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REPRESENTATIONS OF MULTICATEGORIES OF PLANAR DIAGRAMS AND TENSOR CATEGORIES

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We shall discuss how the notions of multicategories and their linear representations are related with tensor categories. When one focuses on the ones arizing from planar diagrams, it particularly implies that there is a natural one-to-one correspondence between planar algebras and singly generated bicategories.

1. Multicategories

Multicategory is a categorical notion which concerns a class of objects and morphisms so that morphisms are enhanced to admit multiple objects as inputs, whereas outputs are kept to be single. The operation of composition can therefore be performed in a ramified way, which is referred to as **plugging** in what follows. The associativity axiom for plugging and the neutrality effect of identity morphisms enable us to visualize the result of repeated pluggings as a rooted tree (Figure 1).

As in the case of ordinary category, defined are functors as well as natural transformations and natural equivalences between them. We say that two multicategories $\mathcal{M}$ and $\mathcal{N}$ are **equivalent** if we can find functors $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$ so that their compositions $F \circ G$ and $G \circ F$ are naturally equivalent to identitiy functors.

Example 1.1. The multicategory $\mathcal{M}\mathcal{S}et$ of sets (and maps) and the multicategory $\mathcal{M}\mathcal{V}ec$ of vector spaces (and multilinear maps).

Given a (strict) monoidal category $\mathcal{C}$, we define a multicategory $\mathcal{M}$ so that $\mathcal{C}$ and $\mathcal{M}$ have the same class of objects and $\mathrm{Hom}(X_1 \times \cdots \times X_d, X) = \mathrm{Hom}(X_1 \otimes \cdots \otimes X_d, X)$.

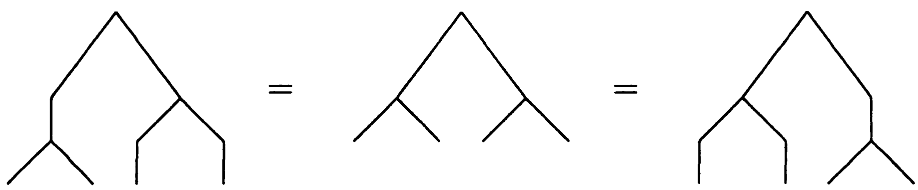


FIGURE 1

Proposition 1.2. *Let $\mathcal{C}'$ be another monoidal category with $\mathcal{M}'$ the associated multicategory. Then a multicategory-functor $\mathcal{M} \to \mathcal{M}'$ is in a one-to-one correspondence with a weakly monoidal functor $\mathcal{C} \to \mathcal{C}'$. Here by a weakly monoidal functor we shall mean a functor $F : \mathcal{C} \to \mathcal{C}'$ with a natural family of morphisms $m_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ satisfying the hexagonal identities for associativity.*

Proof. Given a weakly monoidal functor $F : \mathcal{C} \to \mathcal{C}'$, we extend it to a multicategory-functor $\widetilde{F} : \mathcal{M} \to \mathcal{M}'$ by the composition

$$\widetilde{F}(T) = \Big(F(X_1) \otimes \cdots \otimes F(X_l) \xrightarrow{m} F(X_1 \otimes \cdots \otimes X_l) \xrightarrow{F(T)} F(X)\Big)$$

with $T \in \mathrm{Hom}_{\mathcal{M}}(X_1, \ldots, X_l; X) = \mathrm{Hom}_{\mathcal{C}}(X_1 \otimes \cdots \otimes X_l, X)$. Then $\widetilde{F}$ is multiplicative: Let $T_j : X_{j,1} \otimes \cdots \otimes X_{j,d_j} \to X_j$ $(j = 1, \ldots, l)$ and consider the composition $T \circ (T_1, \ldots, T_l)$. By definition, $\widetilde{F}(T) \circ (\widetilde{F}(T_1), \ldots, \widetilde{F}(T_l))$ is given by

$$\begin{array}{c}
F(X_{1,1}) \otimes \cdots \otimes F(X_{1,d_1}) \otimes \cdots \otimes F(X_{l,1}) \otimes \cdots \otimes F(X_{l,d_l}) \\
\Big\downarrow {\scriptstyle m \otimes \cdots \otimes m} \\
F(X_{1,1} \otimes \cdots \otimes X_{1,d_1}) \otimes \cdots \otimes F(X_{l,1} \otimes \cdots \otimes X_{l,d_l}) \\
\Big\downarrow {\scriptstyle F(T_1) \otimes \cdots \otimes F(T_l)} \\
F(X_1) \otimes \cdots \otimes F(X_l) \\
\Big\downarrow {\scriptstyle m} \\
F(X_1 \otimes \cdots \otimes X_l) \\
\Big\downarrow {\scriptstyle F(T)} \\
F(X)
\end{array}$$

By the commutativity of

$$\begin{array}{ccc}
F(X_{1,1} \ldots X_{1,d_1}) \ldots F(X_{l,1} \ldots X_{l,d_l}) & \xrightarrow{m} & F(X_{1,1} \ldots X_{1,d_1} \ldots X_{l,1} \ldots X_{l,d_l}) \\
\Big\downarrow {\scriptstyle F(T_1) \otimes \cdots \otimes F(T_l)} & & \Big\downarrow {\scriptstyle F(T_1 \otimes \cdots \otimes T_l)} \\
F(X_1) \ldots F(X_l) & \xrightarrow[m]{} & F(X_1 \ldots X_l)
\end{array}$$

and the identity $m \circ (m \otimes \cdots \otimes m) = m$, $\widetilde{F}(T) \circ (\widetilde{F}(T_1), \ldots, \widetilde{F}(T_l))$ is identical with the composition

$$\begin{array}{c} F(X_{1,1}) \otimes \cdots \otimes F(X_{1,d_1}) \otimes \cdots \otimes F(X_{l,1}) \otimes \cdots \otimes F(X_{l,d_l}) \\ \downarrow \\ F(X_{1,1} \otimes \cdots \otimes X_{1,d_1} \otimes \cdots \otimes X_{l,1} \otimes \cdots \otimes X_{l,d_l}) \\ \downarrow \\ F(X_1 \otimes \cdots \otimes X_l) \\ \downarrow \\ F(X) \end{array},$$

which is equall to $\widetilde{F}(T \circ (T_1, \ldots, T_l))$.

