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Delay Business Cycle Model with Nonlinear Accelerator*

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Abstract

It has been well-known that nonlinearity, time delay and local instability are significant sources for a birth of cyclical dynamics since the pioneering work of Goodwin (1951). In particular, a nonlinear business cycle model with the nonlninear acceleration principle and delay is constructed and shown to give rise to cyclic oscillations when its stationary state is locally unstable. However very little is known about time delay effects caused by investment lags in Goodwin's cyclic dynamics, furthermore global dynamics in the locally stable case has not been considered yet. This study draws attentions to these unexplored aspects of Goodwin's business model and shows two main results. It is demonstrated, first, that continuously distributed time lag has the stronger stabilizing effect than fixed time lag and, second, that multiple limit cycles may coexist when the stationary state is locally stable.

Keywords: fixed time delay, continuously distributed time delay, S-shaped investment function, coexistence of multiple limit cycles.

1 Introduction

The contributions of Goodwin (1951) are reconsidered and further developed in this study. Goodwin (1951) introduced a nonlinear accelerator business cycle model with an investment lag, numerically specified it and graphically showed that it could generate a stable limit cycle when a stationary point is locally unstable. Since Goodwin's work, it is expected that instability, nonlinearity and delay could be significant sources for the birth of cyclic behavior. In view of the fact that it is difficult to analytically solve nonlinear models with delay, it is a natural way to perform numerical studies or to convert the model to a tractable one by using approximation. Indeed, considerable effort has been devoted to investigate the nonlinear structure of the ordinary differential version of the unstable Goodwin's model. Recently, Sasakura (1996) gives an elegant proof of

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the stability and the uniqueness of Goodwin's cycle. More recently Lorenz and Nusse (2002), based on Lorenz (1987), reconstructs Goodwin's model as a forced oscillator system and demonstrates the emergence of chaos when nonlinearities become stronger. In the existing literature, however, there have been only limited number of analytical works on the delay differential version of Goodwin's model,[1] and, furthermore, very little has yet been revealed with respect to the circumstances under which the stationary point is locally asymptotically stable. The main purpose of this study is to provide an investigation of these unexplored aspects of Goodwin's business cycle model.

We add two new observations to the existing results. First, we reformate the model in terms of a nonlinear differential equation with the explicit treatment of time delay (i.e., fixed time delay and distributed time delay) and find out how the time delay affects the stability of Goodwin's model: the model with continuously distributed time delay is shown to be more stable than the one with fixed time lag. Second, we demonstrate that the nonlinear delay Goodwin's model may have multiple limit cycles when a stationary point is locally stable. That is, the model is stable and trajectories return to the stationary state for smaller disturbances but is unstable and exhibits persistent fluctuations for larger disturbances.

In what follows, Section 2 overviews the basic structure of Goodwin's nonlinear accelerator model and introduces time delay to see its effect on cyclic dynamics. Section 3 shows the coexistence of a stable stationary point, an unstable limit cycle and a stable limit cycle. Section 4 concludes the paper. The outline of proofs and some mathematical details are given in the appendices.

2 Goodwin's Business Cycle Model

This section is divided into two parts. We recapitulate the basic elements of Goodwin's model and derive the local stability/instability conditions in Section 2.1. We, then, adopt a prototype investment function and, in Section 2.2, take a look at what kind of dynamics is produced by the delay Goodwin model.

2.1 Fixed Delayed Model

Goodwin (1951) presents five different versions of the nonlinear accelerator model. The first version assumes a piecewise linear function with three levels of investment, which can be thought as the crudest or simplest version of the non-linear accelerator. This is a text-book model that can give a simple exhibition on how nonlinearities give rise to endogenous cycles without relying on structurally unstable parameters, exogenous shocks, etc. The second version replaces the piecewise linear investment function with a smooth nonlinear investment function. Although persistent cyclical oscillations of the output are shown to exist, the second version includes a unfavorable phenomenon, namely, discontinuous investment jump, which is not realistic in the real economic world. "In order to come close to reality" (p.11, Goodwin (1951)), the time lag between

[1] Yoshida and Asada (2007) investigates the impact of delayed government stabilization policy on the dynamic behavior of a Keynes-Goodwin model. Also see Bischi, Chiarella, Kopel and Szidarovszky (2007) for applications of the delay differential method to oligopoly models.

decision to invest and the corresponding outlays is introduced in the third version. However, no analytical considerations are given to this third version. The existence of a business cycle is confirmed in the fourth version, which is a linear approximation of the third version with respect to the investment lag. Finally alternation of autonomous expenditure over time is taken into account in the fifth version.

To find out how nonlinearity and time lag can generate endogenous cycles, we review the third version of Goodwin's model,

$$\begin{cases} \varepsilon \dot{y}(t) = \dot{k}(t) - (1-\alpha)y(t), \\ \dot{k}(t) = \varphi(\dot{y}(t-\theta)). \end{cases} \tag{1}$$

Here k is the capital stock, y the national income, α the marginal propensity to consume, which is positive and less than unity, θ the investment lag and the reciprocal of ε a positive adjustment coefficient. The dot over variables stands for time differentiation. The first equation of (1) defines an adjustment process of the national income. Accordingly, national income rises or falls if investment is larger or smaller than savings. The second equation, in which $\varphi(\dot{y}(t-\theta))$ denotes the induced investment, describes an accumulation process of capital stock based on the acceleration principle with time lag θ. According to this principle, investment depends on the rate of delayed changes in the national income. A distinctive feature of Goodwin' model is to introduce a nonlinearity into the investment function in such a way that the investment is proportional to the change in the national income in the neighborhood of the stationary income but becomes inflexible (i.e., less elastic) for extremely larger or smaller values of the income. This *delay nonlinear* acceleration principle is crucial in obtaining endogenous cycles in Goodwin's model. We will retain this nonlinear assumption and specify its explicit form in the following analysis. On the other hand, we depart from Goodwin's non-essential assumption of positive autonomous expenditure and will work with zero autonomous expenditure for the sake of simplicity. A direct consequence of this alternation is that an equilibrium solution or a stationary point of (1) is $y(t) = \dot{y}(t) = 0$ for all t.

Inserting the second equation of (1) into the first one and arranging the terms provides a single dynamic equation for the national income y,

$$\varepsilon \dot{y}(t) - \varphi(\dot{y}(t-\theta)) + (1-\alpha)y(t) = 0. \tag{2}$$

This is a *neutral delay nonlinear* differential equation, which we call the *fixed delay model.* Goodwin (1951) did not analyze dynamics generated by this fixed delay model. Furthermore, to the best of our knowledge, no analytical solutions of the delayed model are available yet. However, it is possible to investigate the dynamics of the delayed model by using linearization for local dynamics and by specifying the investment function and performing numerical simulations for global dynamics.

We first focus on local dynamics. The fixed delay model is autonomous and its special solution is constant (i.e., $y(t) = 0$) so that its linearized version takes the form of a *linear neutral autonomous delay* differential equation,

$$\varepsilon \dot{y}(t) - v\dot{y}(t-\theta) + (1-\alpha)y(t) = 0. \tag{3}$$

It is well known that if the characteristic polynomial of a linear neutral equation has roots with only negative real parts, then the stationary state is locally asymptotically stable. Applying the method used by Kuang (1993), we have the following results.

