Higher depth regularized products and
zeta functions of Milnor type*

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1 Introduction

For a complex sequence $a = \{a_n\}_{n \in I}$, the (zeta) regularized product of $a$ is defined by

$$
\prod_{n \in I} a_n := \exp\left( \frac{d}{ds} \zeta_a(s) \bigg|_{s=0} \right),
$$

where $\zeta_a(s) := \sum_{n \in I} a_n^{-s}$ is the zeta function attached to $a$. Here, we assume that $\zeta_a(s)$ converges absolutely in some right half plane, admits a meromorphic continuation to some region containing the origin and is holomorphic at the origin. This gives a kind of generalization of the usual product. In fact, if $a$ is a finite sequence, then one can see that $\prod_{n \in I} a_n = \prod_{n \in I} a_n$. The most important and fundamental example of the regularized product is the following Lerch formula;

$$
\prod_{n \geq 0} (n + z) = \exp\left( \frac{d}{ds} \zeta(s, z) \bigg|_{s=0} \right) = \frac{\sqrt{2\pi}}{\Gamma(z)},
$$

where $\Gamma(z)$ is the gamma function and $\zeta(s, z) := \sum_{n \geq 0} (n + z)^{-s}$ is the Hurwitz zeta function. In particular, letting $z = 1$, we have $\prod_{n \geq 1} n (= \infty!) = \sqrt{2\pi}$. Notice that, if $\prod_{n \in I} (a_n + z)$ exists, then, as a function of $z$, it defines an entire function whose zeros are located at $z = -a_n$ for $n \in I$.

Let $\zeta(s) := \sum_{n \geq 1} n^{-s}$ be the Riemann zeta function and $\mathcal{R}$ the set of all non-trivial zeros of $\zeta(s)$. The following formula was obtained by Deninger [D, Theorem 3.3] (see also [SS, V]);

$$
\Xi(z) := \prod_{\rho \in \mathcal{R}} \left( \frac{z - \rho}{2\pi} \right) = 2^{-\frac{1}{2}} \pi^{-\frac{3}{2}} \pi^{-\frac{1}{2}} \Gamma\left( \frac{z}{2} \right) \zeta(z) z(z - 1) = \frac{1}{2\pi^2} \Lambda(z),
$$

where $\Lambda(z) := \frac{1}{2} z(z - 1) \Gamma(\frac{z}{2}) \zeta(z)$ is the complete Riemann zeta function. The aim of this note is to give “higher depth” generalizations of the formula (1.2) for Hecke $L$-functions. Namely, we explicitly calculate “higher depth regularized products” for the zeros of Hecke $L$-functions.

We here explain the higher depth regularized products above. In [Mi], from the viewpoint of the Kubert identity which plays an important role in the study of Iwasawa theory, Milnor introduced a “higher depth gamma function” $\Gamma_r(z)$ defined by

$$
\Gamma_r(z) := \exp\left( \frac{d}{ds} \zeta(s, z) \bigg|_{s=1-r} \right)
$$

and studied, for examples, special values, a Stirling formula (that is, an asymptotic formula as $z \to +\infty$) and functional relations among them (see also [KOW]). Notice that, by the Lerch

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formula (1.1), we have \( \Gamma_1(z) = \frac{\Gamma(z)}{\sqrt{2\pi}} \), whence \( \Gamma_r(z) \) indeed gives a generalization of \( \Gamma(z) \). Based on the study of Milnor, we define a higher depth (or depth \( r \)) regularized product of the sequence \( a \) by

\[
\prod_{n \in I}^{[r]} a_n := \exp \left( -\frac{d}{ds} \zeta_a(s) \big|_{s=1-r} \right),
\]

where we further assume that \( \zeta_a(s) \) admits a meromorphic continuation to some region containing \( s = 1 - r \) and is holomorphic at the point. It is clear that the case \( r = 1 \) reproduces the usual regularized product; \( \prod_{n \in I}^{[1]} a_n = \prod_{n \in I} a_n \). Note that it can be written as \( \Gamma_r(z)^{-1} = \prod_{n \geq 0} (n + z)^1 \).

