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Modification of Pre-closure Spaces as Closure/Interior Operations on the Lattice of Pre-closure Operators

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Abstract. Closure space formalisms appear in a wide range of mathematical theories providing a common conceptual framework for building models which require crossing of interdisciplinary borders. However, the task of building a closure space combining structures from different mathematical theories is more difficult than it could be expected. Classical example of such combination can be found in a closure space description of the topological algebra developed by Frink and Graetzer. Unfortunately, their approach does not allow for generalization being too much restricted by the specific properties of the structures involved. In the present paper generalization is achieved by the inclusion into consideration more general pre-closure operators, which form much richer structure. On the lattice of such operators modifications are considered as poset closure or interior operators. This allows for a generalization of the concept of operator modification opening way for the study of combinations of closure spaces with arbitrary properties.

1. Introduction

Direct motivation for the present paper comes from author's interest in the use of closure spaces in the modeling of information integration processing. For this purpose a generalized Venn gate has been proposed in which closed subsets of a closure space form input, while a subset of the space called a frame serve as output. [1]

One of main advantages in using closure spaces for modeling information integration follows from the fact that virtually all types of information processed by human brain have acquired formalization in mathematics in terms of such structures. Thus, we have closure space description of, for instance geometric, topological, logical characteristics of the carriers of information. However, this seemingly fortunate abundance of closure space formulations in mathematics does not solve automatically the problem of identifying structures suitable for modeling of the integration of actual information input. The issue is that mathematical formalisms are usually being developed with the interest in abstraction of particular type of characteristics.

For instance, the same domain of spatial relations has several closure space formalisms of geometry, convex geometry, and topology. Defining properties of these structures are contradictory, in the sense that only trivial closure space can have them all. For instance, the first and second above have so called finite character (closure of an infinite set can be obtained by unions of closures of finite subsets,) while the only additive closure (closure of the union of sets is the union of closures required in topology) having such property is the trivial one with all subsets closed. Thus, compounding properties of closure spaces describing different types of spatial relations does not produce the closure space which describes them all. Similar difficulty can be encountered when we want to combine geometric closure systems characterized by Steiner's exchange property with those of convex geometries characterized by anti-exchange property.

There is a legitimate question whether it is possible to find a closure space formalism unifying two other closure space formalisms of such contradictory characteristics. The answer is that there are several examples of such closure spaces, the most prominent is the closure space of topological algebra combining algebraic closure operator of finite character and topological closure. The original construction of the "topologization" of subalgebra closure space by Frink and Graetzer has utilized an expansion of the Moore family of closed subsets (subalgebras) by inclusion of the finite unions of these sets.[2]

Similar procedure of modifying a closure operator into one of finite character by expanding the Moore family of closed subsets by adding unions of all directed subfamilies has been used by Mayer-Kalkschmidt and Steiner. [3] These results have been generalized for the arbitrary type of modifications in the remote past by the author in his unpublished doctoral dissertation providing the foundation for the general description of the modifications in terms of closure/interior operations on the lattice of pre-closure operators presented here as an introduction to the methods allowing for combination of closure spaces with arbitrary properties. Material presented below is heavily dependent on author's results on the structure of the set of all pre-closure operators, where the conceptual framework is introduced and more detailed explanation going beyond needs of the present paper can be found.[4] In the following, it is assumed that the reader is familiar with the conventions, notation and standard concepts of the literature of the subject.

2. Preliminaries

Although our ultimate goal is to develop methods of combining closure spaces defined as sets equipped with a mapping f of their subsets called a closure operator which for all subsets A, B satisfies the conditions: $A \subseteq f(A)$, $A \subseteq B \Rightarrow f(A) \subseteq f(B)$, $f(f(A)) = f(A)$, we need more general conceptual framework of pre-closure operators, here called simply operators which assume only first two conditions. Operators satisfying the third condition are called transitive operators. F(S) stands for the set of all operators on a given set S, while I(S) for the set of all transitive operators.