Theorem 1 *For any $\theta > 0$, the linearized fixed delay model is locally unstable if $v > \varepsilon$ and asymptotically locally stable if $v \leq \varepsilon$.*

Proof. See a proof given in Appendix A. ■

If there is no time lag (i.e., $\theta = 0$), (2) is reduced to the second version of Goodwin's model. In the same way, (3) with no time lag becomes a first-order ordinary differential equation that corresponds to the linearized second version. Applying separation of variables gives its complete solution,

$$y(t) = y_0 e^{\lambda t} \text{ with } \lambda = \frac{1-\alpha}{v-\varepsilon},$$

where y_0 is an initial condition. It is clear that the stationary point of the linear differential equation is locally stable if $v < \varepsilon$ and unstable if $v > \varepsilon$. This result and Theorem 1 imply that the stability condition of the third version (i.e., fixed delay model) is the same as that of the second version. Introducing fixed time lag does not affect the stability condition. It is, however, numerically confirmed that the fixed investment lag has distinctive effects on the global dynamics if the investment function possesses strong nonlinearities as we will see shortly.

2.2 Two Examples

We proceed to examine global dynamics. Goodwin (1951) expands the fixed delay model with respect to θ to obtain the following second-order nonlinear differential equation,

$$\varepsilon\theta\ddot{y}(t) + [\varepsilon + (1-\alpha)\theta]\dot{y}(t) - \varphi(\dot{y}(t)) + (1-\alpha)y(t) = 0. \tag{4}$$

This is Goodwin's fourth version. It has been demonstrated that (4) gives rise to cyclic dynamics. For the present, it may be useful to look at more closely at some of the interesting dynamic features of (4). In particular, it causes a relaxation oscillation for infinitesimally small θ and a coexistence of attractors for sufficiently large θ. With these points in mind, we reduce the second-order differential equation to a two-dimensional differential system, following Sasakura (1996). We first change time scale and variable, respectively, by

$$t_1 = \sqrt{\frac{1-\alpha}{\varepsilon\theta}}t \text{ and } x = \sqrt{\frac{1-\alpha}{\varepsilon\theta}}y.$$

to convert (4) into a differential equation of the form,

$$\ddot{x}(t) + \chi(\dot{x}(t)) + x(t) = 0, \tag{5}$$

where

$$\chi(\dot{x}(t)) = \frac{1}{\sqrt{\varepsilon\theta(1-\alpha)}.}\left([\varepsilon + (1-\alpha)\theta]\dot{x}(t) - \varphi(\dot{x}(t))\right).$$

We then redefine $y = -x$ and $t_1 = t$ to further convert (5) into the following $2D$ differential system,

$$\begin{cases} \dot{x}(t) = y(t) - \sigma F(x(t)), \\ \dot{y}(t) = -x(t), \end{cases} \tag{6}$$

where

$$\sigma = \frac{1}{\sqrt{\varepsilon\theta(1-\alpha)}}$$

and

$$F(x) = [\varepsilon + (1-\alpha)\theta]x - \varphi(x).$$

Needless to say, (4), (5) and (6) are different expressions of the same equation and hence mathematically equivalent. Note that σ is inversely proportional to θ, given α and ε.

We present two numerical examples in which the limit cycle of (6) converges to a relaxation oscillation as θ approaches to zero and with a sufficiently large θ, (6) may have multiple limit cycles. To this end, we assume $\varphi(x) = v\tan^{-1}(x)$ for a while. Since $\varphi(x)$ is an odd function, it follows that (6) has exactly one stable limit cycle if $\varepsilon + (1-\alpha)\theta - v < 0$ is imposed on $F(x)$.[2] To proceed further, we replace $\tan^{-1}(x)$ by its truncated Tayler series.[3] First we take up to the third-order term, $x - \frac{1}{3}x^3$, to approximate $\tan^{-1}(x)$, which approximates (6) as,

$$\begin{cases} \dot{x}(t) = y(t) - \sigma\left\{[\varepsilon + (1-\alpha)\theta - v]x(t) + \dfrac{v}{3}x(t)^3\right\}, \\ \dot{y}(t) = -x(t). \end{cases} \tag{7}$$

This is a variant of the van der Pol equation. We let $\sigma z = y$ and replace $\varepsilon + (1-\alpha)\theta - v$ with $\varepsilon - v$ to transform (7) to the following 2D system:

$$\begin{cases} \dot{x}(t) = \sigma\left\{z(t) + (v-\varepsilon)x(t) - \dfrac{v}{3}x(t)^3\right\}, \\ \dot{z}(t) = -\dfrac{x(t)}{\sigma}. \end{cases} \tag{8}$$

Suppose that σ is fairly large (i.e., θ is fairly small). When a trajectory of (8) is away from the curve $z = \frac{v}{3}x^3 - (v-\varepsilon)x$, then $|\dot{x}| >> |\dot{z}| = O(\sigma^{-1})$ makes the trajectory move rapid in the horizontal direction in the (x, z) space. When it enters the region in which $\left|z + (v-\varepsilon)x - \frac{v}{3}x^3\right| = O(\sigma^{-2})$, the trajectory goes slowly along the curve because $\dot{x}$ and $\dot{y}$ are comparable in the sense that $|\dot{x}| = |\dot{z}| = O(\sigma^{-1})$. This rapid-slow oscillation is depicted as the bold line in Figure 1 in which a trajectory jumps from A to B and from C to D.[4] The

[2] It is shown in the next section that $\varepsilon + (1-\alpha)\theta - v < 0$ is the locally instability condition of (4).

[3] This expansion of $\tan^{-1}(x)$ are considered in Puu (1986).

[4] It can be proved that the limit cycle generated by (8) with a linear change of time scale $\sigma\tau = t$ approaches the closed curve consisting of the two horizontal line segments on $z = \pm\frac{2}{3}\frac{(v-\varepsilon)^{3/2}}{\sqrt{v}}$ and the two arces of the cubic function $\frac{v}{3}x^3 - (v-\varepsilon)x$. See Appendix IV in Stoker (1950).

special casae of $\theta = 0$ reduces the fixed delay model (2) to the second version of Goodwin's model. Although the second version is a 1D differential equation, the nonlinearity of the investment function prevents us from deriving an explicit solution. This example implies that the limit cycle is asymptotic to a relaxation oscillation if $\sigma \to \infty$, that is, if $\theta \to 0$. The point made so far applies in principle to any $\varphi(x)$ having a S-shaped curve. With this result, we can say with fair certainty that the second version has a periodic solution but with discontinuous jumps.

