数理解析研究所講究録
第 1710 巻 2010 年 93-100

The arithmetic function in 3 complex variables closely related to L-functions of global fields *

Yasutaka Ihara †

1 Introduction

First, recall that the Riemann zeta function $\zeta(s)$ has the Euler product expansion

$$\zeta(s)=\prod_{p}\zeta_{p}(s) \tag{1.1}$$

on $\mathrm{Re}(s)>1$, where

$$\zeta_{p}(s)=(1-p^{-s})^{-1}, \tag{1.2}$$

and also the Riemann-Hadamard decomposition

$$\zeta(s)=\epsilon(s)^{-1}\prod_{\rho}\left(1-\frac{s}{\rho}\right)e^{\frac{s}{\rho}}, \tag{1.3}$$

where $\epsilon(s)$ is of the form $s(s-1)e^{Bs}\Gamma(s/2)$ and ρ runs over all non-trivial zeros of $\zeta(s)$. As is well-known, comparison of the two decompositions (1.1) and (1.3) leads to various identities connecting "$\{p\}$" with "$\{\rho\}$".

The function in the title, called $\tilde{M}(s;z_1,z_2)$, in which complex powers of $\zeta(2s)$ are comprised, also has two types of product decompositions, each of which having some common features with *both* (1.1) and (1.3) (see §6). Let us recall the definition. First, the local factor $\tilde{M}_p(s;z_1,z_2)$ for each prime p. Consider the power series expansion in p^{-s} of the complex x-th power of $\zeta_p(s)$:

$$\zeta_{p}(s)^{x}=(1-p^{-s})^{-x}=1+\sum_{n=1}^{\infty}a_{n}(x)p^{-ns}, \tag{1.4}$$

$$a_{n}(x)=(x)_{n}=\frac{x(x+1)\cdots(x+n-1)}{n!}. \tag{1.5}$$

It is convenient to use the complex variables z_1, z_2 defined by

$$x_{\nu}=iz_{\nu}/2 \qquad (\nu=1,2), \tag{1.6}$$

*For details, cf. [3] which is a continuation of [1] and of the joint articles [5, 6] with K. Matsumoto (cf. also [2, 4, 7]).

†RIMS, Kyoto University (P.E.), Kitashirakawa-Oiwakecho, Kyoto 606-8502, Japan ihara@kurims.kyoto-u.ac.jp

where $i = \sqrt{-1}$. Then $\tilde{M}_p(s; z_1, z_2)$ is defined by

$$\tilde{M}_p(s; z_1, z_2) = 1 + \sum_{n=1}^{\infty} a_n(x_1)a_n(x_2)p^{-2ns} = F(x_1, x_2; 1; p^{-2s}), \qquad (1.7)$$

where

$$F(a, b; c; t) = 1 + \frac{a.b}{1.c}t + \frac{a(a+1)b(b+1)}{1.2c(c+1)}t^2 + \cdots \qquad (1.8)$$

($|t| < 1$) denotes the Gauss hypergeometric series. It is clear that $\tilde{M}_p(s; z_1, z_2)$ is a holomorphic function of s, z_1, z_2 on $\mathrm{Re}(s) > 0$, symmetric in z_1, z_2. The zero divisor of $\tilde{M}_p(s; z_1, z_2)$ is non-trivial (see below §5). The global holomorphic function $\tilde{M}(s; z_1, z_2)$ of s, z_1, z_2 on the domain $\mathrm{Re}(s) > 1/2$ is defined by

$$\tilde{M}(s; z_1, z_2) = \prod_p \tilde{M}_p(s; z_1, z_2) \qquad (1.9)$$

which is absolutely convergent in the following sense. Fix any $\sigma_0 > 1/2$ and $R > 0$. Then $|\tilde{M}_p(s; z_1, z_2) - 1| < 1$ holds on $\mathrm{Re}(s) \geq \sigma_0$ and $|z_1|, |z_2| \leq R$ for almost all p (depending on σ_0, R), and the sum of $\log \tilde{M}_p(s; z_1, z_2)$ (the principal branch) over these p is absolutely convergent; thus $\tilde{M}(s; z_1, z_2)$ is defined as the product of finitely many local factors and the exponential of a holomorphic function on this domain. In particular, the zero divisor of $\tilde{M}(s; z_1, z_2)$ is the sum of those of local factors. Note that $\tilde{M}(s; -2i, -2ix) = \zeta(2s)^x$ ($x \in \mathbf{C}$).