To state our main result, let us recall Hecke \( L \)-functions. Let \( K \) be an algebraic number field of degree \( n \) and of discriminant \( d_K \), \( \mathcal{O}_K \) the ring of integers of \( K \), and \( \tau_1 \) and \( \tau_2 \) the number of real and complex places of \( K \), respectively. Let \( \chi \) be a Hecke grössencharacter with conductor \( f \) and

\[
L_K(s; \chi) := \prod_{p} \left( 1 - \frac{\chi(p)}{N(p)^s} \right)^{-1} = \sum_a \frac{\chi(a)}{N(a)^s} \quad (\text{Re}(s) > 1)
\]

the Hecke \( L \)-function associate with \( \chi \). Here, \( p \) runs over all prime ideals of \( \mathcal{O}_K \) and \( a \) over all integral ideals of \( \mathcal{O}_K \) (we understand that \( \chi(p) = 0 \) if \( p \) and \( f \) are not coprime). It is well known that \( L_K(s; \chi) \) admits a meromorphic continuation to the whole complex plane \( \mathbb{C} \) with a possible simple pole at \( s = 1 \) and has a functional equation \( \Lambda_K(1-s; \overline{\chi}) = W_K(\chi)\Lambda_K(s; \chi) \) where \( W_K(\chi) \) is a constant with \( |W_K(\chi)| = 1 \) and \( \Lambda_K(s; \chi) \) is the entire function defined by

\[
\Lambda_K(s; \chi) := \left( \frac{1}{2}(s-1) \right)^{\epsilon} \left( \frac{N(f)|d_{A'}|}{2^{2r}} \right)^{\frac{1}{2}} L_{K}(s; \chi) \prod_{v \in S_{\infty}(K)} \Gamma \left( \frac{N_v(s+i\varphi_v)+|\iota r_{v}|}{2} \right)
\]

Here, \( S_{\infty}(K) \) is the set of all archimedean places of \( K \), \( \varepsilon = 1 \) if \( \chi \) is principal and 0 otherwise. Moreover, for \( v \in S_{\infty}(K) \), \( N_v = 1 \) if \( v \) is real and 2 otherwise, and \( \varphi_v = \varphi(\chi) \in \mathbb{R} \) with \( \sum_{v \in S_{\infty}(K)} N_v \varphi_v = 0 \) and \( m_v = m(\chi) \in \mathbb{Z} \) are uniquely determined by

\[
\chi(\alpha) = \prod_{v \in S_{\infty}(K)} |\alpha_v|^{-tN_v \varphi_v} \left( \frac{\alpha_v}{|\alpha_v|} \right)^{m_v} \quad (\alpha \in \mathcal{O}_K \text{ with } \alpha \equiv 1 \ mod \chi \text{ and } f),
\]

where \( \text{mod } \chi \) indicates the multiplicative congruence and \( \alpha_v \) is the image of \( \alpha \) with respect to the embedding \( K \hookrightarrow K_v \) with \( K_v = \mathbb{R} \) or \( \mathbb{C} \). We remark that, if \( \varphi_v = m_v = 0 \) for all \( v \in S_{\infty}(K) \), then \( \chi \) is called a class character.

Let \( \mathcal{R}_K(\chi) \) be the set of all non-trivial zeros of \( L_K(s; \chi) \) and \( \xi_K(s, z; \chi) \) the zeta function attached to the sequence \( \{ \frac{z-r}{2\pi} \}_{r \in \mathcal{R}_K(\chi)} \) that is,

\[
\xi_K(s, z; \chi) := \sum_{\rho \in \mathcal{R}_K(\chi)} \left( \frac{z-r}{2\pi} \right)^{-s} \quad (\text{Re}(s) > 1, \text{Re}(z) > 1).
\]

Moreover, let

\[
\Xi_{K,r}(z; \chi) := \prod_{\rho \in \mathcal{R}_K(\chi)} \left( \frac{z-r}{2\pi} \right)^{[r]} = \exp \left( -\frac{d}{ds} \xi_K(s, z; \chi) \big|_{s=1-r} \right).
\]

Remark that, when \( \text{Re}(z) > 1 \), the function \( \Xi_{K,r}(z; \chi) \) can be defined because it will be shown that \( \xi_K(s, z; \chi) \) admits a meromorphic continuation to the whole plane \( \mathbb{C} \) as a function of \( s \) and, in particular, is holomorphic at \( s = 1 - r \) for any \( r \in \mathbb{N} \) (Proposition 2.2). Now our main result is given as follows.