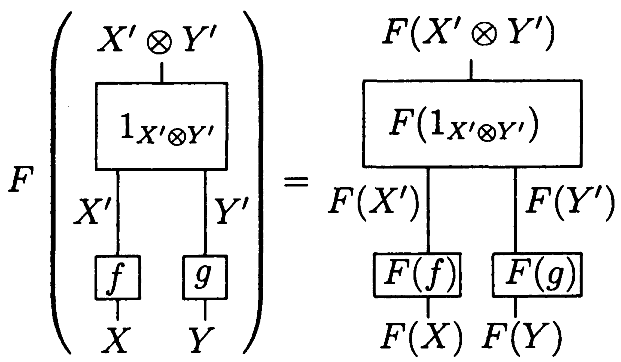
Conversely, starting with a multicategory-functor $\widetilde{F} : \mathcal{M} \to \mathcal{M}'$, let $F : \mathcal{C} \to \mathcal{C}'$ be the restriction of $\widetilde{F}$ and set

$$m_{X,Y} = \widetilde{F}(1_{X \otimes Y}) : F(X) \otimes F(Y) \to F(X \otimes Y).$$

Here $1_{X\otimes Y}$ in the argument of $\widetilde{F}$ is regarded as a morphism in $\mathrm{Hom}_{\mathcal{M}}(X, Y; X \otimes Y) = \mathrm{End}_{\mathcal{C}}(X \otimes Y)$. The commutativity of

$$\begin{tikzcd} F(X) \otimes F(Y) \arrow[r, "F(f)\otimes F(g)"] \arrow[d, "m_{X,Y}"'] & F(X') \otimes F(Y') \arrow[d, "m_{X',Y'}"] \\ F(X \otimes Y) \arrow[r, "F(f\otimes g)"'] & F(X' \otimes Y') \end{tikzcd}$$

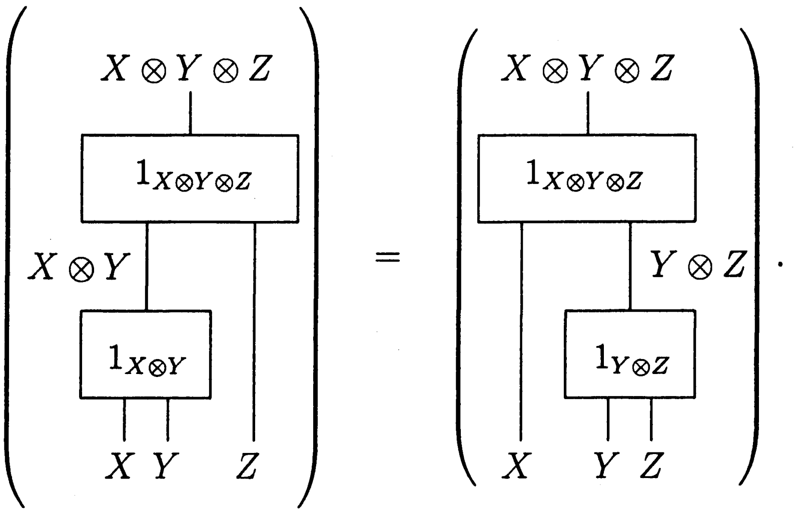
follows from



and the associativity of m (the hexagonal identities), i.e., the commutativity of

$$\begin{tikzcd} F(X) \otimes F(Y) \otimes F(Z) \arrow[r, "m_{X,Y}"] \arrow[d, "m_{Y,Z}"'] & F(X \otimes Y) \otimes F(Z) \arrow[d, "m_{X\otimes Y,Z}"] \\ F(X) \otimes F(Y \otimes Z) \arrow[r, "m_{X,Y\otimes Z}"'] & F(X \otimes Y \otimes Z) \end{tikzcd}$$

is obtained if we apply $\widetilde{F}$ to the identity



□

Definition 1.3. A (linear) **representation** of a multicategory $\mathcal{M}$ is just a functor $F : \mathcal{M} \to \mathcal{MV}ec$. A representation is equivalently described in terms of a family $\{V_X\}$ of vector spaces indexed by objects of $\mathcal{M}$ together with a family of multilinear maps $\{\pi_T : V_{X_1} \times \cdots \times V_{X_d} \to V_X\}$ indexed by morphisms in $\mathcal{M}$ (satisfying certain relations for multiplicativity).

An intertwiner between two representations $\{\pi_T, V_X\}$ and $\{\pi'_T, V'_X\}$ is defined to be a natural linear transformation, which is specified by a family of linear maps $\{\varphi_X : V_X \to V'_X\}$ making the following diagram commutative for each morphism $T : X_1 \times \cdots \times X_d \to X$ in $\mathcal{M}$:

$$\begin{tikzcd}
V_{X_1} \times \cdots \times V_{X_d} \arrow[r, "\pi_T"] \arrow[d, "\varphi_{X_1}\times\cdots\times\varphi_{X_d}"'] & V_X \arrow[d, "\varphi_X"] \\
V'_{X_1} \times \cdots \times V'_{X_d} \arrow[r, "\pi'_T"'] & V'_X
\end{tikzcd}$$

If $\mathcal{M}$ is a small multicategory (i.e., objects of $\mathcal{M}$ form a set), representations of $\mathcal{M}$ constitute a category $\mathcal{R}ep(\mathcal{M})$ whose objects are representations and morphisms are intertwiners.

Let $F : \mathcal{M} \to \mathcal{N}$ be a functor between small multicategories. By pulling back, we obtain a functor $F^* : \mathcal{R}ep(\mathcal{N}) \to \mathcal{R}ep(\mathcal{M})$; given a representation (π, V) of $\mathcal{N}$, $F^*(\pi, V) = (F^*\pi, F^*V)$ is the representation of $\mathcal{M}$ defined by $(F^*V)_X = V_{F(X)}$ and $(F^*\pi)_T = \pi_{F(T)}$.