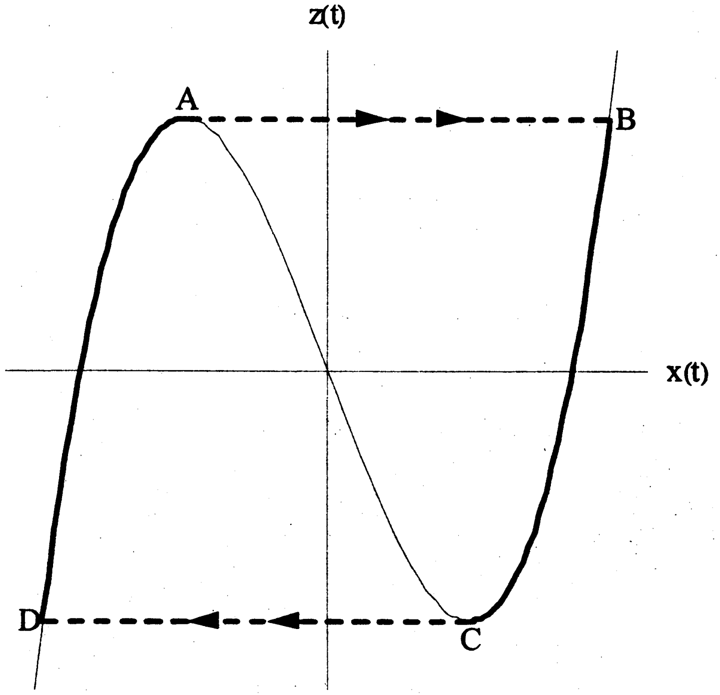


Figure 1. Relaxation oscillation with discotinuous jumps

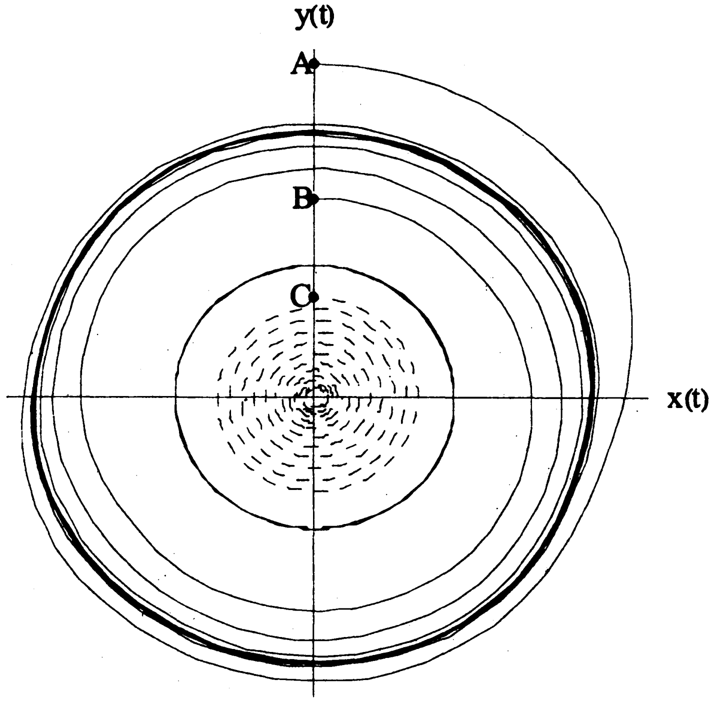
More complicated dynamics may be found in the Goodwin model with a more complicated investment function. If we could assume the form of $\varphi(x)$ in such a way that $F(x)$ in (6) is approximated by $F(x) = a_1x + a_2x^2 + ... + a_{2m+1}x^{2m+1}$ and σ is sufficiently small (namely, θ is sufficiently large), then it can be shown that (6) has at most m limit cycles.[5] For example, (6) has two limit cycles if we use the fifth-order Tayler series approximation and set $\sigma = 0.08$,

$$\begin{cases} \dot{x}(t) = y(t) - \sigma \left\{ x - \dfrac{5}{3}x^3 + \dfrac{2}{5}x^5 \right\}, \\ \dot{y}(t) = -x(t). \end{cases} \tag{9}$$

As shown in Figure 2, a dotted trajectory starting at point C located within the inner limit cycle converges to the stationary state while two trajectories starting

[5] See Theorem 6 in Section 3.8 of Perko (1991).

at points A and B located outside this cycle converge to the outer limit cycle.



Figures 2. Coexistence of two limit cycle in (9)

The second example shows numerically (but with mathematical backgrounds) that the fourth version of the Goodwin model could have multiple limit cycles if the time lag is considerable large and the stationary state is locally stable. The multiplicity is of great interest for studying Goodwin's cycles. It is, however, problematic for two reasons. One is the economics reason that the truncated investment function does not have asymptotic upper and lower bounds, which is different from the original assumption of Goodwin (1951). The other is the mathematical reason that θ in (4) and also in (9) is supposed to be sufficiently close to zero while the coexistence of limit cycles has been shown with a large θ. To escape from these deficiencies, we will replace fixed time lag with continuously distributed time lag, specify an appropriate investment function having asymptotic bounds and then demonstrate that the modified delay nonlinear Goodwin model can offer a wide variety of dynamics involving the coexistence of limit cycles.

3 Modified Business Cycle Model

In what follows, we introduce continuously distributed time lag into the third version and derive its stability conditions in Section 3.1. We specify the investment function, select the slope of the investment function evaluated at the stationary point as the bifurcation parameter and investigate the possibility of Hopf bifurcation in Section 3.2. We construct an invariant set in the state space and apply the Poincaré-Bendixson theorem to find a stable limit cycle that encloses the unstable limit cycle in Section 3.3. Lastly, in Section 3.4, we use another forms of the investment function and show that the coexistence of the limit cycles is sensitive to the choice of the investment function .

3.1 Continuously Distributed Delay Model

Continuously distributed time delay is an alternative approach to deal with time lag in investment. If the expected change of national income is denoted by $\dot{y}^e(t)$ at time t and is formed based on the entire history of the actual changes of national income from zero to t, the dynamic system (2) can be written as the system of Volterra-type integro-difference equations:

$$\varepsilon\dot{y}(t) - \varphi(\dot{y}^e(t)) + (1-\alpha)y(t) = 0,$$

$$\dot{y}^e(t) = \int_0^t \frac{1}{\theta} e^{-\frac{t-s}{\theta}} \dot{y}(s) ds, \tag{10}$$

where θ is a positive real parameter and is associated with the length of the delay. We call this dynamic system the *distributed delay model.* It can be seen in the second equation of (10) that the weighting function of the past changes in national income gives the most weight to the most recent income change and is exponentially declining afterwards. Before turning to a closer examination of the distributed delay model, we rewrite it as a system of ordinary differential equations. By doing so, we can use all tools known from the stability theory of ordinary differential equations to analyze its local and global dynamic behavior.

The time-differentiation of the second equation of (10) gives a simple equation for the new variable $z = \dot{y}^e$:

$$\dot{z}(t) = \frac{1}{\theta}\left(\dot{y}(t) - z(t)\right). \tag{11}$$

Solving the first equation for $\dot{y}$, replacing $\dot{y}^e$ with z, replacing $\dot{y}$ in (11) with the new expression of $\dot{y}$ and then adding the new dynamic equation of z will transform the system of the integro-differential equations to the following 2D system of ordinary differential equations:

$$\begin{cases} \dot{y}(t) = -\dfrac{1-\alpha}{\varepsilon}y(t) + \dfrac{1}{\varepsilon}\varphi(z(t)), \\[2ex] \dot{z}(t) = \dfrac{1}{\theta}\left(-\dfrac{1-\alpha}{\varepsilon}y(t) + \dfrac{1}{\varepsilon}\varphi(z(t)) - z(t)\right). \end{cases} \tag{12}$$

By linearizing the system at the stationary state, $y = z = 0$, we obtain the Jacobian matrix,

$$J = \begin{pmatrix} -\dfrac{1-\alpha}{\varepsilon} & \dfrac{\nu}{\varepsilon} \\[2ex] -\dfrac{1-\alpha}{\varepsilon\theta} & \dfrac{1}{\theta}\left(\dfrac{\nu}{\varepsilon} - 1\right) \end{pmatrix}, \tag{13}$$

where $\nu = \varphi'(0)$ is the marginal investment rate at the stationary point. The corresponding characteristic equation is quadratic in λ,[6]

$$\lambda^2 + \frac{\varepsilon + (1-\alpha)\theta - \nu}{\varepsilon\theta}\lambda + \frac{1-\alpha}{\varepsilon\theta} = 0.$$

[6] Notice that this characteristic equation is identical with the characteristic equation that can be derived from the fourth version (4). This means that both equations generate exactly the same dynamics in the neighborhood of $\theta = 0$. See Szidarovszky and Matsumoto (2007) for the similarities and dissimilarities between the two dynamic equations.

