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WORD PROBLEMS FOR SEMIGROUPS *

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Duncan and Gilman introduced the concept "word problems" for finitly generated semigroups in terms of formal langauges. In this paper, we show that finitly generated semigroups with a regular [resp. context-free] word problem has a has a presentation with regular [resp. context-free] congruence classes.

1. Word problems and semigroups with a presentation with spcial congruence classes

We recall regular, context-fee and one counter languages (see [4]).

Definition 1 . *Let X be finite alphabets. X^* is the set of all words over X. X^+ is the set of all non-empty words over X. A subset of X^* is called a* **language** *over X*

Definition 2 . *An automaton is defined as a 5-tuple:*

$M = (Q, X, \delta, q_0, F)$ where Q is a finite set of states, q_0 is the start state, X is a finite set of the input alphabets, $\delta : Q \times X_\epsilon \to Q$ is the transition function, $X_\epsilon = X \cup \{\epsilon\}$, $F \subset Q$ is the set of accepting states.

For q_0 and $w = x_1x_2 \cdots x_{n-1}x_n$ $(\in X^*)$, let $q = \delta(\cdots\delta(\delta(q_0, x_1), x_2)\cdots), x_n)$. Then we define $\delta(q_0, w) = q$.

The accepted language by the M

$$L(M) = \{w \in \Sigma \mid \delta((q_0, w) = q \ (q \in F)\}$$

is called a regular language.

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*This is an absrtact and the paper will appear elsewhere.

Definition 3 . *A Pushdown automaton is defined as a 7-tuple:*

$M = (Q, X, \Gamma, \delta, q_0, Z_0, F)$ *where* Q *is a finite set of states,* q_0 *is the start state,* X *is a finite set of the input alphabets,* Γ *is a finite set of the stack alphabets,* Z_0 *is the initial stack symbol,* $\delta : Q \times X_\epsilon \times \Gamma_\epsilon \to P(Q \times X_\epsilon)$ *is the transition function, where* $P(S)$ *denotes the power set of* S*,* $X_\epsilon = X \cup \{\epsilon\}$*,* $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$*.* $F \subset Q$ *is the set of accepting states.*

For $q \in Q$*,* $x \in X_\epsilon, w \in X^*)$ *and* $Z \in \Gamma_\epsilon, \alpha \in \Gamma^*$*, we define a relation* $\Rightarrow_M$ *by* $(q, xw, Z\alpha) \Rightarrow_M (p, w, \beta\alpha')$ *if* $(p, \beta) \in \delta((q, x, Z))$*. The transitive closure of the relation* $\Rightarrow_M$ *is written as a relation* $\overset{*}{\Rightarrow}_M$*. The accepted language by* M

$$L(M) = \{w \in \Sigma \mid (q_0, w, Z_0) \overset{*}{\Rightarrow}_M (q, \epsilon, \epsilon) \ (q \in F)\}$$

is called a contex-free language.

Definition 4 . *A one counter machine is a pushdown automaton with a single stack alphabet. The accepted language by the one counter machine is called a one counter language.*

Definition 5 . ([7]) *A semigroup* S *has a presentation with regular [resp. context-free] congruence classes if there exists a finite set* X *and there exists a surjective homomorphism* ϕ *of* X^+ *to* M *such that for each word* $w \in X^+$*,* $\phi^{-1}(\phi(w))$ *is a regular [resp. context-free] language.*

Let X be finite alphabets. For a word $w = x_1 x_2 \cdots x_{n-1} x_n \in X^*$, $x_n x_{n-1} \cdots x_2 x_1$ is called the *reverse* of w denoted by w^R.

Definition 6 . ([2]) *Let* S *be a finitely generated semigroup, there exist finite alphabets* X *and there exists a surjective homomorphism* ϕ *of* X^+ *to* S*.*

Then

(1) *for each* $w \in X^+$*, the set* $\{w\sharp v^R \mid \phi(w) = \phi(v)\}$ *is called a word problem for the semigroup* S*.*

(2) S *has a regular [* resp. *regular context-free] word problem if the set* $\{w\sharp v^R \mid \phi(w) = \phi(v)\}$ *is a regular [* resp. *context-free] language over* $X \cup \{\sharp\}$*.*

Theorem 1 . *A finitly generated semigroup with a regular [resp. context-free, one counter] word problem has a presentation with regular [resp. context-free, one counter] congruence classes.*

2. Regular word problems and semigroups with a presentation with regular congruence classes

The property of that a semigroup has a regular word problem is very strong. Actually, the following is known.

Result 1 . ([3], Proposition 3.1). *A semigroup S has a regular word problem if only if S is finite.*

On the other hand, we have

Definition 7 . *A semigroup S is called residually finite if for each pair of elements $s, t \in S$ there is a homomorphism ϕ of S such that $\phi(S)$ is a finite semigroup and $\phi(s)$ and $\phi(t)$ are not equal.*

Result 2 . ([7], Theorem 5) *If a finitely generated semigroup S has a presentation with regular congruence classes, then S is residually finite.*

Result 3 . ([3], Proposition 4.2) *If S is a finitely generated semigroup with a one counter word problem, then there exist finitely many elements $a_i, b_i, c_i \in S \cup \{\epsilon\}$ such that every element of S is represented by a word of the form $a_i b_i^n c_i$ for some i and some $n \geq 0$.*

3. Context-free word problems and semigroups with a presentation with context-free congruence classes

Definition 8 . *Let L be a language over A. The syntactic congruence σ_L on A^+ is defined by $w\sigma_L w'$ if and only if*

$$\{(x, y) \in A^+ \times A^+ \mid xwy \in L\} = \{(x, y) \in A^+ \times A^+ \mid xw'y \in L\}.$$

Then the factor semigroup A^+/σ_L is called syntactic semigroup of L denoted by $Syn(L)$.