\[1\] For \( r \geq 2 \), if \( \prod_{n \in I}^{[r]} (a_n + z) \) exists, then it defines in general a multivalued function with branch points at \( z = -a_n \) for \( n \in I \). See [KWY] for more precise discussions. In particular, \( \Gamma_r(z) \) is a multivalued function with branch points at \( z = -n \) for \( n \geq 0 \) or defines a holomorphic function in \( \mathbb{C} \setminus \{-\infty, 0\} \).

\[2\] From now on, the sum \( \sum_{\rho \in \mathcal{R}_K(\chi)} \) means \( \lim_{T \to \infty} \sum_{\rho \in \mathcal{R}_K(T; \chi)} \) where \( \mathcal{R}_K(T; \chi) := \{ \rho \in \mathcal{R}_K(\chi) \ | \ \text{Im}(\rho) < T \} \).
**Theorem 1.1.** For $\text{Re}(z) > 1$, it holds that

\[
\Xi_{K,r}(z; \chi) = \left(\frac{z}{2\pi}\right)^{\epsilon_{r}(\frac{z}{2\pi})^{-1}} \left(\frac{z-1}{2}\right)^{\epsilon_{1}(\frac{z}{2\pi})^{-1}} L_{K}^{(r)}(z; \chi)^{(-1)}(r-1)(2\pi)^{1-r},
\]
\[
\times \prod_{v \in \mathcal{S}_{\infty}(K)} \left(1 - \frac{v}{N_{v}}\right)^{B_{r}(\frac{v}{N_{v}})} \Gamma_{r}\left(\sum_{v \in \mathcal{S}_{\infty}(K)} \left|\frac{v}{N_{v}}\right|^{r}\right)
\]

Here, $B_{r}(z)$ is a poly-Hecke $L$-function of degree $r$. Letting $B_{r}(z) := \sum_{n=1}^{\infty} \frac{\lambda_{r,n}}{n^{r}}$, we have the following Euler product;

\[
L_{K}^{(r)}(s; \chi) := \prod_{p} H_{r}\left(\frac{\chi(p)}{N(p)^{s}}\right)^{-(\log N(p))^{1-r}} \quad (\text{Re}(s) > 1),
\]

where $H_{r}(z) := \exp(-L_{r}(z))$ with $L_{r}(z) := \sum_{m=1}^{\infty} \frac{m^{r}}{m^{s}}$ being the polylogarithm of degree $r$.

We call $L_{K}^{(r)}(s; \chi)$ a "poly-Hecke $L$-function" of degree $r$. Remark that this is a generalization of $L_{K}(s; \chi)$. Actually, since $L_{1}(1) = -\log(1 - z)$ and hence $H_{1}(1) = 1 - z$, we have $L_{K}^{(1)}(s; \chi) = L_{K}(s; \chi)$. Some analytic properties of this new "$L$" function are given in the last section.

As a corollary of this theorem, letting $r = 1$ with noting that $B_{1}(z) = z - \frac{1}{2}$, $\Gamma_{1}(z) = \sqrt{2\pi}$ and $L_{K}^{(1)}(z; \chi) = L_{K}(z; \chi)$, we obtain the following regularized product expressions of Hecke $L$-functions.

**Corollary 1.2.** It holds that

\[
\prod_{\rho \in \mathcal{R}_{K}(\chi)} \left(\frac{z - \rho}{2\pi}\right) = \frac{(N(\rho)|d_{K}|)^{-\frac{1}{2}}}{2\pi^{1/2} + \frac{1}{2}m_{\rho}^{2}2z_{\rho} + m}, \quad \Lambda_{K}(z; \chi),
\]

where $\varphi_{\rho} := \sum_{v \in \mathcal{S}_{\infty}(K)} |m_{v}|$, and $m := \sum_{v \in \mathcal{S}_{\infty}(K)} |m_{v}|$. In particular, if $\chi$ is a class character, that is, $\chi_{v} = v = 0$ for all $v \in \mathcal{S}_{\infty}(K)$, then we have

\[
\prod_{\rho \in \mathcal{R}_{K}(\chi)} \left(\frac{z - \rho}{2\pi}\right) = \frac{(N(\rho)|d_{K}|)^{-\frac{1}{2}}}{2\rho_{1}^{1/2} + \frac{1}{2}m_{\rho}^{2}2\rho_{1}} \Lambda_{K}(z; \chi).
\]

Furthermore, letting $\chi = 1$ (of course $1$ is a class character) and writing $\zeta_{K}(s) := L_{K}(s; 1)$, that is, $\zeta_{K}(s)$ is the Dedekind zeta function of $K$, $\mathcal{R}_{K} := \mathcal{R}_{K}(1)$ and $\Lambda_{K}(s) := \Lambda_{K}(s; 1)$ in (1.7), respectively, one obtains the regularized product expression of the Dedekind zeta function.