The product of the eigenvalues is equal to the determinant of J and is positive,

$$\lambda_1 \lambda_2 = \frac{1-\alpha}{\varepsilon\theta} > 0 \tag{14}$$

due to the assumptions imposed on parameters, $0 < \alpha < 1$ and $0 < (\varepsilon, \theta)$. These parametric restrictions ensure that the stationary point is not a saddle point. The sum of the eigenvalues is equal to the trace of J and is either positive or negative,

$$\lambda_1 + \lambda_2 = \frac{\nu - [\varepsilon + (1-\alpha)\theta]}{\varepsilon\theta} \gtrless 0. \tag{15}$$

We then have the following result on the local stability of the distributed delay model:

Theorem 2 *For $\theta > 0$, the linearized distributed delay model is locally asymptotically stable if $\nu < \varepsilon + (1-\alpha)\theta$ and locally unstable otherwise.*

To confirm the dependency of the stability on the parameters θ and ν, we define the parameter region by $\Omega = \{(\theta, \nu) \mid \theta > 0 \text{ and } \nu > 0\}$, taking the values of the other two parameters α and ε given. Comparing Theorem 2 with Theorem 1 reveals that the linearized distributed delay model is more stable than the linearized fixed delay model in the sense that the stability region of the former model in Ω is larger than the one of the latter model. In the second equation of (10), we assume the exponential kernel function. In the case of the general kernel function

$$\frac{1}{n!}\left(\frac{n}{\theta}\right)^{n+1}(t-s)^n e^{-\frac{n(t-s)}{\theta}},$$

we know that as $n \to \infty$, the function converges to the Dirac-delta function centered at $t - s = \theta$. Therefore, in the limiting case the distributed delay model converges to the fixed delay model, which then implies that the stability conditions are definitely different for $n < \infty$ and coincide in the limit of n. So far, we have seen the following result:

Theorem 3 *The Goodwin model with continuously distributed time lag having the exponential kernel function is more stable than the Goodwin model with the fixed time lag.*

Since the stability of the stationary point depends on the sign of $\nu - [\varepsilon + (1-\alpha)\theta]$, the partition line that divides Ω into two regions, stable and unstable regions, is defined by

$$\nu = \varepsilon + (1-\alpha)\theta, \tag{16}$$

which is positive-sloping. It depends on the value of the discriminant of the characteristic equation whether the local dynamics is oscillatory or monotonic. The curve when the discriminant is zero is determined by

$$\nu = \varepsilon + (1-\alpha)\theta \pm 2\sqrt{(1-\alpha)\varepsilon\theta}, \tag{17}$$

which distinguishes the parameter region for real roots from that for complex roots. It is tangent to the vertical line at $\nu = \varepsilon$ and the horizontal line at $\theta = \varepsilon/(1-\alpha)$. The partition line is located in the region in which the discriminant is negative and is identical with the zero-discriminant locus for $\theta = 0$. The eigenvalues are complex and their real parts change their signs from positive to negative or vise versa when either of parameters θ and ν in Ω crosses the partition line. This indicates a possibility of a Hopf bifurcation, which we will consider in the next section.

3.2 Hopf Bifurcation

By applying the Hopf bifurcation theorem, we investigate whether there is a limit cycle in the distributed delay model. According to the theorem, a Hopf bifurcation occurs if the complex conjugate eigenvalues as functions of the bifurcation parameters cross the imaginary axis. We have already seen that the eigenvalues are complex conjugates with zero real part if $\nu = \varepsilon + (1-\alpha)\theta$. As there are no other eigenvalues in the two-dimensional system, a limit cycle exists if the eigenvalues cross the imaginary axis with non-zero speed at the bifurcation point. Though there may exist several possibilities to parametrize the distributed delay model, it seems interesting to choose the marginal investment rate evaluated at the stationary point as the bifurcation parameter.

It can be easily seen that there exists a value ν_0 for which

$$\nu_0 = \varepsilon + (1-\alpha)\theta, \tag{18}$$

implying that the complex conjugate roots cross the imaginary axis. Since $\frac{\partial R(\lambda_{1,2})}{\partial v} > 0$, the real parts are positive or negative if $\nu > \nu_0$ or $\nu < \nu_0$, respectively. Therefore, v_0 is indeed a bifurcation value of the distributed delay model. Since the requirements of the Hopf bifurcation theorem are fulfilled, it is guaranteed that a limit cycle exists in a neighborhood of the stationary point $(0,0)$ at $\nu = \nu_0$.[7]

The Hopf theorem, however, has no indication about the nature of the limit cycle. There are two possibilities, one is that orbits spiral outward from the stationary point toward a limit cycle, called the *supercritical Hopf bifurcation*, and the other is that all orbits starting inside the cycle spiral in toward the stationary point and becomes explosive outside the cycle, called the *subcritical Hopf bifurcation*. To make the distinction between the sub- and super-critical Hopf bifurcation, we calculate the stability index of (12).

The distributed delay model can be written as

$$\begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = J \begin{pmatrix} y \\ z \end{pmatrix} + \begin{pmatrix} g^1(z) \\ g^2(z) \end{pmatrix}$$

where J is the Jacobian matrix defined above, and $g^i(z)$ for $i = 1, 2$ are nonlinear terms that can be derived as

$$\begin{aligned} g^1(z) &= \frac{1}{\theta}\left(\varphi(z) - \nu_0 z\right), \\ g^2(z) &= \frac{1}{\varepsilon\theta}\left(\varphi(z) - \nu_0 z\right). \end{aligned}$$

In order to transform the Jacobian matrix of the distributed delay model into the normal form, we introduce the coordinate transformation

[7] See Lorenz (1993) for the Hopf bifurcation theorem, the stability index to be considered soon with suitable coordinate transformations.

$$\begin{pmatrix} y \\ z \end{pmatrix} = D \begin{pmatrix} u \\ v \end{pmatrix} \text{ with } D = \begin{pmatrix} 0 & 1 \\ d_{21} & d_{22} \end{pmatrix},$$

where

$$d_{21} = \frac{\varepsilon}{\nu_0 \theta}\sqrt{\frac{\nu_0 - \varepsilon}{\varepsilon}} \text{ and } d_{22} = \frac{\nu_0 - \varepsilon}{\nu_0 \theta}.$$