This function $\tilde{M}(s; z_1, z_2)$ has a Dirichlet series expansion on $\mathrm{Re}(s) > 1/2$ whose coefficients are polynomials of z_1, z_2, formally arising from the Euler product expansion (1.9). It is absolutely convergent also as Dirichlet series on the same domain. We recall [6]§4 that (again for $\mathrm{Re}(s) > 1/2$) it has an everywhere absolutely convergent power series expansion in z_1, z_2:

$$\tilde{M}(s; z_1, z_2) = 1 + \sum_{a,b \geq 1} \mu^{(a,b)}(s)\frac{x_1^a x_2^b}{a!b!} = 1 - \frac{1}{4}\mu(s)z_1 z_2 + \cdots, \qquad (1.10)$$

where each $\mu^{(a,b)}(s)$ is a certain Dirichlet series and

$$\mu(s) = \mu^{(1,1)}(s) = \sum_p (\sum_{n=1}^{\infty} \frac{1}{n^2 p^{2ns}}). \qquad (1.11)$$

2 Connection with the value-distribution of the logarithm of Dirichlet L-functions (Review of joint work with K. Matsumoto)

We briefly recall the connection. For details, cf. [5, 6] and/or the survey article [7]. Our function $\tilde{M}(s; z_1, z_2)$ is denoted there as $\tilde{M}_s(z_1, z_2)$, and is a special case corresponding to "Case 2 over $\mathbf{Q}$" of [6]. A similar but different function corresponding to "Case 1" is related to the value-distribution of the logarithmic derivative of L-functions, which was defined and studied earlier [1, 2]. While in [1, 6] the base field K is any global field, here, we restrict our attention to $K = \mathbf{Q}$.

The connection may be formulated in two ways. The first formulation asserts that for each $x_\nu = iz_\nu/2 \in \mathbf{C}$ $(\nu = 1, 2)$,

$$\mathrm{Avg}_\chi \left(\overline{L(s,\chi)}^{x_1} L(s,\chi)^{x_2} \right) = \tilde{M}(\sigma; z_1, z_2) \tag{2.1}$$

holds on $\sigma = \mathrm{Re}(s) > 1/2$ under some assumption, where χ runs over all Dirichlet characters with prime conductors and Avg_χ denotes some average. The assumption is that either (i) $z_2 = \bar{z}_1$ with the average in a weaker sense [5], or (ii) under GRH (the Generalized Riemann Hypothesis) with the average in a stronger sense [6]. Let $\sigma > 1/2$ and

$$M_\sigma(w) = \int_{\mathbf{C}} \tilde{M}(\sigma; z, \bar{z}) e^{-i\mathrm{Re}(\bar{z}w)} |dz| \tag{2.2}$$

($|dz| = dxdy/2\pi$ for $z = x + iy$) denote the inverse Fourier transform of $\tilde{M}(\sigma, z, \bar{z})$. Then $M_\sigma(w)$ is a non-negative real valued continuous (in fact, C^∞-) function on $\mathbf{C}$ satisfying

$$\int_{\mathbf{C}} M_\sigma(w)|dw| = 1; \tag{2.3}$$

hence it can be regarded as a density function on $\mathbf{C}$. It also satisfies

$$\int_{\mathbf{C}} M_\sigma(w) w |dw| = 0, \tag{2.4}$$

that is, the center of gravity is the origin $\mathbf{0}$. Its variance μ_σ is

$$\begin{aligned} \mu_\sigma &= \int_{\mathbf{C}} M_\sigma(w)|w|^2|dw| \\ &= \mu(\sigma) > 0, \end{aligned} \tag{2.5}$$