Lemma 1 . ([7]). *Let S be a finitely generated semigroup and ϕ a surjective homomorphism of A^+ to S.*

For an element s of S, let $L = \phi^{-1}(s)$.

Define a congruence $\sigma_s = \{(a, b) \in S \times S \mid xay = s$ if and if $xby = s$ $(x, y \in S^1)\}$ which is called a syntacticcongruence on S.

Then the syntactic semigroup A^+/σ_L of L is isomorphic to the factor semigroup S/σ_s of S. (S/σ_s is called the syntactic semigroup of S at s.)

Theorem 2 . *A semigroup S has a presentation with context-free* [resp. *one counter*] *congruence classes if and only if for each $s \in S$, the syntactic semigroup S/σ_s has context-free* [resp. *one counter*] *congruence classes.*

Example 1 . *Let $A = \{a, b\}$ and a context-free language $L = \{a^n b^n, b^n a^n | n \in \mathbb{N}\}$. Then all of σ_L-classes are $\overline{ab} = \{a^p b^p | p \in \mathbb{N}\}$, $\overline{a^n} = \{a^n\}$, $\overline{b^n} = \{b^n\}$, $\overline{a^{n+1}b} = \{a^{p+n}b^p | p \in \mathbb{N}\}$, $\overline{ab^{n+1}} = \{a^q b^{q+n} | q \in \mathbb{N}\}$, $\overline{ba} = \{b^p a^p | p \in \mathbb{N}\}$, $\overline{ba^{n+1}} = \{b^p a^{p+n} | p \in \mathbb{N}\}$, $\overline{b^{n+1}a} = \{b^{q+n} a^q | q \in \mathbb{N}\}$. $0 = A^*bA^*aA^*bA^* \cup A^*aA^*bA^*aA^*$. Then*

(1) *L is a one counter language.*

(2) *$Syn(L)$ has a linear growth function. Actually, $\left(|\overline{A^n}| \leq 6n\right)$.*

(3) *$Syn(L)$ have a context-free word problem but does not have a one counter word problem.*

(4) *$Syn(L)$ has a presentation with one counter congruence classes.*

Example 2 . *Let $A = \{a, b\}$ and $L = \{a^n b^m a^m b^n | m, n \in \mathbb{N}\}$. Then all of σ_L-classes are $\overline{a^i} = \{a^i\}$ $(i \in \mathbb{N})$, $\overline{b^i} = \{a^i\}$ $(i \in \mathbb{N})$, $\overline{a^i b^j} = \{a^i b^j\}$ $(i, j \in \mathbb{N})$, $\overline{b^i a^j} = \{b^i a^j\}$ $(i, j \in \mathbb{N})$, $\overline{a^i b^j a^k} = \{a^i b^j a^k\}$ $(i, j, k \in \mathbb{N})$, $\overline{b^i a^j b^k} = \{b^i a^j b^k\}$ $(i, j, k \in \mathbb{N})$, $\overline{a^i b^j a^j b} = \{a^i b^j a^j b\}$ $(i, j \in \mathbb{N})$, $\overline{ab^j a^j b^i} = \{ab^j a^j b^i\}$ $(i, j \in \mathbb{N})$. $0 = \cup_{i \neq j} A^* ab^i a^j bA^*$.*

Then

(1) *L is a context-free language.*

(2) *$Syn(L)$ does not have a presentation with context-free congruence classes.*

Example 3 . *Let* $A = \{a, b, c\}$ *and* $L = \{a^n b^n c^n | n \in N\}$. *Then all of* σ_L*-classes are* $\{1\}$, $\overline{a^q b^r} = \{a^q b^r\}$, $(q, r \in \mathbb{N})$, $\overline{b^q c^r} = \{b^q c^r\}$, $(q, r \in \mathbb{N})$, $\overline{ab^q c^r} = \{a^i b^{q+i} c^{r+i} | q, r, i \in \mathbb{N}, 2 \leq r \leq q\}$, $\overline{a^p b^q c} = \{a^{p+i} b^{q+i} c^i | p, q, i \in \mathbb{N}, 2 \leq p \leq q\}$, $\overline{abc} = \{a^n b^n c^n | n \in \mathbb{N}\}$, $0 = X^*baX^* \cup X^*cbX^* \cup X^*caX^* \cup X^*acX^* \cup \{a^p b^q c^r | p, q, r \in \mathbb{N}, \text{either } q \leq p \text{ or } q \leq r\}$. *Then*

(1) *Syn*(L) *has a polynomial growth function. Actually,* $\left(|\overline{A^n}| = \dfrac{n(n+1)}{2} \right)$.

Therefore,

(2) *Syn*(L) *does not have a onecunter word problem.*

(3) *Syn*(L) *has a presentation with one counter congruence classes.*

Finally we pose the following two open problems.

Problem 1 . *Characterize a finitely generated semigroup* S *with a one counter word problem.*

Problem 2 . *Characterize a finitely generated semigroup* S *which has a presentation with one counter congruence classes.*

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