Since matrix D transforms the coordinate system (y, z) into a new coordinate system (u, v), the distributed delay model becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\frac{1-\alpha}{\varepsilon\theta}} \\ \sqrt{\frac{1-\alpha}{\varepsilon\theta}} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mathbf{a}g(u,v) \\ \mathbf{b}g(u,v) \end{pmatrix}$$

where

$$g(u,v) = \varphi(d_{21}u + d_{22}v) - \nu_0(d_{21}u + d_{22}v)$$

and

$$\mathbf{a} = \frac{1}{\sqrt{\varepsilon(\nu_0 - \varepsilon)}} \text{ and } \mathbf{b} = \frac{1}{\varepsilon}.$$

It is well-known that the stability of the emerging cycle depends on up to the third-order derivatives of the nonlinear function $g(u, v)$. The stability index is then

$$\begin{aligned} I &= \frac{1}{16}\left[\mathbf{a}\left(g_{uuu} + g_{uvv}\right) + \mathbf{b}\left(g_{uuv} + g_{vvv}\right)\right] \\ &\quad + \frac{1}{16}\sqrt{\frac{\varepsilon\theta}{1-\alpha}}\left[\left(\mathbf{a}^2 - \mathbf{b}^2\right) g_{uv}\left(g_{uu} + g_{vv}\right) + \mathbf{ab}\left(\left(g_{uu}\right)^2 - \left(g_{vv}\right)^2\right)\right] \end{aligned}$$

where the partial derivatives are

$$g_{uu} = \varphi''\left(d_{21}\right)^2, \quad g_{vv} = \varphi''\left(d_{22}\right)^2, \quad g_{uv} = \varphi'' d_{21} d_{22},$$

$$g_{uuu} = \varphi'''(d_{21})^3, \quad g_{vvv} = \varphi'''(d_{22})^3, \quad g_{uvv} = \varphi''' d_{21}(d_{22})^2, \quad g_{uuv} = \varphi'''\left(d_{21}\right)^2 d_{22}.$$

Then

$$I = \frac{\nu_0(\nu_0 - \varepsilon)}{16\varepsilon\theta^3}\varphi'''(0). \tag{19}$$

The sign of the stability index depends only on the sign of the third-derivative of $\varphi(z)$ at $z = 0$. In order to determine this sign, we need to specify $\varphi(z)$. Although Goodwin (1951) assumed a piecewise linear investment function, we, for the sake of analytical convenience, adopt the following smooth investment function of the form

$$\varphi_1(z) = \delta\left(\tan^{-1}(z - \eta) - \tan^{-1}(-\eta)\right) \text{ with } \delta > 0 \text{ and } \eta > 0, \tag{20}$$

This is a prototype function that possess a S-shaped curve. It is seen that it passes through the origin and has endogenous, asymptotic, asymmetric ceiling and floor. Since the inflection points of this function are positive (i.e., $z = \eta$), there is a positive z for which the derivative of the function is larger than the average of the function. In consequence, $\varphi_1(z)$ does not satisfy the uniqueness condition, $\varphi_1(z)/z - \varphi'(z) > 0$ for $z \neq 0$.[8]

Substituting $\varphi_1(z)$ for $\varphi(z)$ in (12), the corresponding dynamic system is obtained. Since $\varphi_1'(0)$ is monotonically increasing in δ, the partition line, the zero-discriminant curve and the bifurcation value of the model with $\varphi_1(z)$ are essentially the same as given in (16), (17) and (18). The sign of the stability index of the distributed delay model is therefore

$$\varphi_1'''(0) = \frac{2\delta(-1+3\eta^2)}{(1+\eta^2)^3}$$

which is positive if $\eta > 1/\sqrt{3}$ and negative if the inequality is reversed. It is followed that either of a unstable limit cycle or a stable limit cycle can bifurcate from the origin as δ departs from δ_0 $(= (1+\eta^2)\nu_0)$, depending on the value of η. To be more accurate, a stable limit cycle bifurcates from the origin (i.e., the stationary point) as δ increases from δ_0 if $\eta < 1/\sqrt{3}$ while an unstable limit cycle bifurcates from the origin as δ decreases from δ_0 if $\eta > 1/\sqrt{3}$. To summarize the above dicsussion, we have the following theorem:

Theorem 4 *A supercritical Hopf bifurcation occurs if $\eta < \frac{1}{\sqrt{3}}$ and a subcritical Hopf bifurcation occurs if $\eta > \frac{1}{\sqrt{3}}$ in the distributed delay model with $\varphi_1(z)$.*

3.3 Coexistence of Limit Cycles

In the previous section, we saw that the Hopf bifurcation theorem could be used to establish the existence of limit cycles. We step forward in this section. The point to notice is that the second example given in the end of Section 2 suggests a possibility of multiple limit cycles when a certain nonlinear planar system has a stable stationary state. We assume $\eta > 1/\sqrt{3}$ and examine this possibility that the distributed delay model with $\varphi_1(z)$ possesses multiple limit cycle. It is worthwhile to notice that $\varphi_1(z)$ has its ceiling three times higher than its floor for $\eta = 1$ as it was the case in Goodwin's model. The coexistence of multiple cycles has been already shown for a multiplier-accelerator model in Puu (1986), for Kaldor's business cycle model in Grasman and Wentzel (1994) and for a Metzlerian inventory cycle model in Matsumoto (1996) using an approach that will be further explored.

Due to Theorem 4, an unstable limit cycle exists for $\delta < \delta_0$ under the assumption that $\eta > 1/\sqrt{3}$. We will further show the existence of a stable limit cycle enclosing this unstable limit cycle. To this end, we will first construct an invariant set in such a way that once an orbit enters the set, it cannot escape from it at any future time and then apply the Poincaré-Bendixon theorem to examine whether a stable cycle can arise in the set. The result obtained is presented in the following theorem.

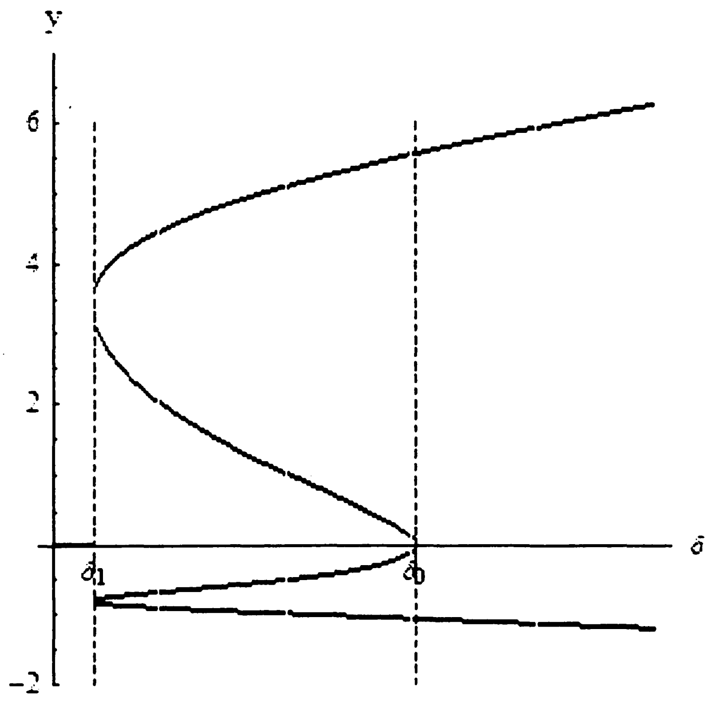
Theorem 5 *If $\eta > \frac{1}{\sqrt{3}}$, then the distributed delay model with $\varphi_1(z)$ has a stable limit cycle.*

Proof. See a proof given in Appendix B. ■

[8] See Sasakura (1996) for this uniqueness condition.