$\mu(s)$ being the Dirichlet series (1.11). Now the second formulation of the connection reads as follows. The equality

$$\mathrm{Avg}_\chi \Phi(\log L(s,\chi)) = \int_{\mathbf{C}} M_\sigma(w)\Phi(w)|dw| \tag{2.6}$$

holds on $\sigma = \mathrm{Re}(s) > 1/2$ under either the above assumption (i) with Φ any continuous and bounded function on $\mathbf{C}$ [5], or (ii) with Φ any continuous function on $\mathbf{C}$ of at most exponential growth [6]. The case where Φ is the characteristic function of a compact set can be included into each case. The equality (2.1) is a special case of (2.6) where $\Phi(w) = e^{x_1\bar{w} + x_2 w}$.

As for connection with the (more classical) value-distribution theory for $\{\log \zeta(\sigma + it)\}_{t \in \mathbf{R}}$, cf. [5]§1 or [7].

3 Limit behaviors at $s = 1/2$ (cf. [3]§1,§2)

It is natural to pay attention to the "variance-normalized" function

$$M_\sigma^\star(w) = \mu_\sigma M_\sigma(\mu_\sigma^{1/2} w) \tag{3.1}$$

which has the variance $= 1$ and the Fourier transform

$$\tilde{M}_\sigma^\star(z) = \tilde{M}(\sigma; \mu_\sigma^{-1/2} z, \mu_\sigma^{-1/2} \bar{z}). \tag{3.2}$$

As in [2, 3], consider the Plancherel volume

$$\nu_\sigma := \int_{\mathbf{C}} M_\sigma(w)^2 |dw| = \int_{\mathbf{C}} |\tilde{M}(\sigma; z, \bar{z})|^2 |dz|. \tag{3.3}$$

The product $\mu_\sigma \nu_\sigma$, which may be expressed as

$$\mu_\sigma \nu_\sigma = \int_{\mathbf{C}} M_\sigma^\star(w)^2 |dw| = \int_{\mathbf{C}} |\tilde{M}_\sigma^\star(z)|^2 |dz|, \tag{3.4}$$

is an interesting object of study. By analytic reasons, we always have $\mu_\sigma \nu_\sigma \geq 8/9$ cf. [2, 3].

Theorem (09A) *As* $s \to 1/2 + 0$,

$$\mu(s)/\log \frac{1}{2s-1} \quad \to \quad 1, \tag{3.5}$$

$$\tilde{M}(s; \mu(s)^{-1/2} z_1, \mu(s)^{-1/2} z_2) \quad \to \quad \exp(-z_1 z_2/4); \tag{3.6}$$

in particular,

$$\tilde{M}_\sigma^\star(z) \to \exp(-|z|^2/4). \tag{3.7}$$

The convergences in (3.6)(3.7) are uniform in the wider sense. These follow from the special case $N = 1$ of Theorem (09B') below. By combining (3.7) with the following rapid decay property of $|\tilde{M}_\sigma(z)|$: *for any* $0 < \epsilon < 1$, *if* $(2\sigma - 1)^{-1} \gg_\epsilon 1$ *then the inequality*

$$|\tilde{M}(\sigma; z, \bar{z})|^2 \leq \exp\left(-\frac{1-\epsilon}{2} \mu_\sigma |z|^{2(1-\epsilon)}\right) \tag{3.8}$$

holds for all $z \in \mathbf{C}$ ([3]§4 Theorem 7C), we obtain

Theorem (09A') *As* $\sigma \to 1/2 + 0$,

$$M_\sigma^\star(w) \quad \to \quad 2\exp(-|w|^2), \tag{3.9}$$

$$\mu_\sigma \nu_\sigma \quad \to \quad 1. \tag{3.10}$$

A private discussion with S. Takanobu was very helpful in obtaining the formula (3.9) in this general form. As for the proofs, and for the limits as $\mathrm{Re}(s) \to +\infty$, cf. [3].