We will consider sets of operators distinguished by some properties, for instance finitely additive operators well known from topology satisfying: $f(A \cup B) = f(A) \cup f(B)$. In such a case the name of the class of operators will consist of a capital letter possibly preceded by some small letters, in the case of the finite additivity the class of such operators will be called fA(S). Classes of operators with multiple properties are indicated by the concatenation of respective symbols in arbitrary order. So, the class of transitive, finitely additive operators is written as IfA(S).

The property of finite character defining class fC(S) mentioned in the introduction is formally expressed as: $\forall A \subseteq S \forall x \in S$: $x \in f(A) \Rightarrow \exists A_0 \in Fin(A)$: $x \in f(A_0)$, where Fin(A) is the set of all finite subsets of A.

One of the most basic tools in the analysis of operators is the partial order defined on F(S) by: $f \leq g$ if $\forall A \subseteq S$: $f(A) \subseteq g(A)$.

Portions of closure or pre-closure space theory have been developed at the higher level of generality where operators are defined not on subsets of a given set (i.e. on the Boolean algebra of its subsets,) but on a partially ordered set (poset). How much of the theory can be recovered in this more general context depends on the properties of the poset. Certainly, all theory can be recovered, if the poset is an atomic, complete Boolean lattice, as it can be represented as the structure of subsets of a set.

We will need some level of increased generality, as for our purpose the closure and interior operators will be considered on the partially ordered set of all operators F(S) defined above. However, this partial order defines surprisingly rich structure allowing the reconstruction of big portions of set theoretical closure space theory. It is this substantial enrichment of the structure defined on F(S) which motivated the author in his decision to go beyond the structure defined on the set of transitive operators I(S), whose poverty caused that in the fifty years after its original study by Ore, very little has been added.[5] Not only it is difficult to look for the tools in the properties of the partial ordering in I(S) (it is a lattice, but not modular,) but it is not closed with respect to composition of operators, as it is obvious that such composition fg of transitive operators f and g is transitive if and only if fg = gf. F(S) is a monoid compatible with its partial order (i.e. $f \leq g \Rightarrow fh \leq gh$ and $hf \leq hg$) with the unity operator defined by $\forall A \subseteq S$: $e(A) = A$, and additionally equipped with zero o defined by $\forall A \subseteq S$: $o(A) = S$.

Another reason to go beyond the structure of transitive operators is the existence of an involution on poset F(S) defined by the dual operator. To define it we have first to recall that there are several different but equivalent ways to define a structure of closure space, as it is done for instance in topology. Instead of defining a closure operator, in the case of transitive operators we can select its Moore family of closed subsets defined as a family of subsets which has all set as its member and which is closed with respect to arbitrary intersections. This way cannot be generalized to the case of not necessarily transitive operators, as many different operators may have identical family of closed subsets. However, we can always use instead of an operator f its associated interior operator Int_f related to f by $\forall A \subseteq S$: $Int_f(A) = [f(A^c)]^c$. For a mapping Int of subsets of the set S, to be an interior operation for some operator f (i.e. $Int = Int_f$) it is necessary and sufficient that for all subsets A, B of S, $Int(A) \subseteq A$ and $A \subseteq B \Rightarrow Int(A) \subseteq Int(B)$.

Yet another way to introduce a structure equivalent to that defined by an operator is using the concept of a derived set operator df familiar from topology: Let f be an operator on S. $\forall A \subseteq S$: $df(A) = A^{df} = \{x \in S: x \in f(A \setminus \{x\})\}$. Then $f(A) = A \cup A^{df}$. For a mapping d of subsets of S to be a derived set operator df for some operator f it is necessary and sufficient that d is monotone ($A \subseteq B \Rightarrow d(A) \subseteq d(B)$) and $\forall A \subseteq S \forall x \in S$: $x \in d(A \setminus \{x\})$ iff $x \in d(A)$.