We are ready to present our second main result. In Figure 3 we show a bifurcation diagram in which the amplitude of the cycle is on the vertical axis and the bifurcation parameter δ on the horizontal axis. δ_0 is the critical value at which the distributed delay model loses its stability, and a limit cycle is born. For $\delta > \delta_0$, the model is destabilized for small perturbations so that any orbit moves away from the stationary point. Nonlinearity of the model prevents it from diverging globally but leads to a unique stable limit cycle. This is essentially the same cycle as the one that Goodwin (1951) demonstrates by applying the Lienard method. On the other hand, for $\delta < \delta_0$, the stationary point is locally stabilized but the model generates an unstable cycle as well as a stable cycle for δ in the interval $[\delta_1, \delta_0]$. It can be seen in Figure 3 that as δ decreases from δ_0, the amplitude of the inner unstable cycle increases and that of the outer stable cycle decreases. δ_1 is the other threshold value for which the two cycles coincide. For $\delta < \delta_1$, limit cycles no longer exit since the invariant set vanishes. To combine the first half of Theorem 2 with Theorem 5, we have the following result:

Theorem 6 *Given the stability of the stationary point, a stable limit cycle coexists with an unstable limit cycle that encloses a stable equilibrium point for the distributed delay model with* $\varphi_1(z)_{|\eta>1/\sqrt{3}}$.



Figures 3. Bifurcation diagram

The upper asymmetry (that is, the upper bound is larger than the lower bound) is the basic assumption of Goodwin (1951). Since it becomes larger as η becomes larger, the shift parameter η of $\varphi_1(z)$ represents the degree of this upper asymmetry. Theorem 6 implies that the strong upper asymmetry can be a source of the coexistence of limit cycles in the case of adopting $\varphi_1(z)$.

3.4 Another Specifications of $\varphi(z)$

We consider another versions of the investment function and pursue the possibility of multiple cycles. We start with the function that takes the following form,

$$\varphi_2(z) = \begin{cases} \delta_2 \tan^{-1}(z) & \text{if } z < 0, \\ \\ \delta_2(1+\eta^2)(\tan^{-1}(z-\eta) + \tan^{-1}(\eta)) & \text{if } z \geq 0. \end{cases} \tag{21}$$

It is a sort of hybrid formed from two different functions. It has the asymmetric S-shaped curve and is one time differentiable at $z = 0$ because the right-hand derivative of the first equation and the left-hand derivative of the second equation are identical. Substituting $\varphi_2(z)$ for $\varphi(z)$ of (12) yields exactly the same forms of the Jacobi matrix and the stability condition as for (13) and (15) in which $\nu = \delta_2$. It is, thus, clear that the requirements of the Hopf bifurcation theorem are fulfilled in the distributed delay model with $\varphi_2(z)$. It then follows that the bifurcation of the limit cycle from the origin occurs at the bifurcation value $\delta_2^0 = \nu_0$. The second and third derivatives of $\varphi_2(z)$ are not defined at the bifurcation point. Therefore, the stability index is not applicable to confirm the nature of the limit cycle. In spite of this, it is numerically confirmed, as shown in Figure 4, that two limit cycles coexist in the distributed delay model with $\varphi_2(z)$. These are alternatively unstable and stable. It can be seen that the trajectory starting at point A or B approaches the stable outer cycle while the trajectory starting at point C converges oscillating to the stationary point.

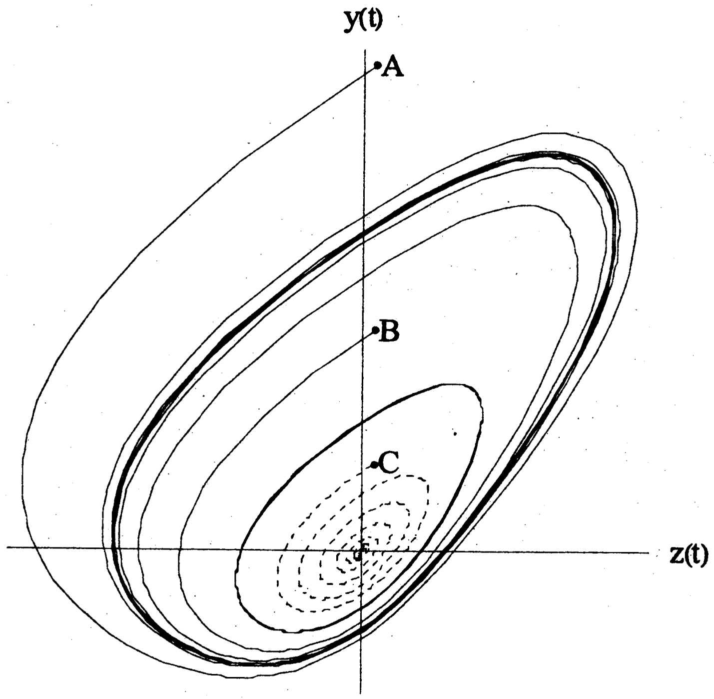


Figure 4. Coexistenc of two limit cycles in the model with $\varphi_2(z)$

Both $\varphi_1(z)$ and $\varphi_2(z)$ induce the distributed delay model to give rise to similar dynamics. Although these functions look similar, these are actually distinct. Indeed, these are identical for $z \geq 0$ if $\delta = 2\delta_2$ and $\eta = 1$. It is, nevertheless, as shown in Appendix C that these are distinct for $z < 0$. We have an alternative form of the investment function if the first equation of $\varphi_2(z)$ is exchanged with the second one. This modified function, which we denote by $\varphi_2^R(z)$, is convex-concave with respect to the origin and satisfies the uniqueness condition of the limit cycle, $\varphi_2^R(z)/z - \varphi_2^{R\prime}(z) > 0$ for $z \neq 0$. In consequence, the fourth version of the Goodwin model with $\varphi_2^R(z)$ produces a unique limit cycle. Furthermore, numerical simulations also indicate that the distributed delay model with $\varphi_2^R(z)$ does not produce multiple limit cycles.

The upper asymmetry of $\varphi_2^R(z)$ is defined only for $0 < \eta < 1$ and its degree does not increase as large as the asymmetry degree of $\varphi_1(z)$ and $\varphi_2(z)$. From these observations, we incline to infer that the multiplicity is due to a S-shaped investment function with the strong asymmetry. However, the following investment function implies that this is not necessary the case:

$$\varphi_3(z) = \frac{3C}{Be^{\delta\beta}}\left(\frac{1}{1+3e^{-\delta x}} - \frac{1}{4}\right) \text{ with } B > 0,\ C > 0,\ \delta > 0 \text{ and } \beta \geq 0. \quad (22)$$

This function has an asymmetric S-shaped curve and is designed to possess exactly the three time higher upper bound than the lower bound. Its asymptotic bounds are the same as those of $\varphi_1(z)$ and $\varphi_2(z)$ if $\eta = 1, \delta = 2\delta_2$ and $\delta\pi = \frac{3C}{B}e^{-\delta\beta}$. Although the three investment functions looks similar, their effects on the dynamics of the distributed delay model are quite different. The third derivative of $\varphi_3(z)$ is

$$\varphi_3'''(0) = -\frac{9C}{128B}\delta^3 e^{-\delta\beta} < 0.$$

This implies that the stability index in the case of adopting $\varphi_3(z)$ is negative. Therefore a subcritical Hopf bifurcation occurs and numerical simulations imply no occurrence of multiplicity of limit cycles in this case.