4 Analytic continuation (cf. [3]§3)

Put

$$\mathcal{D} = \{\mathrm{Re}(s) > 0;\ s \neq \frac{1}{2n}, \frac{\rho}{2n};\ \rho : \textit{nontrivial zeros of}\ \zeta(s),\ n \in \mathbf{N}\}. \quad (4.1)$$

Theorem (09B) *$\tilde{M}(s; z_1, z_2)$ extends to a multivalent analytic function on $\mathcal{D} \times \mathbf{C}^2$.*

This means that it extends to an analytic function on $\tilde{\mathcal{D}} \times \mathbf{C}^2$, where $\tilde{\mathcal{D}}$ is the universal covering of $\mathcal{D}$. Actually, $\tilde{\mathcal{D}}$ can be replaced by the maximal unramified abelian covering of $\mathcal{D}$. Let

$$\ell(t) = -\log(1-t) = t + \frac{1}{2}t^2 + \cdots, \quad (4.2)$$

and $P_n(x_1, x_2)$ $(n = 1, 2, \cdots)$ be the polynomial of degree $\leq n$ in each variable defined by the formal power series equality

$$\log F(x_1, x_2; 1;\ t) = \sum_{n=1}^{\infty} P_n(x_1, x_2)\ell(t^n). \quad (4.3)$$

Then a more descriptive account of Theorem (09B) reads as follows.

Theorem (09B')

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{P_n(x_1, x_2)} \quad (4.4)$$

holds in the following sense; (i) for any $N \geq 0$, the quotient of $\tilde{M}(s; z_1, z_2)$ by the partial product over $n \leq N$ on the right hand side extends to a holomorphic function on $\mathrm{Re}(s) > 1/(2N+2)$*; (ii) the equality (4.4) holds on $|z_1|, |z_2| \leq R$ and* $\mathrm{Re}(s) \geq \sigma_0 > 1/2$*, provided that either R is fixed and σ_0 is sufficiently large, or σ_0 is fixed and R is sufficiently small.*

We have $P_1(x_1, x_2) = x_1 x_2$ and $P_2(x_1, x_2) = -x_1 x_2 (x_1 - 1)(x_2 - 1)/4$. Note that (4.3) already gives the "formal local version"

$$\log \tilde{M}_p(s; z_1, z_2) = \sum_{n=1}^{\infty} P_n(x_1, x_2) \log \zeta_p(2ns) \quad (4.5)$$

of (4.4). To prove the global analytic equality (4.4), we need to justify the commutativity of summations over p and those over the exponents of p^{-2s}, x_1, x_2. This follows from suitable estimations of various summands. If z_1, z_2 are fixed and s encircles a punctured point $s_0 \in \{\mathrm{Re}(s) > 0\} \setminus \mathcal{D}$ in the positive direction, and if, say, s_0 can be expressed in just one way as $s_0 = \rho/2n$ with some $n \geq 1$ and with a simple zero ρ of $\zeta(s)$, then the function $\tilde{M}(s; z_1, z_2)$ is multiplied by

$$\exp(2\pi i P_n(x_1, x_2)).$$

5 Zeros of $\tilde{M}(s; z_1, z_2)$ ([3]§0.4)

One can prove that the zero divisor of the analytic continuation of $\tilde{M}(s; z_1, z_2)$ on $\tilde{\mathcal{D}} \times \mathbf{C}^2$ is well-defined as a divisor on $\mathcal{D} \times \mathbf{C}^2$, and that it is simply the (locally finite) sum over p of the zero divisor of $\tilde{M}_p(s; z_1, z_2)$. The zero divisor of the local factor

$$\tilde{M}_p(s; z_1, z_2) = F(x_1, x_2; 1; t^2) \tag{5.1}$$

($z_\nu = ix_\nu/2$, $t = t_p = p^{-s}$) is smooth because of the Gauss differential equation. Its property has not been analyzed systematically. But the intersection with the hyperplane defined by $x_1 + x_2 = 0$ can be analyzed as follows. For $|t| < 1$, consider the "locally normalized" function