If $\phi : F \to G$ is a natural transformation $\{\phi_X : F(X) \to G(X)\}$ with $G : \mathcal{M} \to \mathcal{N}$ another functor, it induces a natural transformation $\varphi : F^* \to G^*$: Let (π, V) be a representation of $\mathcal{N}$. Then $\varphi_{(\pi,V)} : F^*(\pi, V) \to G^*(\pi, V)$ is an intertwiner between representations of $\mathcal{M}$ defined by the family $\pi(\phi) = \{\pi_{\phi_X} : V_{F(X)} \to V_{G(X)}\}$ of linear maps.

By the multiplicativity of π, the correspondence $\phi \to \varphi$ is multiplicative as well and the construction is summarized to be defining a functor $\mathcal{H}om(\mathcal{M}, \mathcal{N}) \to \mathcal{H}om(\mathcal{R}ep(\mathcal{N}), \mathcal{R}ep(\mathcal{M}))$.

Proposition 1.4. *The family of functors*

$$\mathcal{H}om(\mathcal{M}, \mathcal{N}) \to \mathcal{H}om(\mathcal{R}ep(\mathcal{N}), \mathcal{R}ep(\mathcal{M}))$$

for various multicategories $\mathcal{M}$ and $\mathcal{N}$ defines a anti-multiplicative meta-functor of strict bicategories: $(F \circ G)^* = G^* \circ F^*$ for $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{L} \to \mathcal{M}$.

Corollary 1.5. *If small multicategories $\mathcal{M}$ and $\mathcal{N}$ are equivalent, then so are their representation categories $\mathcal{R}ep(\mathcal{M})$ and $\mathcal{R}ep(\mathcal{N})$.*

Proof. If an equivalence between $\mathcal{M}$ and $\mathcal{N}$ is given by functors $F : \mathcal{M} \to \mathcal{N}$ and $G : \mathcal{N} \to \mathcal{M}$ with $F \circ G \cong \mathrm{id}_{\mathcal{N}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{M}}$, then $G^* \circ F^* = (F \circ G)^* \cong \mathrm{id}_{\mathcal{R}ep(\mathcal{N})}$ and $F^* \circ G^* = (G \circ F)^* \cong \mathrm{id}_{\mathcal{R}ep(\mathcal{M})}$ show the equivalence between $\mathcal{R}ep(\mathcal{M})$ and $\mathcal{R}ep(\mathcal{N})$. $\square$

As observed in [3], the multicategory $\mathcal{MV}ec$ admits a special object; the vector space of the ground field itself, which plays the role of unit when multiple objects are regarded as products. In the multicategory $\mathcal{MS}et$, the special object in this sense is given by any one-point set. Multicategories of planar diagrams to be discussed shortly also admit such special objects; disks or boxes without pins. It is therefore natural to impose the condition that V_S is equal to the ground field for a special object S.

It is quite obvious to introduce other enhanced categories of similar flabor: co-multicategories and bi-multicategories with hom-sets indicated by

$$\mathrm{Hom}(X; X_1, \dots, X_d), \quad \mathrm{Hom}(X_1, \dots, X_m; Y_1, \dots, Y_n)$$

respectively.

2. Planar Diagrams

We introduce several multicategories related with planar diagrams (namely, tangles without crossing points).

2.1. **Disk Type.** Let n be a non-negative integer. By a disk of type n (or simply an n-disk), we shall mean a disk with n pins attached on the peripheral and numbered consecutively from 1 to n anticlockwise.

Our first example of multicategories has n-disks for various n as objects with morphisms given by planar diagrams connecting pins inside the multiply punctured region of the target object (disk), Figure 2. The multicategory obtained in this way is denoted by $\mathcal{D}_\circ$ and called the multicategory of planar diagrams of disk type. The identity morphisms are given by diagrams consisting of spokes (Figure 3).

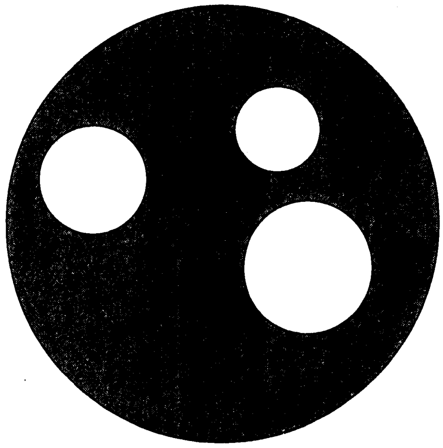


FIGURE 2

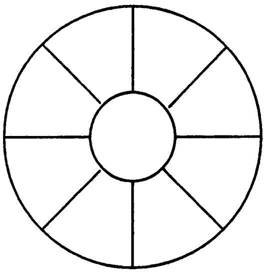
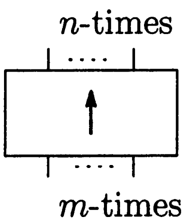


FIGURE 3

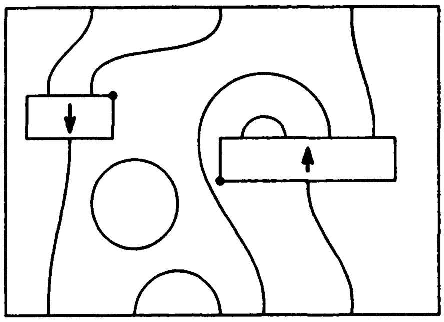
2.2. **Box Type.** Let $m, n \in \mathbb{N}$ be non-negative integers. By a box of type (m, n) or simply an (m, n)-box, we shall mean a rectangular box with m pins and n pins attached on the lower and upper edges respectively. Visually, the distinction of lower and upper edges can be indicated by putting an arrow from bottom to top.



The second example of multicategory has (m, n)-boxes for various m, n as objects. For a pictorial description of morphisms, we distinguish boxes depending on whether it is used for outputs or inputs; outer or inner boxes, where pins are sticking out inward or outward respectively. For outer boxes, arrows are often omitted. When $m = n$, the box is said to be diagonal.

