$. Note that by setting $\sigma(w) = w^{-1}$ and $D = \sum_1^4 a_i$ we have

$$\sigma^* D = D, \quad \sigma \circ f = g; \quad f(z), g(z) \in \mathbf{C}^*.$$

Theorem 1.3. (E.M. Schmid 1971) *Let E be an elliptic curve, and let $a_i \in E, 1 \le i \le 5$, be distinct five points. Let $f, g : \mathbf{C} \to E$ be holomorphic maps. If* Supp $f^*a_i =$ Supp $g^*a_i, 1 \le i \le 5$, *then $f \equiv g$.*

Theorem 1.4. (H. Fujimoto 1975) *Let $f, g : \mathbf{C} \to \mathbf{P}^n(\mathbf{C})$ be holomorphic curves such that at least one of them is linearly non-degenerate. Let $\{H_j\}_{j=1}^{3n+2}$ be hyperplanes of $\mathbf{P}^n(\mathbf{C})$ in general position. If $f^*H_j = g^*H_j, 1 \le j \le 3n+2$ (as divisors, counting multiplicities), then $f \equiv g$.*

Schmid's and Fujimoto's theorems are deduced from some SFT's in the corresponding cases. It is an interesting problem to decrease the number "five" in Theorem 1.1, and the case of "three" is critical:

Theorem 1.5. *Let $a_i \in \hat{\mathbf{C}}$ $(1 \le i \le 3)$ be distinct points. Let f and g be distinct nonconstant meromorphic functions on $\mathbf{C}$ such that $f^*\{a_i\} = g^*\{a_i\}$ as divisors for all $i = 1, 2, 3$. Then there is no meromorphic function h on $\mathbf{C}$ other than f and g, satisfying $h^*\{a_i\} = f^*\{a_i\}$ $(i = 1, 2, 3)$.*

By a linear fractional transformation we may assume $\{a_i\}_{i=1}^3 = \{0, 1, \infty\}$. Imposing f and g to have values in the multiplicative group $\mathbf{C}^* = \mathbf{C} \setminus \{0\}$, we have

Corollary 1.6. *Let $f, g : \mathbf{C} \to \mathbf{C}^*$ be nonconstant and holomorphic. If $f^*\{1\} = g^*\{1\}$, then $f \equiv g$ or $f \equiv \frac{1}{g}$; i.e., with the automorphism $\phi(w) = \frac{1}{w}$ of $\mathbf{C}^*$ fixing 1, "$f = \phi \circ g$" holds.*

N.B. The above Corollary is most relevant to the present talk. By our main Theorem 2.1 which will be stated soon later, the above condition "$f^*\{1\} = g^*\{1\}$" (as divisors) can be replaced by $f^{-1}\{1\} = g^{-1}\{1\}$ (as sets); this special case is already a new result even in the classical setting.

The following is a kind of unicity problem in arithmetic theory, which is sometimes called a "support problem":

Erdös' Problem (1988). *Let x, y be positive integers. Is it true that*

$$\{p; prime, p|(x^n - 1)\} = \{p; prime, p|(y^n - 1)\}, \forall n \in \mathbf{N}$$

$$\Longleftrightarrow x = y \quad ?$$

The answer is Yes:

Theorem 1.7. (Schinzel 1960/75, Corrales-Rodorigáñez and R. Schoof, JNT 1997; cf. [4])

(i) *Suppose that except for finitely many prime $p \in \mathbf{Z}$*

$$y^n \equiv 1 \pmod{p} \text{ whenever } x^n \equiv 1 \pmod{p}, \forall n \in \mathbf{N}.$$

Then, $y = x^h$ with some natural number $h \in \mathbf{N}$.

(ii) *Let E be an elliptic curve defined over a number field k, and let $P, Q \in E(k)$. Suppose that except for finitely many prime $p \in O(k)$*

$$nQ = 0 \text{ whenever } nP = 0 \text{ in } E(k_p).$$

Then either $Q = \sigma(P)$ with some $\sigma \in \mathrm{End}(E)$, or both P, Q are torsion points.