$$f_t(x) = F(x/(2\arcsin(t)), -x/(2\arcsin(t)); 1; t^2). \tag{5.2}$$

Then $f_0(x) = J_0(x)$, the Bessel function of order 0. Let $\pm\{\gamma_\nu\}_{\nu=1}^{\infty}$ with $0 < \gamma_1 < \gamma_2 < \cdots$ denote all the zeros of $J_0(x)$, so that $\gamma_\nu \in ((\nu - 1/2)\pi, \nu\pi)$. Then we can prove that there exists $0 < t_0 < 1$ such that for $|t| \leq t_0$, (i) each γ_ν extends uniquely and holomorphically to a zero $\gamma_\nu(t)$ of $f_t(x)$ satisfying $\mathrm{Re}(\gamma_\nu(t)) \in ((\nu - 1/2)\pi, \nu\pi)$ and $|\mathrm{Im}(\gamma_\nu(t))| < 1$, and (ii) there are no other zeros of $f_t(x)$. These lead directly to the Weierstrass decomposition

$$f_t(x) = \prod_{\nu=1}^{\infty} \left(1 - \frac{x^2}{\gamma_\nu(t)^2}\right) \tag{5.3}$$

of $f_t(x)$; hence we obtain the second infinite product decomposition

$$\tilde{M}(s; z, -z) = \prod_p \prod_{\nu=1}^{\infty} \left(1 + \left(\frac{\arcsin(p^{-s})}{\gamma_\nu(p^{-s})}\right)^2 z^2\right) = \prod_{\mu=1}^{\infty} (1 + \theta_\mu(s)^2 z^2) \tag{5.4}$$

of $\tilde{M}(s; z, -z)$, $\{\theta_\mu(s)\}_\mu$ being a reordering of $\{\arcsin(p^{-s})/\gamma_\nu(p^{-s})\}_{p,\nu}$ according to the absolute values. (Here, in order to assure that each $\gamma_\nu(p^{-s})$ makes clear sense, we need to assume that $\mathrm{Re}(s)$ is sufficiently large. On the other hand, (5.3) itself holds for each fixed t if we simply let $\pm\gamma_\nu(t)$ denote all the zeros of $f_t(x)$. So, (5.4) remains valid for each fixed s with $\mathrm{Re}(s) > 1/2$ after suitable modifications of local factors for small p's. We might add here that $\lim_{t\to 1} f_t(x) = \sin x/x$.)

We shall indicate here the main ingredients for the proofs of the above statements on the zeros of $f_t(x)$, in order to supplement [3]§0.4 and explain why $\arcsin(t)$ should appear. First we need:

Key lemma A *The function $f_t(x)$ admits a Neumann series expansion*

$$\sum_{n=0}^{\infty} a_{2n}(t) J_{2n}(x), \tag{5.5}$$

where $J_{2n}(x)$ is the Bessel function of order $2n$, and $a_{2n}(t)$ is a holomorphic function of t^2 on $|t| < 1$ divisible by t^{2n}, with $a_0(t) = 1$ and $a_{2n}(t) \ll |t|^{2n}$, with $\ll$ independent of n (depending only on the compact subdomain of $|t| < 1$ considered).

To prove this lemma, we may assume that t is positive real. Then the key parameter $\arcsin(t)$ appears as the maximal value of $|\mathrm{Arg}(1 - te^{-i\theta})|$ for $\theta \in \mathbf{R}/2\pi$. By using the new argument θ' defined via

$$\mathrm{Arg}(1 - te^{-i\theta})/\arcsin(t) = \sin\theta', \tag{5.6}$$

we may express $f_t(x)$ as

$$\begin{aligned} f_t(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\sin\theta'}(d\theta/d\theta')d\theta' \\ &= \frac{2}{\pi}\int_0^{\pi/2} K_\tau(\theta')\cos(x\sin\theta')\cos(\tau\sin\theta')d\theta', \end{aligned} \tag{5.7}$$

where $\tau = \arcsin(t)$ and

$$\begin{aligned} K_\tau(\theta') &= \frac{\tau\cos\theta'}{\sqrt{\sin^2\tau - \sin^2(\tau\sin\theta')}} \\ &= \sum_{\mu=0}^{\infty} \alpha_{2\mu}(\tau)\cos(2\mu\theta'), \end{aligned} \tag{5.8}$$

with $\alpha_{2\mu}(\tau)$ given explicitly and divisible by $\tau^{2\mu}$. We thus obtain

$$f_t(x) = \frac{1}{2}\sum_{\mu=0}^{\infty} \alpha_{2\mu}(\tau)(J_{2\mu}(x+\tau) + J_{2\mu}(x-\tau)), \tag{5.9}$$

from which follows the lemma by the addition formula for Bessel functions.