By a **planar** (m, n)**-diagram** or simply an (m, n)**-diagram**, we shall mean a planar arrangement of inner boxes and curves (called strings) inside an outer (m, n) box with each endpoint of strings connected to exactly one pin sticking out of inner or outer boxes so that no pins are left free. We shall not ditinguish two (m, n)-diagrams which are planar-isotopic.

$n=3$



$m=5$

If inner boxes are distinguished by numbers $1,\dots,d$, we have a sequence of their types $((m_1,n_1),\dots,(m_d,n_d))$. When all relevant boxes are diagonal, the diagram is said to be diagonal.

Multicategory morphisms are then given by planar diagrams with the following operation of **plugging** (or nesting): Let T be a planar (m,n)-diagram containing boxes of inner type $((m_j,n_j))_{1\leq j\leq d}$ and T_j be an (m_j,n_j)-diagram $(1\leq j\leq d)$. Then the plugging of T_j into T results in a new (m,n)-diagram, which is denoted by $T\circ(T_1\times\cdots\times T_d)$. Note that the plugging produces diagonal planar diagrams out of diagonal ones.

The plugging operation satisfies the associativity and we obtain a multicategory $\mathcal{D}_\square$, which is referred to as the **multicategory of planar diagrams** of box type. Here identity morphisms are given by parallel vertical lines. Note that two objects (boxes) are isomorphic if and only if they have the same number $m+n$ of total pins.

When objects are restricted to disks or boxes having even number of pins, we have submulticategories $\mathcal{D}_\circ^{\mathrm{even}}$ and $\mathcal{D}_\square^{\mathrm{even}}$.

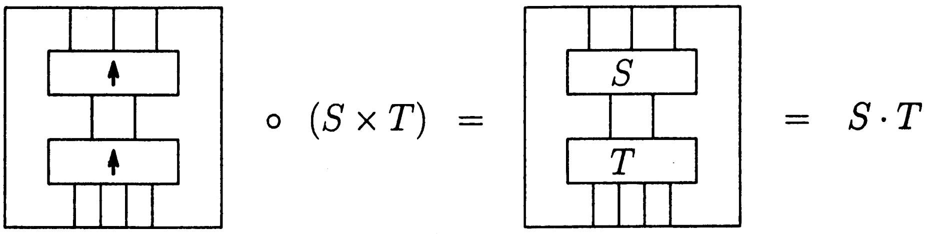
If boxes (objects) are further restricted to diagonal ones in $\mathcal{D}_\square^{\mathrm{even}}$, then we obtain another submulticategory $\mathcal{D}_\triangle$ as a subcategory of $\mathcal{D}_\square$.

Proposition 2.1. *Two multicategories $\mathcal{D}_\circ$, $\mathcal{D}_\square$ are equivalent. Three multicategories $\mathcal{D}_\circ^{\mathrm{even}}$, $\mathcal{D}_\square^{\mathrm{even}}$ and $\mathcal{D}_\triangle$ are equivalent, whence they produce equivalent representation categories.*

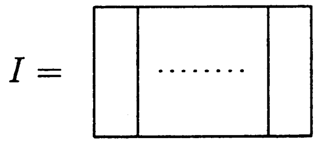
Proof. The obvious functors $\mathcal{D}_\square\to\mathcal{D}_\circ$, $\mathcal{D}_\square^{\mathrm{even}}\to\mathcal{D}_\circ^{\mathrm{even}}$ and $\mathcal{D}_\triangle\to\mathcal{D}_\square^{\mathrm{even}}$ are fully faithful. For example, to see the essential surjectivity of $\mathcal{D}_\triangle\to\mathcal{D}_\square^{\mathrm{even}}$ on objects, given an object of $\mathcal{D}_\square^{\mathrm{even}}$labeled by (m,n), let $S:(m,n)\to((m+n)/2,(m+n)/2)$ and $T:((m+n)/2,(m+n)/2)\to(m,n)$ be morphisms in $\mathcal{D}_\square$ obtained by bending strings in the right vacant space. Then $S\circ T=1_{(m+n)/2,(m+n)/2}$ and $T\circ S=1_{m,n}$ show that (m,n) and $((m+n)/2,(m+n)/2)$ are isomorphic as objects. $\square$

Here are three special plugging operations of special interest in $\mathcal{D}_\square$; composition, juxtaposition and transposition.

Composition (or product) produces an (l,n)-diagram ST from an (m,n)-diagram S and an (l,m)-diagram T: $((l,m,n)=(4,2,3)$ in the figure)

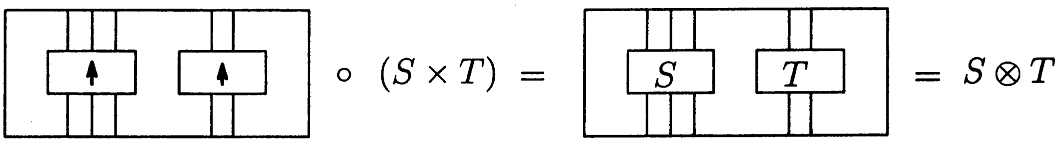


The composition satisfies the associativity law and admits the identity diagrams for multiplication.



In this way, we have found another categorical structure for planar diagrams of box type; the category $\mathcal{M}$ has natural numbers $0, 1, 2, \dots$ as objects with hom sets $\mathcal{M}(m, n)$ consisting of (m, n)-diagrams.

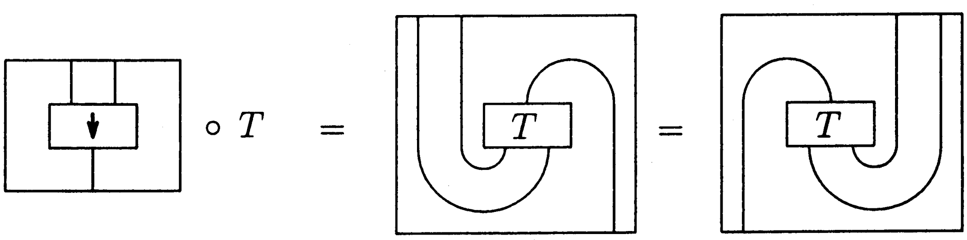
Juxtaposition (or tensor product) produces an $(k + m, l + n)$-diagram $S \otimes T$ from an (l, k)-diagram S and an (m, n)-diagram T.