**Corollary 1.3.** It holds that

\[
\prod_{\rho \in \mathcal{R}_{K}} \left(\frac{z - \rho}{2\pi}\right) = \frac{|d_{K}|^{-\frac{1}{2}}}{2\rho_{1}^{1/2} + \frac{1}{2}} \Lambda_{K}(z).
\]
2 Sketch of the proof of Theorem 1.1

In this section, we give a brief proof of Theorem 1.1. Remark that the proof is completely based on that of the equation (1.2) due to Deninger [D]. To do that, we first recall the Weil explicit formula refined by Barner [Ba]. For a function $F$ of bounded variation (i.e., $V_{\mathbb{R}}(F) < +\infty$ where $V_{\mathbb{R}}(F)$ is the total variation of $F$ on $\mathbb{R}$), we define the function $\Phi_{F}(s)$ ($s \in \mathbb{C}$) by

$$\Phi_{F}(s) := \int_{-\infty}^{\infty} F(x) e^{(s-\frac{1}{2})x} dx.$$ 

Moreover, for a Hecke character $\chi$ and $v \in S_{\infty}(K)$, we put $F_{v}(x; \chi) := F(x) e^{-i\varphi_{v}x}$. Then, the Weil explicit formula is given as follows.

**Lemma 2.1 ([Ba, Theorem 1]).** Let $\chi$ be a Hecke character and $F : \mathbb{R} \rightarrow \mathbb{C}$ a function of bounded variation satisfying the following three conditions$^3$:

(a) There is a positive constant $b$ such that $V_{\mathbb{R}}(F(x) e^{(\frac{1}{2}+b)|x|}) < +\infty$.

(b) $F$ is “normalized”, that is, $2F(x) = F(x+0) + F(x-0)$ ($x \in \mathbb{R}$).

(c) For any $v \in S_{\infty}(K)$, it holds that $F_{v}(x; \chi) + F_{v}(-x; \chi) = 2F(0) + O(|x|)$ as $|x| \rightarrow 0$.

Then, the following equation holds:

$$(2.1) \sum_{\rho \in \mathcal{R}_{K}(\chi)} \Phi_{F}(\rho) = \epsilon_{\chi}(\Phi_{F}(0) + \Phi_{F}(1)) + F(0) \log \frac{N(f)|d_{K}|}{2\pi} \approx -\sum_{p} \sum_{l=1}^{\infty} \frac{\log N(\mathfrak{p})}{N(\mathfrak{p})^{\frac{l}{2}}} (\chi(\mathfrak{p}^{l})F(\log N(\mathfrak{p})^{l}) + \overline{\chi}(\mathfrak{p}^{l})F(-\log N(\mathfrak{p})^{l})) + \sum_{v \in S_{\infty}(K)} W_{v}(F; \chi),$$

where

$$W_{v}(F; \chi) := \int_{0}^{\infty} \left( \frac{N_{v}F(0)}{x} - (F_{v}(x; \chi) + F_{v}(-x; \chi)) e^{\frac{2\cdot|m_{v}|}{N_{v}}} \right) e^{-\frac{x}{N_{v}}} dx.$$ 

For $\text{Re}(z) > 1$ and $\text{Re}(s) > 1$, let

$$F(z) := \begin{cases} x^{s-1}e^{-(s-\frac{1}{2})x} & (x \geq 0), \\ 0 & (x < 0). \end{cases}$$

Then, one can easily check that the function $F(z)$ satisfies the conditions (a), (b) and (c) in Lemma 2.1 and see that $\Phi_{F}(w) = \frac{\Gamma(s)}{(s-w)^{s}}$, whence $\Phi_{F}(0) = \frac{\Gamma(s)}{z^{s}}$ and $\Phi_{F}(1) = \frac{\Gamma(s)}{(z-1)^{s}}$. Therefore, using the explicit formula (2.1) with this $F$ (together with the integral representations of $\zeta(s, z)$ and the gamma function), we obtain the following expression of $\xi_{K}(s, z; \chi)$.