By this lemma, $f_t(x)$ and $df_t(x)/dx$ are "close to" $J_0(x)$ and $-J_1(x)$ (respectively) of which the asymptotic behaviors away from zeros are well-understood ([8]§7.21). A quantitative closeness is guaranteed by:

Key lemma B

$$|J_n(x)| \ll_{abs.} (n+1)^{1/2}|x|^{-1/2}e^{|\mathrm{Im}(x)|} \qquad (n = 0, 1, 2, \cdots; x \in \mathbf{C}). \tag{5.10}$$

This proof is parallel to that of Lemma 3.3.4 of [1] which was for $x \in \mathbf{R}$; just replace $J_n(x)$ there by $e^{-|\mathrm{Im}(x)|}J_n(x)$.

6 Comparisons

We thus have two decompositions related to $\tilde{M}(s; z_1, z_2)$: The first one

$$\tilde{M}(s; z_1, z_2) = \prod_{n=1}^{\infty} \zeta(2ns)^{P_n(x_1, x_2)} \tag{6.1}$$

is similar to the Riemann-Hadamard decomposition (1.3) of $\zeta(s)$ in the sense that it is related to analytic continuation with respect to s, but is similar to the Euler product decomposition (1.1) of $\zeta(s)$ in the sense that it tells us nothing about the zeros. The second,

$$\tilde{M}(s;z,-z)=\prod_{p}\prod_{\nu=1}^{\infty}\left(1+\left(\frac{\arcsin(p^{-s})}{\gamma_\nu(p^{-s})}\right)^2 z^2\right)=\prod_{\mu=1}^{\infty}(1+\theta_\mu(s)^2z^2), \quad (6.2)$$

is similar to (1.1) in the sense that it is firstly the product over p, but in the sense that it is the Weierstrass decomposition according to its zeros, it is similar to (1.3). It is still mysterious, but we hope that the comparison of these two decompositions will bring us some new insight.

References

[1] Y. Ihara, On "M-functions" closely related to the distribution of L'/L-values, Publ. RIMS Kyoto Univ. **44** (2008), 893-954.

[2] Y. Ihara, Some density functions and their Fourier transforms arising from number theory, in "Number Theory and Probability Theory", H. Sugita (ed.), RIMS Kôkyûroku, Kyoto Univ., 2008, pp.28-42.

[3] Y. Ihara, On certain arithmetic functions $\tilde{M}(s;z_1,z_2)$ associated with global fields: Analytic properties, RIMS Preprint 1685 (2009).

[4] Y. Ihara and K. Matsumoto, On L-functions over function fields: Power-means of error-terms and distribution of L'/L-values, in "Algebraic Number Theory and Related Topics", H.Nakamura et al. (eds.), RIMS Kôkyûroku Bessatsu, to appear.

[5] Y. Ihara and K. Matsumoto, On certain mean values and the value-distribution of logarithms of Dirichlet L-functions, The Quarterly J. Math. 2010; doi: 10.1093/qmath/haq002 .

[6] Y. Ihara and K. Matsumoto, On $\log L$ and L'/L for L-functions and the associated "M-functions": Connections in optimal cases, RIMS Preprint 1667 (2009).

[7] Y. Ihara and K. Matsumoto, On the value distribution of $\log L$ and L'/L, in "New Directions in Value-Distribution Theory of Zeta and L-Functions", J. Steuding et al. eds., (Proceedings of the Würzburg Conference 2008), Shaker Publication, (2009) 85-97.

[8] G.N. Watson, Theory of Bessel functions, Second Edition, Cambridge at the University Press (1962).