With this operation, $\mathcal{M}$ becomes a strict monoidal category ($m \otimes n = m + n$). The unit object is 0 with the identity morphism in $\mathcal{M}(0, 0)$ given by the empty diagram (neither inner boxes nor strings).

Warning: monoidal categories connote multicategory structure as observed before, which is, however, different from $\mathcal{D}_\square$: They have different classes of objects.

Transposition is an involutive operation on planar diagrams of box type, which produces an (n, m)-diagram ${}^tT$ out of an (m, n)-diagram T.



Notice the last equality holds by planar isotopy. Here are some obvious identities:

$$ {}^t(ST) = {}^tT\,{}^tS, \qquad {}^t(S \otimes T) = {}^tT \otimes {}^tS. $$

With this operation, our monoidal category $\mathcal{M}$ is furnished with a **pivotal** structure.

From the definition, a representation of the multicategory $\mathcal{D}_\Box$ means a family of vector spaces $\{P_{m,n}\}_{m,n\geq 0}$ together with an assignment of a linear map

$$\pi_T : P_{m_1,n_1} \otimes \cdots \otimes P_{m_d,n_d} \to P_{m,n}$$

to each morphism T in $\mathcal{D}_\Box$, which satisfies

$$\pi_T(\pi_{T_1}(x_1) \otimes \cdots \otimes \pi_{T_d}(x_d)) = \pi_{T\circ(T_1\times\cdots\times T_d)}(x_1 \otimes \cdots \otimes x_d).$$

According to V. Jones, this kind of algebraic structure is referred to as a **planar algebra**. In what follows, we use the word 'tensor category' to stand for a linear monoidal category.

Proposition 2.2. *A representation $P = \{P_{m,n}\}$ of $\mathcal{D}_\Box$ gives rise to a strict pivotal tensor category $\mathcal{P}$ generated by a single self-dual object X:* $\mathrm{Hom}(X^{\otimes m}, X^{\otimes n}) = P_{m,n}$*, composition of morphisms is given by $ab = \pi_C(a\otimes b)$, tensor product of morphisms is $a\otimes b = \pi_J(a\otimes b)$ and pivotal structure is given by transposition operation. (The identity morphisms are π_I.) The construction is functorial and an intertwiner $\{f_{m,n} : P_{m,n} \to P'_{m,n}\}$ between representations induces a monoidal functor $F : \mathcal{P} \to \mathcal{P}'$ preserving pivotality.*

Conversely, given a pivotal tensor category $\mathcal{P}$ generated by a self-dual object X, we can produce a representation so that $P_{m,n} = \mathrm{Hom}(X^{\otimes m}, X^{\otimes n})$.

Proof. Since the monoidal structure is defined in terms of special forms of plugging, an intertwiner induces a monoidal functor.

Conversely, suppose that we are given a pivotal tensor category with a generating object X. Let $\epsilon : X \otimes X \to I$ and $\delta : I \to X \otimes X$ give a rigidity pair satisfying $\epsilon = {}^t\delta$. Given a planar diagram T, let π_T be a linear map obtained by replacing vertical parts, upper and lower arcs of strings with the identity, δ and ϵ respectively. Then the rigidity identities ensure that π_T is unchanged under planar isotopy on strings if relevant boxes are kept unrotated, while the pivotality witnesses the planar isotopy for rotation of boxes (see Figure 4). The multiplicativity of π for plugging is now obvious from the construction. $\Box$

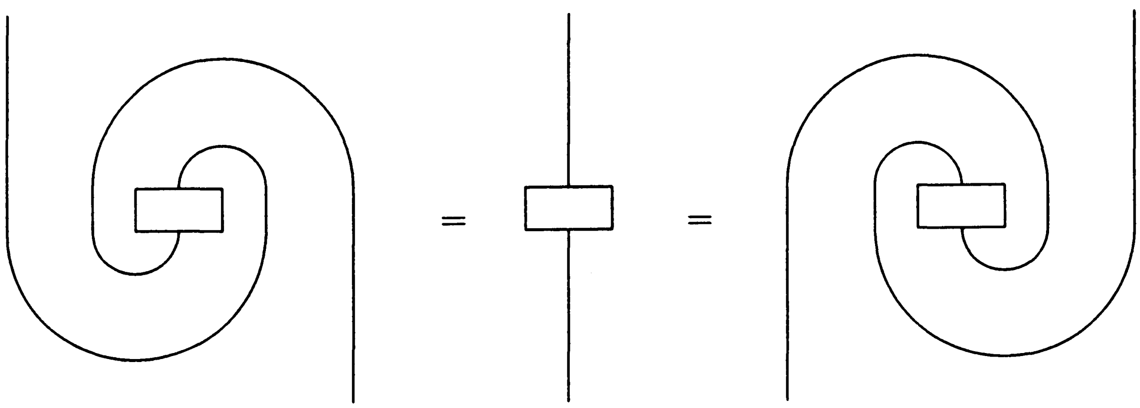


FIGURE 4

Remark .

(i) The condition $\dim P_{0,0} = 1$ is equivalent to the simplicity of the unit object of the associated tensor category.

(ii) If one starts with a representation P and make $\mathcal{P}$, then the pivotal category $\mathcal{P}$ produces P itself. If one starts with a pivotal tensor category $\mathcal{P}$ with P the associated representation and let $\mathcal{Q}$ be the pivotal category $\mathcal{Q}$ constucted from P, then the obvious monoidal functor $\mathcal{Q} \to \mathcal{P}$ so that $n \mapsto X^{\otimes n}$ gives an equivalence of pivotal tensor categories (it may happen that $X^{\otimes m} = X^{\otimes n}$ in $\mathcal{P}$ for $m \neq n$ though).