Now, we can define for every operator f on S its dual operator f^* by $\forall A \subseteq S$: $f^*(A) = A \cup A^{cdfc}$, where $A^c = S \setminus A$.

It is easy to show that $f^{**} = f$ and $f \leq g$ *iff* $g^* \leq f^*$.

Using this duality of operators we can introduce the duality of classes of operators defined by their properties by $f \in X^*(S)$ *iff* $f^* \in X(S)$. One of the most surprising dualities of properties appeared in matroid theory $I^*(S) = E(S)$, where the class of operators E(S) is defined by the property: $\forall A,B \subseteq S \forall x,y \in S$: $x \notin f(A \setminus B)$ & $x \in f(A) \Rightarrow \exists y \in B$: $y \in f(A \setminus \{y\} \cup \{x\})$ which is slightly stronger than Steiner's exchange property defining wE(S):

$\forall A \subseteq S \forall x,y \in S$: $x \notin f(A)$ & $x \in f(A \cup \{y\}) \Rightarrow y \in f(A \cup \{x\})$, but which is equivalent with it in the presence of finite character, and therefore in finite closure spaces. Therefore a matroid or a geometry usually defined by an operator belonging to IfCwE(S), can be defined as member of IfCI*(S).

The existence of selfdual operators ($f = f^*$) on every set with more than one element shows that the involution is not an orthocomplementation in F(S).[4]

In the following there will be frequent use of the following theorem about the structure of operators on a set S [4]:

F(S) with the partial ordering of operators is a bounded, complete, completely distributive, atomistic and dual atomistic lattice L_F with an involution. The set of atoms $At(L_F)$ consists of transitive operators, each defined for a nonempty subset A of S and element x of S\A by:
$\forall B \subseteq S$: $f_{A,x}(B) = B \cup \{x\}$, if $A \subseteq B$ and $f_{A,x}(B) = B$ otherwise. It can be shown that $f_{C,x} = f_{D,y}$ iff $C = D$ and $x = y$. The lattice is atomistic, which follows from the fact that the set of atoms is identical with the set of of all join-irreducible elements.

One of the central concepts of this study is a modification of an operator which has been introduced by Koutski and Sekanina and studied by Slapal in the study of generalized topological spaces obtained by elimination of its basic axioms while more specific conditions (for instance separation axioms) remain.[6] A modification of an operator f of a type designated by some condition X is the smallest operator greater than f which satisfies this condition (upper modification $f^{(X)}$), or the largest smaller than f (lower modification $f_{(X)}$). The original literature of the subject is focused on the mutual relationship of the axioms of topology and its methods do not contribute to our study beyond the introduction of the concept of modification.[7]

The most natural example is the I-(upper) modification. For every operator f, not necessarily transitive, the family of f-closed subsets (satisfying the condition $f(A) = A$) is a Moore family of sets defining a transitive operator $f^{(I)}$. It is obvious that this operator is the least transitive one (i.e. I-operator) greater than f. In the studies carried out in completely different context and without any reference to the concept of modification Moore families for transitive operators have been altered to construct transitive operators satisfying desired conditions, as it was mentioned in the introduction. The constructions involved expansions of the Moore families (by finite unions of their members or by directed unions) and therefore produced lower modifications ($f_{(fA)}$ and $f_{(fC)}$, respectively). Our goal will be to generalize the study to not necessarily transitive operators and for any type of modifications.

This concludes the material about the structure of the set of operators F(S) necessary for our study of operator modifications. Since we want to consider closure and interior operations on F(S) our next step is to select necessary facts about closure operations defined on posets.