Apart from the multiplicity issue, $\varphi_3(z)$ unveils a new dynamic aspect of the Goodwin model, that is, extinction of cyclic oscillations in the case of a strong nonlinearity or large time delay. The point is that the derivative of $\varphi_3(z)$ at the stationary point is not monotonic but has a hump with respect to δ if β of $\varphi_3(z)$ is positive,

$$\varphi_3'(0) = \left(\frac{3}{4}\right)^2 \frac{C}{B}\frac{\delta}{e^{\delta\beta}}.$$

We will delineate the instability region of the model with $\varphi_3(z)_{|\beta>0}$ in the (δ, θ) space to see how the stationary state loses stability and what kind of bifurcation occurs when the boundaries of such instability region is crossed. From the partition line (16) and the zero-dicsriminant locus (17) having $\varphi_3'(0)$, we can illustrate the instability region that is half-oval shaped and shaded in gray in Figure 5(A). It is further divided into two parts by the zero-discriminant locus, above which the eigenvalues are complex and below which the eigenvalues are real. The lower area is depicted in darker gray. A more detailed analysis of Figure 5(A) leads to the following two important points.

(1) Parameter δ has both a stabilizing effect and a destabilizing effect for $\theta < \theta_0$.[9] This actually occurs, for example, on the horizontal line passing

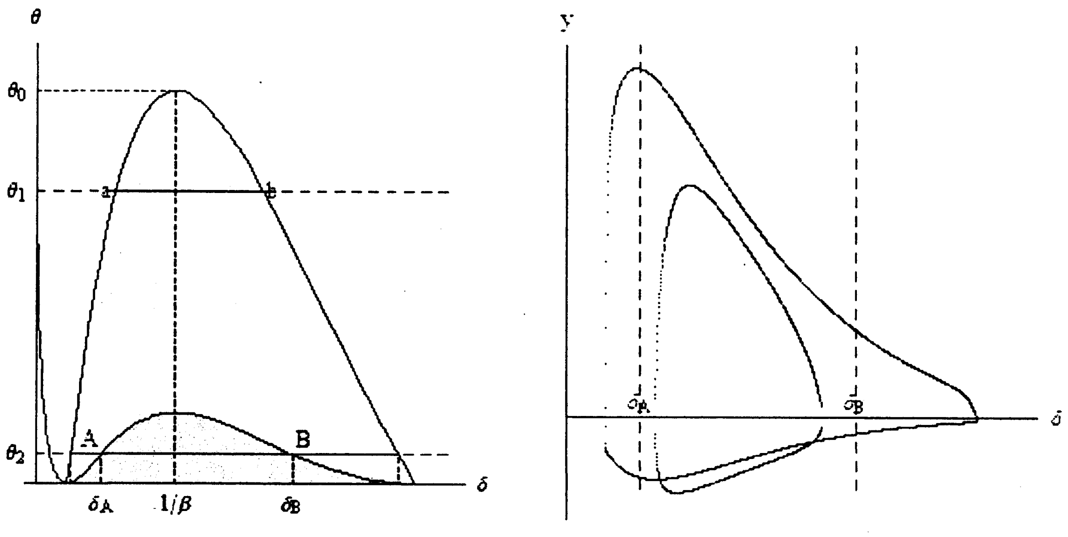
[9] The partition line with $\varphi_3(z)$ has double roots for $\delta = 1/\beta$ and

$$\theta_0 = \frac{1}{1-\alpha}\left(\frac{9C}{16B\beta e} - \varepsilon\right).$$

through θ_1 of Figure 5(A). As δ increases from zero along this line, the model is stable for a smaller value, becomes unstable and generates a limit cycle via a supercritical Hopf bifurcation when δ crosses the left-side part of the partition line at point a and finally becomes stable again when δ crosses the right-side part at point b. A birth and extinction of the limit cycle can be seen in the bifurcation diagram with respect to δ in Figure 5(B). It is also observed that the limit cycle disappears for larger value of θ.

(2) For a very small value of θ, the Hopf bifurcation theorem may not guarantee the existence of a limit cycle. This occurs along the horizontal line passing through θ_2 of Figure 5(A) that crosses the zero-discriminant locus twice at points A and B. For $\delta_A < \delta < \delta_B$ and θ_2, the discriminant is positive and hence the Hopf bifurcation theorem is inapplicable. However, as seen in Figure 5(B), the model still generates a limit cycle. Another mathematical treatment is necessary to confirm an existence of a limit cycle in this interval of δ. The application of the Poincaré-Bendixon theorem is a reasonable approache because the model has a single unstable stationary point and a S-shaped investment function, which can be a typical application of the theorem.[10]

The location and size of the instable region are dependent on a choice of the parameter values but the half-oval shape is independent of it. The same observations as mentioned above apply for the different configurations of the parameters.



Figures 5(A) and (B). Instability region and bifurcation diagram

4 Concluding Remarks

We reconsider Goodwin's nonlinear accelerator model of business cycle and demonstrate two new features. Adopting two different approaches for dealing

[10] See Chaper 2.2 of Lorenz (1993) for example.

with a lag in investment, we first show that the Goodwin model with continuously distributed time lag is more stable than the one with fixed time lag (that is, Theorems 1, 2 and 3). Specifying the form of the investment function, we then confirm the coexistence of multiple business cycles when the stationary state is locally stable and the investment function has the strong degree of asymmetry between the lower bound and the upper bound of investment (that is, Theorems 4, 5 and 6). The multiplicity, however, remains as a matter to be discussed further. It is sensitive to the choice of the investment function as we have seen that the model with $\varphi_1(z)$ produces multiple limit cycles and the model with $\varphi_3(z)$ does not although both functions have the similar nonlinerities.

Concerning the second feature, we show, by combining the result obtained from the Hopf bifurcation theorem with the one using Poincaré-Bendixon theorem, that two limit cycles can coexist with the stable stationary state: one cycle is unstable and surrounds the stationary state, and the other is stable and encloses the unstable limit cycle. This finding indicates that a damping force dominates and makes trajectories approach the stationary state for small disturbances but an anti-damping force dominates and makes trajectories converge to the outer stable limit cycle for larger disturbances. The result implies global stability of Goodwin's model regardless of the local dynamic properties.

Appendix A: Proof of Theorem 1.

Substituting $y(t) = y_0 e^{\lambda t}$ into (3), dividing both sides of the resultant equation by $\varepsilon y_0 e^{\lambda t}$ and introducing the new variables $A = \frac{1-\alpha}{\varepsilon}$ and $B = -\frac{\nu}{\varepsilon}$, we obtain the corresponding characteristic equation:

$$\lambda + A + B\lambda e^{-\lambda\theta} = 0. \tag{A-I}$$

Kuang (1993) derives explicit conditions for stability/instability of the n-th order linear real scalar neutral differential difference equation with a single delay. (A-I) is a special case of the n-th order equation. Applying his methods, we can examine the stability of the linearized equation (3) in the following three cases.

(i) If $v > \varepsilon$, then applying the result of Kuang (1993, Theorem 1.2) implies that the real parts of the solutions of equation (A-I) are positive for all θ.