Example 2.3. Let $K(m,n)$ be the set of Kauffman diagrams, i.e., planar (m,n)-diagrams with neither inner boxes nor loops. Recall that $|K(m,n)|$ is the $(m+n)/2$-th Catalan number if $m+n$ is even and $|K(m,n)| = 0$ otherwise. Let $\mathbb{C}[K(m,n)]$ be a free vector space of basis set $K(m,n)$. Given a complex number d, we define a representation of $\mathcal{D}$ by extending the obvious action of planar diagrams on $K(m,n)$ with each loop replaced by d. The resultant tensor category is the so-called **Temperley-Lieb category** and denoted by $\mathcal{K}_d$ in what follows. (See [6] for more information.)

Example 2.4. Let $\mathcal{T}an(m,n)$ be the set of tangles and let $\mathbb{C}[\mathcal{T}an(m,n)]$ be the free vector space generated by the set $\mathcal{T}an(m,n)$. By extending the obvious action of planar diagrams on $\mathcal{T}an$ to $\mathbb{C}[\mathcal{T}an]$ linearly, we have a representation of $\mathcal{D}$. Note that $\mathbb{C}[\mathcal{T}an(0,0)]$ is infinite-dimensional.

3. Decoration

The previous construction allows us to have many variants if one assigns various attributes to strings and boxes. We here discuss two kinds of them, coloring and orientation, which can be applied independently (i.e., at the same time or seperately).

To be explicit, let C be a set and call an element of C a color. By a **colored planar diagram**, we shall mean a planar diagram T with a color assigned to each string. For colored planar diagrams, plugging is allowed only when color matches at every connecting point.

As before, colored planar diagrams constitute a multicategory $\mathcal{D}_C$ whose objects are disks or boxes with pins decorated by colors. For colored planar diagrams of box type, a strict pivotal category $\mathcal{M}_C$ is associated so that objects in $\mathcal{M}_C$ are the words associated with the letter set C, which are considered to be upper or lower halves of decorations of boxes. In other words, objects in $\mathcal{D}_C$ are labeled by pairs of objects in $\mathcal{M}_C$.

Example 3.1. Let $K(v,w)$ ($v \in C^m, w \in C^n$ with $m,n \in \mathbb{N}$) be the set of colored Kauffman diagrams. Then, given a function $d : C \to \mathbb{C}$, we obtain a representation of $\mathcal{D}_C$ by making free vector spaces $\mathbb{C}[K(v,w)]$

as in the Temperley-Lieb category. The resultant tensor category is denoted by $\mathcal{K}_d$ and referred to as the **Bisch-Jones category**.

Given a colored planar diagram T, we can further decorate it by assigning orientations to each string in T. We call such a stuff a (planar) **oriented diagram** (simply pod). The operation of plugging works here for colored pods and we obtain again a multicategory $\mathcal{OD}_C$ of pods colored by C, where objects are disks or boxes with pins decorated by colors and orientations.

Associated to colored pods of box type, we have a pivotal monoidal category $\mathcal{OM}_C$ whose objects are words consisting of letters in $\{c_+, c_-; c \in C\} = C \times \{+, -\}$ (for a pictorial display, we assign $+$ (resp. $-$) to an upward (resp. downward) arrow on boundaries of boxes). The product of objects is given by the concatenation of words with the monoidal structure for morphisms defined by the same way as before.

Given a representation of $\mathcal{D}_C$ or $\mathcal{OD}_C$, we can construct a pivotal tensor category as before.

Example 3.2. For an object x in $\mathcal{OD}_C$, let P_x be the free vector space (over a field) generated by the set

$$\bigsqcup_{d, x_1, \dots, x_d} \mathcal{OD}_C(x_1 \times \cdots \times x_d, x)$$

of all colored pods having x as a decoration of the outer box. If the plugging operation is linearly extended to these free vector spaces, we obtain a representation of $\mathcal{OD}_C$, which is referred to as the **universal representation** because any representation of $\mathcal{OD}_C$ splits through the universal one in a unique way.

Question: Is it possible to extract analytic entities out of the universal representation?

4. Half-Winding Decoration

Related to the orientation, we here explain another kind of decoration on planar diagrams of box type according to [2]. To this end, we align directions of relevant boxes horizontally and every string (when attached to a box) perpendicular to the horizontal edges of a box. Let p_0 and p_1 be two end points of such a string and choose a smooth parameter $\varphi : [0,1] \to \mathbb{R}^2$ so that $\varphi(0) = p_0$ and $\varphi(1) = p_1$. By the assumption, $\frac{d\varphi}{dt}(0)$ and $\frac{d\varphi}{dt}(1)$ are vertical vectors. The **half-winding number** of the string from p_0 to p_1 is then an integer w defined by

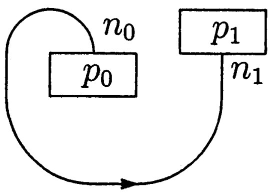
$$w = \frac{\theta(1) - \theta(0)}{\pi},$$

where a continuous function $\theta(t)$ is introduced so that $\varphi(t) = \left|\frac{d\varphi}{dt}(t)\right| (\cos\theta(t), \sin\theta(t))$. Thus w is even or odd according to $\frac{d\varphi}{dt}(0) \cdot \frac{d\varphi}{dt}(1) > 0$ or not.

We now decorate boxes by assigning an integer to each pin. A diagram framed by such boxes is said to be **winding** if it contains no loops and each string with end points p_0 and p_1 satisfies

$$w = n_1 - n_0,$$

where n_0 and n_1 are integers attached to pins at p_0 and p_1 respectively.



$$n_1 = n_0 + 2$$

FIGURE 5

A diagram colored by a set C is said to be winding if integers are assigned to relevant pins in such a way that the diagram is winding. It is immediate to see that winding diagrams in $\mathcal{D}_C$ are closed under the operation of plugging (particularly, plugging does not produce loops out of winding diagrams) and we obtain a multicategory $\mathcal{WD}_C$ of colored winding diagrams.

By the following identification of left and right dual objects

$$(X, n) = \begin{cases} X^{*\cdots *} & \text{if } n > 0, \\ X & \text{if } n = 0, \\ {}^{*\cdots *}X & \text{if } n < 0, \end{cases}$$

we have a one-to-one correspondence between representations of $\mathcal{WD}_C$ and rigid tensor categories generated by objects labeled by the set C as an obvious variant of the previous construction.