(b) Yamanoi's Unicity Theorem. K. Yamanoi proved in Forum Math. 2004 the following striking unicity theorem:

Theorem 1.8. *Let $A_i, i = 1, 2$, be abelian varieties, and let $D_i \subset A_i$ be irreducible divisors such that*

$$\mathrm{St}(D_i) = \{a \in A_i; a + D_i = D_i\} = \{0\}.$$

Let $f_i : \mathbf{C} \to A_i$ be (algebraically) nondegenerate entire holomorphic curves. Assume that $f_1^{-1}D_1 = f_2^{-1}D_2$ as sets. Then there exists an isomorphism $\phi : A_1 \to A_2$ such that

$$f_2 = \phi \circ f_1, \quad D_1 = \phi^* D_2.$$

N.B. (i) The new point is that we can determine not only f, but the moduli point of a polarized abelian variety (A, D) through the distribution of $f^{-1}D$ by a nondegenerate $f : \mathbf{C} \to A$.

(ii) The assumptions for D_i to be irreducible and the triviality of $\mathrm{St}(D_i)$ are not restrictive. There is a way of reduction.

(iii) For simplicity we assume them here.

2 Main Results

We want to uniformize the results in the previous section. Therefore we deal with semi-abelian varieties.

Let $A_i, i = 1,2$ be semi-abelian varieties:

$$0 \to (\mathbf{C}^*)^{t_i} \to A_i \to A_{0i} \to 0,$$

where A_{0i} are abelian varieties. Let $D_i \subset A_i, i = 1,2,$ be irreducible divisors such that

$$\mathrm{St}(D_i) = \{0\} \qquad \text{(for simplicity)}.$$

For real-valued functions $\phi(r)$ and $\psi(r)$ $(r > 1)$, we write $\phi(r) \leq \psi(r)||_E$ if there is a Borel subset $E \subset [1, \infty)$ such that $m(E) < \infty$, and $\phi(r) \leq \psi(r), r \notin E$. We set

$$\phi(r) \sim \psi(r)|| \iff \exists E, \exists C > 0,\ C^{-1}\phi(r) \leq \psi(r) \leq C\phi(r)||_E.$$

Main Theorem 2.1. ([4]) *Let $f_i : \mathbf{C} \to A_i$ $(i = 1,2)$ be non-degenerate holomorphic curves. Assume that*

$$\underline{\mathrm{Supp}\ f_1^* D_1}_{\infty} \subset \underline{\mathrm{Supp}\ f_2^* D_2}_{\infty} \quad (\textit{germs at } \infty), \tag{2.2}$$

and

$$N_1(r, f_1^* D_1) \sim N_1(r, f_2^* D_2)||. \tag{2.3}$$

Here $N_1(r, f_1^* D_1) = N(r, \mathrm{Supp}\ f_1^* D_1))$. Then there is a finite étale morphism $\phi : A_1 \to A_2$ such that

$$\phi \circ f_1 = f_2, \quad D_1 \subset \phi^* D_2.$$

If equality holds in (2.2), *then ϕ is an isomorphism and $D_1 = \phi^* D_2$.*

N.B. Assumption (2.3) is necessary (see Example below).

The following corollary follows immediately from the Main Theorem 2.1.

Corollary 2.4. (i) *Let $f : \mathbf{C} \to \mathbf{C}^*$ and $g : \mathbf{C} \to E$ with an elliptic curve E be holomorphic and non-constant. Then*

$$\underline{f^{-1}\{1\}}_{\infty} \neq \underline{g^{-1}\{0\}}_{\infty}.$$

(ii) *If $\dim A_1 \neq \dim A_2$ in the Main Theorem 2.1, then*

$$\underline{f_1^{-1} D_1}_{\infty} \neq \underline{f_2^{-1} D_2}_{\infty}.$$

N.B.

(i) The first statement means that the difference of the value distribution property caused by the quotient $\mathbf{C}^* \to \mathbf{C}^*/\langle\tau\rangle = E$ cannot be recovered by any later choice of f and g, even though they are allowed to be *arbitrarily transcendental.*

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{f} & \mathbf{C}^* \\ & \searrow_{g} & \downarrow /\langle\tau\rangle \\ & & E \end{array}$$

(ii) The second statement implies that the distribution of $f_i^{-1}D_i$ about ∞ contains the topological informations such as $\dim A_i$ and the compactness or non-compactness of A_i. It is already interesting to observe that this works even for one parameter subgroups with Zariski dense image.