Closure operator on a poset $<P, \leq>$ is naturally defined by the analogue to closure space conditions: $\forall p,q \in P$: $p \leq f(p)$ & $p \leq q \Rightarrow f(p) \leq f(q)$ & $f(f(p)) = f(p)$.
In posets, the concept of the Moore family of closed subsets Baer has replaced by the concept of a partial ordinal as a subset M of P such that $\forall p \in P$: $\leq(p) \cap M \neq \emptyset$ and $\leq(p) \cap M$ has the least element, where $\leq(p) = \{q \in P: p \leq q\}$.[8] Then the relationship between closure operators and Moore families of closed subsets is restored as a relationship between closure operators on posets and partial ordinals.

On posets with an involution we can reintroduce the concept of an interior operator associated with the closure by $i = {}^c f^c$, where the symbol of complementation is re-interpreted as the symbol of the involution. Such operators satisfy the dual conditions: $\forall p,q \in P$: $i(p) \leq p$ & $p \leq q \Rightarrow i(p) \leq i(q)$ & $i(i(p)) = i(p)$, and they are associated with the dual partial ordinal (defined by reversal of the partial order) generalizing the family of open sets.

As in the case of closure spaces, we can introduce a partial order on closure operators on a given poset: $f \leq g$ if $\forall p \in P$: $f(p) \leq g(p)$.

We will need in the following a simple lemma which belongs to the early study of closure operations on posets:

Let f and g be transitive closure operators on a poset. Then the following statements are equivalent:
a) fg is a transitive closure operator,
b) $gf \leq fg$,
c) $gfg = fg$.

In this study, the concept of closure or interior operators defined in posets is applied to the poset F(S) of all operators on a set S which satisfies very strong conditions allowing to recover quite large portions of the conceptual framework of closure spaces. For instance, we can define a derived set operator as follows:
$\forall a \in L_F$: $a^{df} = \vee\{p \in At(L_F)$: $p \leq f(\vee\{q \in At(L_F)$: $q \leq a$ and $q \neq p\})\}$.

3. Generalization of Finite Aditivity and Finite Character Modifications

The constructions of Frink and Graetzer, and of Mayer-Kalkschmidt and Steiner are based on the modifications of the Moore family of closed subsets. Since non-transitive operators are not determined by their closed subsets, their constructions cannot be used in the general case. In this section, the constructions for any operator will be proposed which allow to recover the results of earlier authors without the assumption of transitivity.

We will start from listing without proof some simple facts regarding the transitive modification of operators which assigns to every operator f the transitive operator defined by the Moore family of f-closed subsets.

PROPOSITION 3.1 *Let f be an operator on a set S. Then:*
a) If B is f-closed, then $\forall A \subseteq S$: $A \subseteq f(B) \Rightarrow f(A) \subseteq f(B)$,
b) $\forall A \subseteq S$: $f[f^{(I)}(A)] = f^{(I)}(A)$ and $f^{(I)}[f(A)] = f^{(I)}(A)$,
c) $f \leq f^{(I)}$,
d) $f \in I(S)$ iff $f = f^{(I)}$,
e) $f \leq g \Rightarrow f^{(I)} \leq g^{(I)}$, but this implication cannot be reversed in general,
f) $f \in fC(S) \Rightarrow f^{(I)} \in fC(S)$,
g) $f \in fA(S) \Rightarrow f^{(I)} \in fA(S.$,

Now we can move to constructions of other modifications.

PROPOSITION 3.2 *Let f be an operator on a set S. Then the greatest operator of finite character less than f is given by: $\forall A \subseteq S$: $f_{(fC)}(A) = \cup\{f(B)$: $B \in Fin(A)\}$.*

Proof: Obviously : $f_{(fC)} \in fC(S)$ and $f_{(fC)} \leq f$. Now, suppose $g \leq f$ and $g \in fC(S)$. Then $x \in g(A) \Rightarrow \exists B \in Fin(A)$: $x \in g(B)$, but then $x \in f(B)$, and therefore $x \in f_{(fC)}(A)$, which shows that $g \leq f_{(fC)}$.

It can be easily shown that for all operators f and g on S we have: $f \leq g \Rightarrow f_{(fC)} \leq g_{(fC)}$ and $f \in fC(S) \Rightarrow f_{(fC)} = f$.