**Proposition 2.2.** For $\text{Re}(z) > 1$, we have

$$(2.2) \xi_{K}(s, z; \chi) = \epsilon_{\chi} \left( \frac{2\pi}{z} \right)^{s} + \left( \frac{2\pi}{z-1} \right)^{s} + \frac{(2\pi)^{s}}{2\pi i} \int_{L_{-}}^{L_{+}} \frac{L_{K}^{t}(z-t; \chi)}{L_{K}} t^{-s} dt - \sum_{v \in S_{\infty}(K)} (N_{v}^{s}) \zeta \left( s, \frac{N_{v}(z+i\varphi_{v}) + |m_{v}|}{2} \right).$$

$^3$These are called the “Barner conditions".
where $L_-$ is the contour consisting of the lower edge of the cut from $-\infty$ to $-\delta$, the circle $t = \delta e^{i\psi}$ for $-\pi \leq \psi \leq \pi$ and the upper edge of the cut from $-\delta$ to $-\infty$. This gives a meromorphic continuation of $\xi_K(s, z; \chi)$ as a function of $s$ to the whole plane $\mathbb{C}$ with a simple pole at $s = 1$. \hfill \square

As stated below, the theorem is obtained by directly calculating the derivatives of $\xi_K(s, z; \chi)$ at $s = 1 - r$ from the expression (2.2).

**Proof of Theorem 1.1.** Write $\xi_K(s, z; \chi) = A_1(s, z) + A_2(s, z) + A_3(s, z)$ where

$$
A_1(s, z) := \varepsilon_{\chi}\left(\frac{2\pi}{z}^s + \frac{2\pi}{z-1}^s\right),
$$

$$
A_2(s, z) := \frac{(2\pi)^s}{2\pi i} \int_{L_-} \frac{L'_K}{L_K}(z-t; \chi)t^{-s}dt,
$$

$$
A_3(s, z) := - \sum_{v \in \mathcal{S}_{\infty}(K)} (N_v\pi)^s \zeta(s, \frac{N_v(z+i\varphi_v) + |m_v|}{2}).
$$

At first, it is easy to see that

$$
-\frac{d}{ds}A_1(s, z)\big|_{s=1-r} = \varepsilon_{\chi}\left(\frac{z}{2\pi}\right)^{r-1}\log \frac{z}{2\pi} + \varepsilon_{\chi}\left(\frac{z-1}{2\pi}\right)^{r-1}\log \frac{z-1}{2\pi}.
$$

The derivative of $A_2(s, z)$ at $s = 1 - r$ is calculated as

$$
-\frac{d}{ds}A_2(s, z)\big|_{s=1-r} = \frac{(2\pi)^{1-r}}{2\pi i} \int_{L_-} \frac{L'_K}{L_K}(z-t; \chi)t^{r-1}\log \frac{t}{2\pi}dt
$$

$$
= (-1)^r (2\pi)^{1-r} \int_0^\infty \frac{L'_K}{L_K}(z+x; \chi)x^{r-1}dx
$$

$$
= (-1)^{r-1}(r-1)!(2\pi)^{1-r}\log L_K^{(r)}(z; \chi).
$$

In the second equality, we have calculated the integral by dividing the contour $L_-$ into three parts; $L_- = (-\infty e^{-\pi i}, -\delta e^{-\pi i}) \cup \{\delta e^{i\psi} | -\pi \leq \psi \leq \pi\} \cup (-\infty e^{\pi i}, -\delta e^{\pi i})$ (and letting $\delta \to 0$) and, in the last equality, we have used the formula

$$
\frac{L'_K}{L_K}(z; \chi) = - \sum_p \sum_{l=1}^\infty \log N(p) \cdot \chi(p)^l \cdot N(p)^{-lz} \quad (\text{Re}(z) > 1)
$$

and the Euler product expression (1.6) of the poly-Hecke $L$-function $L_K^{(r)}(z; \chi)$. Finally, using the well-known formula $\zeta(1-r, z) = -\frac{B_r(z)}{r}$, we have

$$
-\frac{d}{ds}A_3(s, z)\big|_{s=1-r}
$$

$$
= - \sum_{v \in \mathcal{S}_{\infty}(K)} (N_v\pi)^{1-r} \left[ \frac{\log(N_v\pi)}{r} B_r\left(\frac{N_v(z+i\varphi_v) + |m_v|}{2}\right) - \log \Gamma_r\left(\frac{N_v(z+i\varphi_v) + |m_v|}{2}\right) \right].
$$