Example. Set $A_1 = \mathbf{C}/\mathbf{Z}$ ($\cong \mathbf{G}_{\mathrm{m}}$) and let $D_1 = 1$ be the unit element of A_1. Let $f_1 : \mathbf{C} \to A_1$ be the covering map. Take a number $\tau \in \mathbf{C}$ with $\Im\tau \neq 0$. Set $A_2 = \mathbf{C}/(\mathbf{Z}+\mathbf{Z}\tau)$, which is an elliptic curve. Let $D_2 = 0 \in A_2$ and $f_2 : \mathbf{C} \to A_2$ be the covering map.

Then $f_1^{-1}D_1 = \mathbf{Z} \subset \mathbf{Z} + \tau\mathbf{Z} = f_2^{-1}D_2$: assumption (2.2) of the Main Theorem 2.1 is satisfied. There is, however, no non-constant morphism $\phi : A_1 \to A_2$. Note that

$$N_1(r, f_1^*D_1) \sim r, \quad N_1(r, f_2^*D_2) \sim r^2.$$

Thus, $N_1(r, f_1^*D_1) \not\sim N_1(r, f_2^*D_2)||$: assumption (2.3) fails.

3 SFT for semi-abelian varieties

For a closed subscheme $Z \subset X$ of a compact complex space X and an entire holomorphic curve $f : \mathbf{C} \to X$, $f(\mathbf{C}) \not\subset \mathrm{Supp}\, Z$, we write

$$\begin{aligned} T_f(r, \omega_Z) &= \int_1^r \tfrac{dt}{t} \int_{\Delta(t)} f^*\omega_Z, \\ \underline{f^*Z}_{k,a} &= \min\{\mathrm{ord}_a f^*Z, k\} \ (k \leq \infty), \\ N_k(r, f^*Z) &= \int_1^r \tfrac{dt}{t} \left(\textstyle\sum_{a\in\Delta(t)} \underline{f^*Z}_{k,a}\right), \\ N(r, f^*Z) &= N_\infty(r, f^*Z) < T_f(r, \omega_Z) + O(1). \end{aligned}$$

The last equation is referred as Nevanlinna's inequality which is a direct consequence of the First Fundamental Theorem (FFT), also called First Main Theorem (FMT). The FFT for holomorphic curves into complex algebraic varieties is established (cf. [9])

Let A be a semi-abelian variety, and let $f : \mathbf{C} \to A$ be an entire holomorphic curve. Set

- $J_k(A) \cong A \times \mathbf{C}^{nk}$: the k-jet bundle over A;
- $J_k(f) : \mathbf{C} \to J_k(A)$: the k-jet lift of f;
- $X_k(f)$: the Zariski closure of the image $J_k(f)(\mathbf{C})$ in $J_k(A)$.

The following is the SFT for holomorphic curves into semi-abelian varieties.

Theorem 3.1. (Nog.-Winkelmann-Yamanoi, Acta 2002 & [9] & Yamanoi Forum Math. 2004) *Let $f : \mathbf{C} \to A$ be algebraically non-degenerate.*

(i) *Let Z be an algebraic reduced subvariety of $X_k(f)$ ($k \geqq 0$). Then there exists a compactification $\bar{X}_k(f)$ of $X_k(f)$ such that*

$$T_{J_k(f)}(r; \omega_{\bar{Z}}) = N_1(r; J_k(f)^*Z) + o(T_f(r))||. \tag{3.2}$$

(ii) *Moreover, if* $\operatorname{codim}_{X_k(f)} Z \geqq 2$, *then*

$$T_{J_k(f)}(r;\omega_{\bar{Z}}) = o(T_f(r))||. \tag{3.3}$$

(iii) *If* $k = 0$ *and* Z *is an effective reduced divisor* D *on* A, *then* $\bar{A}$ *is smooth, equivariant, and independent of* f; *furthermore,* (3.2) *takes the form*

$$T_f(r; L(\bar{D})) = N_1(r; f^*D) + o(T_f(r; L(\bar{D})))||. \tag{3.4}$$

4 Proof of the Main Theorem

Let me first recall

Theorem 4.1. (Log Bloch-Ochiai, Nog. 1977 Hiroshima Math.J./ '81 Nagoya Math. J.) *Let* $f : \mathbf{C} \to A$ *be an entire holomoprhic curve into a semi-abelian variety* A. *Then the Zariski closure* $\overline{f(\mathbf{C})}^{\mathrm{Zar}}$ *is a translate of a subgroup.*