Now the color set C is chosen to consist of objects in a small linear category $\mathcal{L}$ and we shall introduce a representation $\{P_x\}$ of $\mathcal{WD}_C$ (x runs through objects of $\mathcal{WD}_C$) as follows: $P_x = 0$ if the number of pins in x is odd. To describe the case of even number pins, we consider a diagram of Temperley-Lieb type with its boundary decorated in a winding way and objects of $\mathcal{L}$ assigned to the pins of the diagram, which is said to be admissible. To an admissible diagram D, we associate the vector space

$$\mathcal{L}(D) = \bigotimes_j \mathcal{L}_j,$$

where j indexes strings of the diagram and the vector space $\mathcal{L}_j$ is determined by the following rule: If the j-th string connects a pin (colored by a) on a lower boundary and a pin (colored by b) on a upper boundary, then $\mathcal{L}_j = \mathcal{L}(a,b)$. When the j-th string connects pins on upper boundaries which are decorated by (a,n) and $(b,n+1)$, we set

$$\mathcal{L}_j = \begin{cases} \mathcal{L}(a,b) & \text{if } n \text{ is odd,} \\ \mathcal{L}(b,a) & \text{if } n \text{ is even.} \end{cases}$$

When the j-th string connects pins on lower boundaries which are decorated by (a,n) and $(b,n+1)$, we set

$$\mathcal{L}_j = \begin{cases} \mathcal{L}(a,b) & \text{if } n \text{ is even,} \\ \mathcal{L}(b,a) & \text{if } n \text{ is odd.} \end{cases}$$

Now set

$$P_{(a,k),(b,l)} = \bigoplus_D \mathcal{L}(D).$$

Here D runs through winding diagrams having $(a,k) = \{(a_j,k_j)\}$ and $(b,l) = \{(b_j,l_j)\}$ as upper and lower decorations respectively.

The rule of composition is the following:

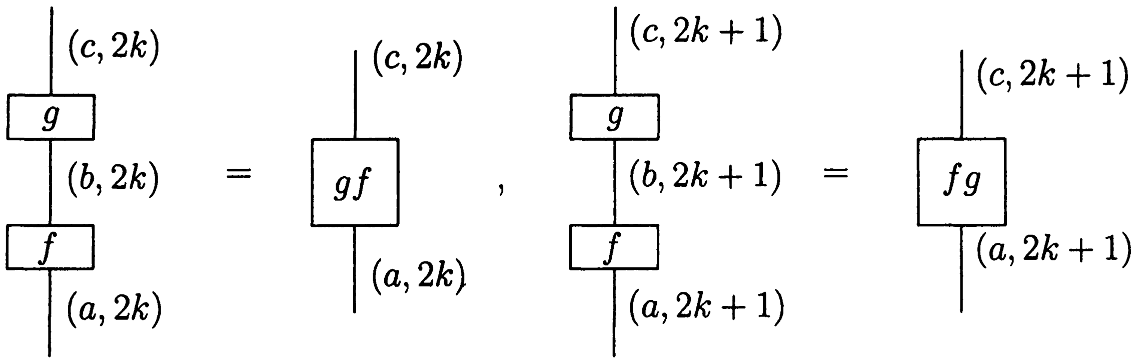


FIGURE 6

The figure 7 indicates that, though restrictive, the boundary decorations do not determine possible diagrams in a unique way.

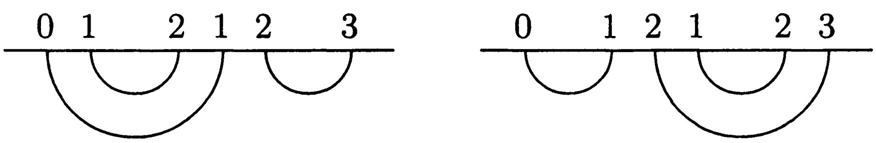


FIGURE 7

Example 4.1. If $\mathcal{L}$ consists of one object $*$, then $\{P_{(a,k),(b,l)}\}$ is the wreath product of the Temperley-Lieb category by the algebra $\mathcal{L}(*,*)$ discussed in [4].

The representation of $\mathcal{WD}_C$ defined so far, in turn, gives rise to a rigid tensor category, which is denoted by $\mathcal{R}[\mathcal{L}]$. Note that $\mathcal{R}[\mathcal{L}]$ is not pivotal by the way of half-winding decoration.

Proposition 4.2 ([2], Theorem 3.8). *Let $\mathcal{R}$ be a rigid tensor category and $F : \mathcal{L} \to \mathcal{R}$ be a linear functor. Then, F is extended to a tensor-functor of $\mathcal{R}[\mathcal{L}]$ into $\mathcal{R}$ in a unique way.*

If the half-winding number indices are identified modulo 2, we are reduced to the situation decorated by oriantation, i.e., a representation of $\mathcal{OM}_C$. Let $\mathcal{P}[\mathcal{L}]$ be the associated pivotal tensor category.

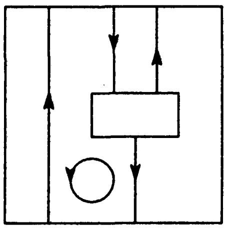
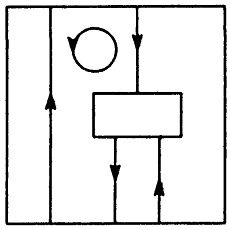
Proposition 4.3 ([2], Theorem 4.4). *Let $\mathcal{P}$ be a pivotal tensor category and $F : \mathcal{L} \to \mathcal{P}$ be a linear functor. Then, F is extended to a tensor-functor of $\mathcal{P}[\mathcal{L}]$ into $\mathcal{P}$ in a unique way.*

Remark . If one replaces planar diagrams with tangles, analogous results are obtained on braided categories ([2], Theorem 3.9 and Theorem 4.5).

5. Positivity

We here work with planar diagrams of box type and use v, w and so on to stand for an object in the associated monoidal category, whence any object of the multicategory is described by a pair (v, w). Thus a representation space $P_{v,w}$ can be viewed as the hom-vector space of a tensor category.