PROPOSITION 3.3 Let f be an operator on a set S. Then the greatest finitely additive operator less than f is given by: *$\forall A \subseteq S$: $f_{(fA)}(A) = \cap \{f(A_1) \cup f(A_2) \cup \ldots \cup f(A_n)$: over all finite partitions $\{A_1, A_2, \ldots, A_n\}$ of $A\}$.*

Proof: Certainly $f_{(fA)} \leq f$. We will show that $f_{(fA)} \in fA(S)$, for which it is enough to show $f_{(fA)}(A \cup B) \subseteq f_{(fA)}(A) \cup f_{(fA)}(B)$ for disjoint sets A and B. Suppose $x \in f_{(fA)}(A \cup B)$, but $x \notin f_{(fA)}(A) \cup f_{(fA)}(B)$. Then there exist finite partitions $\{A_1, A_2, \ldots, A_n\}$ of A and $\{B_1, B_2, \ldots, B_m\}$ of B, such that $x \notin f(A_1) \cup f(A_2) \cup \ldots \cup f(A_n)$ and $x \notin f(B_1) \cup f(B_2) \cup \ldots \cup f(B_m)$. But these two partitions combined give us a finite partition of $A \cup B$, so $x \in f(A_1) \cup f(A_2) \cup \ldots \cup f(A_n) \cup f(B_1) \cup f(B_2) \cup \ldots \cup f(B_m)$, contradiction. It remains to show that $f_{(fA)}$ is the greatest such operator. This follows directly from the fact that $f \leq g \Rightarrow f_{(fA)} \leq g_{(fA)}$ and $f \in fA(S) \Rightarrow f_{(fA)} = f$.

PROPOSITION 3.4 *If $f \in I(S)$, then also $f_{(fC)}, f_{(fA)} \in I(S)$.*

Proof: The proof for both modifications is the same, so we will prove only transitivity of the finite additive modification. We have $f_{(fA)} \leq f$, so by Proposition 3.1 we have $f_{(fA)}{}^{(I)} \leq f^{(I)} = f$. Now, by the same proposition $f_{(fA)}{}^{(I)} \in fA(S)$, and therefore $f_{(fA)} \leq f_{(fA)}{}^{(I)} \leq \vee\{g \in I(S)$: $g \in fA(S)$ and $g \leq f\}$. But $\vee\{g \in I(S)$: $g \in fA(S)$ and $g \leq f\} = \vee\{h \in F(S)$: $h \in fA(S)$ and $h \leq f\} = f_{(fA)}$, therefore $f_{(fA)} = f_{(fA)}{}^{(I)} \in I(S)$.

COROLLARY 3.5 *For a transitive operator the constructions of Mayer-Kalkschmidt and Steiner, and of Frink and Graetzer produce the same operators as those above.*

COROLLARY 3.6 *For every operator f on a set S:*
a) $f_{(fA)}{}^{(I)} \leq f^{(I)}{}_{(fA)}$,
b) $f_{(fC)}{}^{(I)} \leq f^{(I)}{}_{(fC)}$.

Proof: Since the proofs are identical for both modifications, let's consider the first case.
$f \leq f^{(I)} \Rightarrow f_{(fA)} \leq f^{(I)}{}_{(fA)} \Rightarrow f_{(fA)}{}^{(I)} \leq f^{(I)}{}_{(fA)}{}^{(I)} = f^{(I)}{}_{(fA)}$.

Frink and Greatzer considered finite character and finite additivity modifications of transitive operators in terms of the Moore families of closed subsets in the context of a characterization of topologically closed subalgebras of a topological algebra. Their main result was: *A Moore family $\mathcal{M}$ is the family of all topologically closed subalgebras of some algebra which is also a topological space iff $\mathcal{M} = Alg(\mathcal{M}) \cap Top(\mathcal{M})$, where $Alg(\mathcal{M})$ is the expansion of the Moore family to produce finite character property characterizing algebraic closures and $Top(\mathcal{M})$*

is the expansion of the Moore family to produce finite additivity characterizing topological closures. We can achieve the same goal using our conceptual framework of operator modifications.