Proof of Main Theorem 2.1. With the given $f_i : \mathbf{C} \to A_i$ $(i = 1, 2)$ we set $g = (f_1, f_2) : \mathbf{C} \to A_1 \times A_2$. Then $A_0 = \overline{g(\mathbf{C})}^{\mathrm{Zar}}$ is a semi-abelian variety by the above Log Bloch-Ochiai's Theorem; $p_i : A_0 \to A_i$ be the projections; $E_i = p_i^* D_i$. It follows that

$$T_{f_1}(r) \sim T_{f_2}(r) \sim T_g(r) = T(r).$$

By Nog. Math. Z. (1998) and a translation we may assume $g(0) = 0 \in E_1$. Let $E_i = \sum_\nu (F_i + a_{i\nu})$ be the irreducible decomposition and $F_i \ni 0$.

If $F_1 \neq F_2$, then $\operatorname{codim}_{A_0} F_1 \cap F_2 \geq 2$. It follows from our SFT, Theorem 3.1 that

$$T(r) \sim N_1(r, f_1^* D_1) \sim N_1(r, g^*(F_1 \cap F_2)) = o(T(r))||.$$

This is a contradiction. Therefore we see that $F_1 = F_2$. Moreover, we deduce that

(i) $E_1 \subset E_2$,

(ii) $\mathrm{St}(E_1) \subset \mathrm{St}(E_2)$, and they are finite,

(iii) p_i are isogenies,

(iv) $A_1 \cong A_0/\mathrm{St}(E_1) \xrightarrow{\phi} A_0/\mathrm{St}(E_2) \cong A_2$.

□

5 Arithmetic Recurrences

Due to the well-known correspondence between Number Theory and Nevanlinna Theory, it is tempting to give a number-theoretic analogue of Theorem 2.1 as Pál Erdös Problem–Schinzel-Corrales-Rodorigáñez&Schoof Theorem.

A related problem asks to classify the cases where $x^n - 1$ divides $y^n - 1$ for infinitely many positive integers n. The natural generalization to several variables is represented by Pisot's problem, asking to characterize the pairs of linear recurrent sequences $(n \mapsto \mathbf{f}_1(n)), (n \mapsto \mathbf{f}_2(n))$ such that $\mathbf{f}_1(n)$ divides $\mathbf{f}_2(n)$ for every integer n (or for infinitely many integers n).

We would like to deal with the case of a semi-abelian variety with a given divisor, i.e., a polarized semi-abelian variety. As it often happens, the complex-analytic theory is more advanced, and we dispose only of partial results in the *number theoretic case.* In the present situation, we can prove an analogue of the Main Theorem 2.1 only in the *linear toric case*, but not in the general case of *semi-abelian varieties*, that is left to be a *Conjecture*. Here is our result in the number theoretic case.

Theorem 5.1. ([4]) *Let $\mathcal{O}_S$ be a ring of S-integers in a number field k. Let $\mathbf{G}_1$, $\mathbf{G}_2$ be linear tori, let $g_i \in \mathbf{G}_i(\mathcal{O}_S)$ be elements generating Zariski-dense subgroups, and let D_i be reduced divisors defined over k, with defining ideals $\mathcal{I}(D_i)$, such that each irreducible component has a finite stabilizer and* $\mathrm{St}(D_2) = \{0\}$.

Suppose that for infinitely many $n \in \mathbf{N}$,

$$(g_1^n)^*\mathcal{I}(D_1) \supset (g_2^n)^*\mathcal{I}(D_2). \tag{5.2}$$

Then there exist an étale morphism $\phi : \mathbf{G}_1 \to \mathbf{G}_2$, defined over k, and $h \in \mathbf{N}$ such that $\phi(g_1^h) = g_2^h$ and $D_1 \subset \phi^*(D_2)$.

N.B.

(i) Theorem 5.1 is deduced from the main results of Corvaja-Zannier, Invent. Math. 2002.

(ii) By an example we cannot take $h = 1$ in general.

(iii) By an example, the condition on the stabilizers of D_1 and D_2 cannot be omitted.

(iv) Note that inequality (inclusion) (5.2) of ideals is assumed only for an infinite sequence of n, not necessarily for all large n. On the contrary, we need the inequality of ideals, not only of their *supports*, i.e. of the primes containing the corresponding ideals.