(ii) If $v < \varepsilon$, (A-I) has at most finitely many eigenvalues with negative real part. We will demonstrate that a change in θ induces no stability switching in the fixed delayed model. Kuang (1993, Theorem 1.4) shows that if the stability switches at $\theta = \bar{\theta}$, then (A-I) must have a pair of pure conjugate imaginary roots with $\theta = \bar{\theta}$. To find the critical value of $\bar{\theta}$, we assume that $\lambda = i\omega$, with $\omega > 0$, is a root of (A-I) for $\theta = \bar{\theta}$, $\bar{\theta} \geq 0$. Substituting $\lambda = i\omega$ into (A-I), we have

$$A + B\omega \sin \omega\theta = 0$$

and

$$\omega + B\omega \cos \omega\theta = 0.$$

Moving A and ω to the right hand side and adding the squares of the resultant equations, we obtain

$$A^2 + (1 - B^2)\omega^2 = 0.$$

Since $A > 0$ and $1 - B^2 > 0$ as $|B| < 1$ is assumed, there is no ω that satisfies the above equation. In other words, there are no roots of (A-I) crossing the imaginary axis when θ increases. Therefore, there are no stability switches for any θ.

(iii) If $\varepsilon = \nu$, the characteristic equation becomes

$$\lambda(1 - e^{-\lambda\theta}) + A = 0. \tag{A-II}$$

It is clear that $\lambda = 0$ is not a solution of (A-II) since $A > 0$. Thus we can assume that a root of (A-II) has a non-negative real part, $\lambda = u + iv$ with $u \geq 0$ for some $\theta > 0$. From (A-II), we have

$$(u + A)^2 + v^2 = e^{-2u\theta}(u^2 + v^2) \leq (u^2 + v^2),$$

in which the last inequality is due to $e^{-2u\theta} \leq 1$ for $u \geq 0$ and $\theta > 0$. This implies that

$$2uA + A^2 \leq 0,$$

in which the direction of inequality contradicts the assumption that $u \geq 0$ and $A > 0$. Hence it is impossible that the characteristic equation has roots with nonnegative real parts. Therefore, all roots of (II) must have negative real parts for all $\theta > 0$.■

Appendix B: Proof of Therem 3

Theorem 2 implies that the distributed delay model with $\varphi_1(z)_{|\eta > 1/\sqrt{3}}$ has an unstable limit cycle surrounding the stationary point. It is denoted by Γ_1. Let $\mathbf{A}$ be the interior domain of Γ_1, which is an open set. Let $\mathbf{x}(t) = (z(t), y(t)) \in R^2$ be a solution of the model and let $\mathbf{x}_1 = \mathbf{x}(t_1)$ and $\mathbf{x}_2 = \mathbf{x}(t_2)$ be two successive points of intersection of the vertical axis y in Figure A in which $\alpha = 0.6$, $\varepsilon = 0.5$, $\theta = 0.8$, $\eta = 1$, $\delta_0 = \varepsilon + (1-\alpha)\theta$ and $\delta = \delta_0 - 0.1$. It is assumed, first that $t_1 < t_2$, second that $\mathbf{x}_1 \notin \mathbf{A}$ and, third, that $\mathbf{x}_1 \neq \mathbf{x}_2$. Then the arc $\{\mathbf{x} \in \mathbf{R}^2 \mid \mathbf{x} = \mathbf{x}(t),\ t_1 < t < t_2\}$ together with the closed segment $\overline{\mathbf{x}_1\mathbf{x}_2}$ comprises a closed curve Γ_2, which corresponds to the line segment $ABCDE$. Γ_2 separates the whole plane into two regions: one is the set of interior of Γ_2 that is bounded and simply connected and the other is its exterior. Let $\mathbf{B}$ be the set of Γ_2 and its interior. Lastly construct a set by deleting $\mathbf{A}$ from $\mathbf{B}$ and denote it by $\mathbf{C} = \mathbf{B}\backslash\mathbf{A}$, which is the shaded region in Figure A. It can be confirmed that $\mathbf{C}$ is a compact set and has no stationary point. It is also confirmed that any trajectory starting inside $\mathbf{C}$ stays within $\mathbf{C}$. Hence, the Poincarè-Bendixon theorem guarantees the existence of one stable limit cycle in $\mathbf{C}$ as illustrated as the bold cycle in Figure A.■

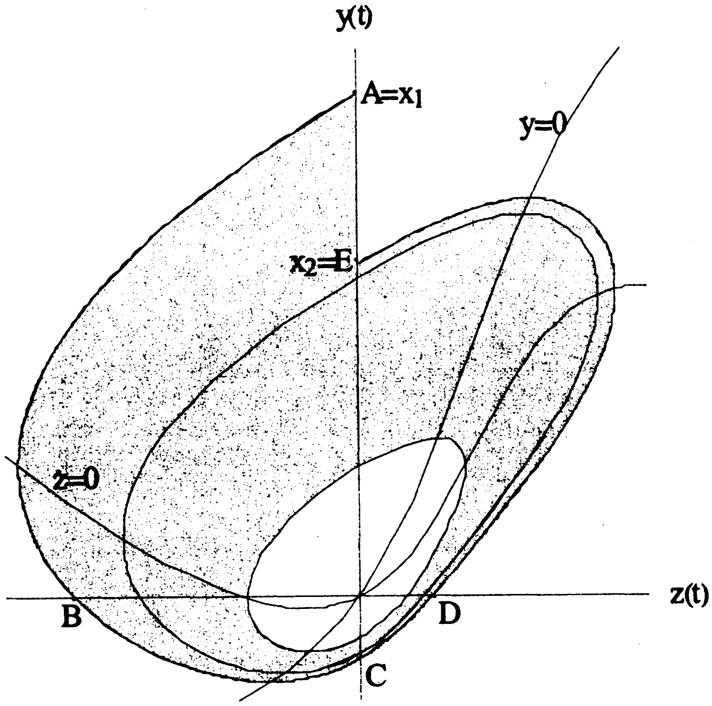


Figure A. Stable limit cycle enclosing the unstable limit cycle

Appendic C

In this appendix we show that the following two equations are distinct for $z < 0$ under the conditions of $\eta = 1$ and $\delta = 2\delta_2$,

$$\delta(\tan^{-1}(z-\eta) + \tan^{-1}(\eta)) \text{ and } \delta_2 \tan^{-1}(z).$$

Suppose in contrary that they are equal and denote $x = -z$ for notational simplicity. Then we have

$$-\tan^{-1}(x) = 2\left(-\tan^{-1}(x+1) + \frac{\pi}{4}\right).$$

Introducing new variables A and B defined by $\tan A = x$ and $\tan B = x + 1$, we can rewrite the last equality as

$$2B - A = \frac{\pi}{2} \text{ or } \tan(2B - A) = \infty.$$

Using the following formulas,

$$\tan(2x) = \frac{2\tan(x)}{1-\tan^2(x)} \text{ and } \tan(x-y) = \frac{\tan(x) - \tan(y)}{1+\tan(x)\tan(y)},$$

we have

$$\tan(2B - A) = \frac{2 + 2x - 2x^2 - x^3}{-4x - 3x^2} = \infty.$$

This implies that two functions are equal only for $z = 0$ or $z = -\infty$.∎

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