Combining these three equations, one obtains the desired result. \hfill \square

### 3 Poly-Hecke $L$-functions

The poly-Hecke $L$-functions, which are naturally appeared in the derivatives of the zeta function $\xi_K(s, z; \chi)$ at non-negative integer points, are mysterious functions at this moment. They are defined by the Euler product (1.6) and, as we have seen before, give generalizations of Hecke $L$-functions. Therefore one may expect that they satisfy similar properties which so-called $L$- or zeta functions.
have, for example, a meromorphic continuation, a functional equation and a "Riemann hypothesis". In this section, as a closing remark, we give an analytic continuation of $L_K^{(r)}(s; \chi)$ for $r \geq 2$ to (not the whole plane $\mathbb{C}$ but) an infinitively many slitted region in $\mathbb{C}$.

Let $\Omega_K(\chi)$ be the set of all complex numbers which are not of the form of $\rho - \lambda$ where $\rho$ is a trivial or a non-trivial zero of $L_K(s; \chi)$ or, if $\chi$ is principal, $1 - \lambda$ for $\lambda \geq 0$ (we show the region $\Omega_K(\chi)$ in Figure 1 in the case where $\chi$ is a principal character). Notice that, from the expression (1.4), all trivial zeros of $L_K(s; \chi)$ are given by $-\frac{i\pi N(p)^{1-r}}{N} - i\varphi_v$ where $v \in S_{\infty}(K)$ and $l \in \mathbb{Z}_{\geq 0}$.

![Figure 1: The region $\Omega_K(\chi)$ (if $\chi$ is principal)](image)

Now let $r \geq 2$. From the differential equation $\frac{d}{dz} \tilde{L}_{K}^{(r)}(z) = \frac{1}{r} \tilde{L}_{K}^{(r-1)}(z)$ of the polylogarithm, one can see that the poly-Hecke $L$-function $L_K^{(r)}(s; \chi)$ satisfies the differential equation

$$\frac{d^{r-1}}{ds^{r-1}} \log L_K^{(r)}(s; \chi) = (-1)^{r-1} \log L_K(s; \chi) \quad (\text{Re}(s) > 1).$$

Using this formula, by induction on $r$, we obtain the following result.

**Theorem 3.1.** Let $\text{Re}(a) > 1$. Then, we have

$$L_K^{(r)}(s; \chi) = Q_K^{(r)}(s, a) \exp \left( \int_{a}^{s} \int_{a}^{x_{r-1}} \cdots \int_{a}^{x_{1}} \log L_K(x_{1}; \chi) d\xi_{1} \cdots d\xi_{r-1} \right)^{-1}. $$

Here $Q_K^{(r)}(s, a) := \prod_{k=0}^{r-2} L_K^{(r-k)}(a; \chi)^{-1} \left(1 - a \right)^{k+1} \left(1 - \frac{1}{2}(s-a)^k \right)$ and the path for each integral is contained in $\Omega_K(\chi)$. The expression gives an analytic continuation of $L_K^{(r)}(s; \chi)$ to the region $\Omega_K(\chi)$. □

It seems to be difficult to continue $L_K^{(r)}(s; \chi)$ to the whole plane $\mathbb{C}$ as a single-valued holomorphic (or meromorphic) function. In fact, from an easy observation, one can prove the following

**Corollary 3.2.** The extended Riemann hypothesis for $L_K(s; \chi)$ is equivalent to say that the function $(s-1)^{-\epsilon} (s-1)^{L_K^{(2)}(s; \chi)}$ is single-valued and holomorphic in $\text{Re}(s) > \frac{1}{2}$. □

**Remark 3.3.** Let

$$\tilde{L}_K^{(r)}(s; \chi) := \prod_{p} H_r \left( \frac{\chi(p)}{N(p)^{s}} \right)^{-1} \quad (\text{Re}(s) > 1)$$

(recall that $L_K^{(r)}(s; \chi) := \prod_{p} H_r \left( \frac{\chi(p)}{N(p)^{s}} \right)^{1-r}$). Then we have $\tilde{L}_K^{(1)}(s; \chi) = L_K(s; \chi)$, whence $\tilde{L}_K^{(r)}(s; \chi)$ also gives a generalization of $L_K(s; \chi)$. It does not, however, seem to have an analytic continuation to the whole plane $\mathbb{C}$. In fact, in [KW], it was shown that $\tilde{L}_K^{(r)}(s; \chi)$ has an analytic continuation to the region $\text{Re}(s) > 0$ but has a natural boundary at $\text{Re}(s) = 0$. □
References