(v) One might ask for a similar conclusion assuming only the inequality of supports. There is some answer for it, but it is of a weaker form.

6 1-parameter Analytic Subgroups

In S. Lang's "Introduction to Transcendental Numbers", Addison-Wesley, 1966, he wrote at the last paragraph of Chap. 3

"Independently of transcendental problem one can raise an interesting question of algebraic-analytic nature, namely given a 1-parameter subgroup of an abelian variety (say Zariski dense), is its intersection with a hyperplane section necessarily non-empty, and infinite unless this subgroup is algebraic?"

In 6 years later, J. Ax (Amer. J. Math. (1972)) took this problem:

Theorem 6.1. *Let θ be a reduced theta function on $\mathbf{C}^m$ with respect to a lattice $\Gamma \subset \mathbf{C}^m$. Let L be a 1-dimensional affine subspace of $\mathbf{C}^m$. Then either $(\theta|L)$ is constant or has an infinite number of zeros; $|\{(\theta|L) = 0\} \cap \Delta(r)| \sim r^2$.*

N.B. In the talk at Kyoto I spoke that it seemed to be still open that $|\{(\theta|L) = 0\}/\Gamma| = \infty$ unless $f(\mathbf{C})$ is algebraic. Later on, I found that it is not difficult to deduce the infinity of $|\{(\theta|L) = 0\}/\Gamma|$ for non-algebraic g from the growth estimate in Theorem 6.1, once it is noticed:

Proof. We necessarily assume $m \geq 2$. Let $\phi : \mathbf{C} \to \mathbf{C}^m/\Gamma$ be a 1-parameter subgroup with dense Zariski image, and $D = \{\theta = 0\}/\Gamma$. If $\phi(\mathbf{C}) \cap D$ is finite, then there would be a point $a_0 \in D$ such that $\limsup_{r\to\infty} |\{z \in \Delta(r); \phi(z) = a_0\}|/r^2 > 0$. By translation we may assume $a_0 = 0$. Since ϕ is a group homomorphism and $\operatorname{Ker}\phi$ is discrete, $\operatorname{Ker}\phi$ had to be a lattice of $\mathbf{C}$ (with compact quotient). Thus ϕ would be factored through an elliptic curve (Contradiction). $\square$

By making use of our SFT, Theorem 3.1 we are able to obtain a more exact growth estimate and detailed geometric property of the intersection $f(\mathbf{C}) \cap D$.

Theorem 6.2. *Let $f : \mathbf{C} \to A$ be a 1-parameter analytic subgroup in a semi-abelian variety A with $v = f'(0)$. Let D be a reduced divisor on A.*

(i) *If A is abelian and $H(\cdot,\cdot)$ denotes the Riemann form associated with D, then we have*

$$N(r; f^*D) = H(v,v)\pi r^2 + O(\log r),$$
$$= (1 + o(1))N_1(r; f^*D).$$

(ii) *Assume that $\dim A \geq 2$. Let f be an arbitrary algebraically non-degenerate holomorphic curve and assume that $\mathrm{St}(D)$ is finite. Then there is an irreducible component D' of D such that then $f(\mathbf{C}) \cap D'$ is Zariski dense in D'; in particular, $|f(\mathbf{C}) \cap D| = \infty$.*

Proof. (i) Note that the first Chern class $c_1(L(D))$ is represented by $i\partial\bar{\partial}H(w,w)$. It follows from our SFT Theorem 3.1 that

$$\begin{aligned} N(r; f^*D) &= T_f(r; L(D)) + O(\log r) \\ &= \int_0^r \frac{dt}{t} \int_{\Delta(t)} iH(v,v)dz \wedge d\bar{z} + O(\log r) \\ &= H(v,v)\pi r^2 + O(\log r) \\ &= (1+o(1))N_1(r, f^*D). \end{aligned}$$

(ii) If the claim does not hold, there exists an algebraic subset E such that $f(\mathbf{C}) \cap D \subset E \subsetneq D$ and $\operatorname{codim}_A E \geq 2$. Then our SFT Theorem 3.1 yields that

$$N(r, f^*E) = o(r^2) = N(r, f^*D) \sim r^2|| \text{ (contradiction)}.$$

□

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