Suppose on a set S we have defined two structures, of an algebra with the algebra generating closure operator f_a, and of topological space with the closure operator f_t. Then the definition of a topological algebra as an algebra in which the topological closure of a subalgebra is a subalgebra can be formulated in terms of operators as a condition $f_a f_t f_a = f_t f_a$, which by the lemma in the section with preliminaries is equivalent to the following two statements $f_a f_t \leq f_t f_a$, or $f_t f_a$ is a transitive operator. Now, for closure spaces (or closure operators defined on posets which are complete lattices) there is another equivalent form of this condition $f_t f_a = f_t \vee f_a$, and therefore we can define a topological algebra as such in which the family of topologically closed subalgebras define a closure operator $f = f_t f_a = f_t \vee f_a$.

Now we can reformulate the result of Frink and Graetzer as follows:

PROPOSITION 3.7 *A transitive closure operator f on a set S is a generating operator for topologically closed subalgebras of topological algebra iff* $f = f_{(fA)} f_{(fC)} = f_{(fA)} \vee f_{(fC)}$.

4. General Modifications

In the section with preliminaries the following definition of a modification of an operator on a set defined by a condition has been introduced.

Let f be an operator on a set S, X(S) be a class of operators satisfying some condition. Then the upper X-modification $f^{(X)}$ *of the operator f is the least of all X-operators greater than f, if one exists. The lower X-modification* $f_{(X)}$ *of the operator f is the largest of all X-operators less than f, if one exists.*

It is easy to recognize that upper modifications satisfy the following conditions.

PROPOSITION 4.1 *Let f be an operator on a set S. Then:*

a) $f \leq f^{(X)}$,

b) $f \leq g \Rightarrow f^{(X)} \leq g^{(X)}$,

c) $f^{(X)(X)} = f^{(X)}$

d) $f \in X(S)$ iff $f = f^{(X)}$,

e) $\forall A \subseteq S$: $f[f^{(I)}(A)] = f^{(I)}(A)$ and $f^{(I)}[f(A)] = f^{(I)}(A)$.

First three statements show that an upper modification is a transitive closure operator on the partially ordered set F(S) of all operators.

COROLLARY 4.2 *The necessary and sufficient condition for given property X of operators to define an upper X-modification for all operators in F(S) is that X(S) is a partial ordinal in the poset F(S).*

Since the poset F(S) is a complete bounded lattice with the greatest element defined by $\forall A \subseteq S$: $o(A) = S$, we get another form of this condition.

COROLLARY 4.3 *The necessary and sufficient condition for a given property X of operators to define an upper X-modification for all operators in F(S) is that the operator defined by* $\forall A \subseteq S$: $o(A) = S$ *has this property and that the set of operators X(S) is closed with respect to arbitrary meets.*

Since duality is introducing an involution in the poset F(S) we have the following.

PROPOSITION 4.4 *An operator f has a lower X-modification* $f_{(X)}$ *iff* f^* *has an upper* X^*-*modification* $f^{*(X^*)}$.

COROLLARY 4.5 *Lower X-modification is an interior operator on the poset F(S) associated with the family of open elements (fixed points of the interior operation). The necessary and sufficient condition for the existence of the modification on F(S) is that the set X(S) is a dual partial ordinal, or equivalently, that the operator defined by* $\forall A \subseteq S$: $e(A) = A$ *has this property and the set X(S) is closed with respect to arbitrary joins.*

PROPOSITION 4.6 *A composition of upper modifications on F(S) is an upper modification on F(S) iff they commute.*

By the duality introduced by involution we have similar condition for lower modifications.

COROLLARY 4.7 *A composition of lower modifications on F(S) is a lower modification iff they commute.*

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