We now introduce two involutive operations on colored pods: Given a colored pod T, let T' be the pod with the orientation of arrows reversed (colors being kept) and T^* be the pod which is obtained as a reflection of T' with respect to a horizontal line (colors being kept while orientaions reflected).

$T =$  , $T^* =$ 

Here are again obvious identities:

$$({}^tT)^* = {}^t(T^*), \quad (ST)^* = T^*S^*, \quad (S \otimes T)^* = S^* \otimes T^*.$$

A representation $(\pi, \{P_{v,w}\})$ of $\mathcal{OD}_C$ is called a ***-representaion** if each $P_{v,w}$ is a complex vector space and we are given conjugate-linear involutions $* : P_{v,w} \to P_{w,v}$ satisfying

$$\pi_T(x_1, \dots, x_l)^* = \pi_{T^*}(x_1^*, \dots, x_l^*).$$

A *-representation is a **C*-representation** if

$$\begin{pmatrix} P_{v_1,v_1} & \dots & P_{v_1,v_n} \\ \vdots & \ddots & \vdots \\ P_{v_n,v_1} & \dots & P_{v_n,v_n} \end{pmatrix}$$

is a C*-algebra for any finite sequence $\{v_1, \dots, v_n\}$.

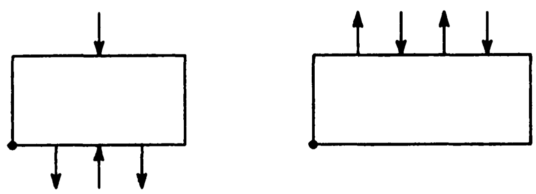
Example 5.1. The universal $\mathbb{C}$-representation of $\mathcal{OD}_C$ is a C*-representation in a natural way.

6. Alternating Diagrams

Consider now the category $\mathcal{OD}$ of pods without coloring (or monochromatic coloring). Thus objects are finite sequences consisting of + and −. We say that the decoration of a disk is **alternating** if even numbers of $\pm$ are arranged alternatingly;

$$(+,-,+,\cdots,+,-) \quad \text{or} \quad (-,+,-,\cdots,-,+).$$

By an alternating pod, we shall mean a pod where all boxes have even number of pins and are decorated by $\pm$ alternatingly and circularly. Thus orientations of strings attached to upper and lower boundaries of a box coincide at the left and right ends. Here are examples of alternating decorations on inner boxes:



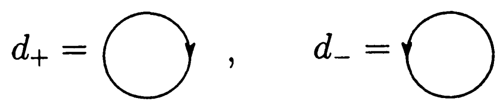
Alternating pods again constitute a multicategory, which is denoted by $\mathcal{AD}$. According to the shape of objects, we have three equivalent categories $\mathcal{AD}_\circ$, $\mathcal{AD}_\square$ and $\mathcal{AD}_\triangle$. So $\mathcal{AD}$ is a loose notation to stand for one of these multicategories.

If we further restrict objects to the ones whose decoration starts with +, then we obtain the submulticategory $\mathcal{AD}^+$, which is the Jones' original form of planar diagrams: A **planar algebra** is, by definition, a representation $\{P_{n,n}\}_{n\geq 0}$ of $\mathcal{AD}_\triangle^+$ satisfying $\dim P_{0,0} = 1$.

We shall now deal with representations of $\mathcal{AD}_+$ satisfying

$$\pi_T = d^l \pi_{T_0},$$

where $d = d_-$ is a scalar, l is the number of anticlockwise loops and T_0 is the pod obtained from T by removing all the loops of anticlockwise orientation.



Lemma 6.1. *Under the assumption that $d \neq 0$, any representation of $\mathcal{AD}_+$ is extended to a representation of $\mathcal{AD}$ and the extension is unique.*

Proof. Assume that we are given a representation (π, P) of $\mathcal{AD}$. According to the parity of label objects, the representation space P is split into two families $\{P_{m,n}^\pm\}$. Let C be a pod in $\mathcal{AD}$ indicated by Figure 8. From the identity $\pi_C(1 \otimes a) = da$ for $a \in P_{m,n}^-$, we see that the map

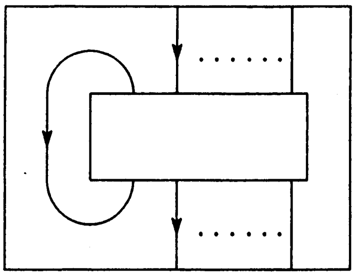


FIGURE 8

$P^-_{m,n} \ni a \mapsto 1 \otimes a \in P^+_{m+1,n+1}$ is injective with its image specified by

$$1 \otimes P^-_{m,n} = \{a \in P^+_{m+1,n+1}; \pi_{1\otimes C}(a) = da\}.$$

If we regard $P^-_{m,n} \subset P^+_{m+1,n+1}$ by this imbedding, π_T for a morphism $T \in \mathcal{AD}$ is identified with

$$\frac{1}{d^e}\pi_{T\circ(C_1^*\times\cdots\times C_d^*)} \quad \text{or} \quad \frac{1}{d^e}\pi_{(1\otimes T)\circ(C_1^*\times\cdots\times C_d^*)}$$

depending on the parity of the output object of T. Here $C_j^* = 1$ or $C_j^* = C$ according to the parity of the j-th inner box and e denotes the number of inner boxes of odd (= negative) parity in T. Note that these reinterpreted T's are morphisms in $\mathcal{AD}_+$. In this way, we have seen that π is determined by the restriction to $\mathcal{AD}_+$.

Conversely, starting with a representation (π^+, P^+) of $\mathcal{AD}_+$, we set

$$P^-_{m,n} = \{\pi^+_{1\otimes C}(a); a \in P^+_{m+1,n+1}\} \subset P^+_{m+1,n+1}$$

and define a multilinear map π_T by the above relation:

$$\pi_T = \frac{1}{d^e}\pi^+_{T\circ(C_1^*\times\cdots\times C_d^*)} \quad \text{or} \quad \frac{1}{d^e}\pi^+_{(1\otimes T)\circ(C_1^*\times\cdots\times C_d^*)}.$$

From the definition, $\pi_T = \pi_T^+$ if T is a morphism in $\mathcal{AD}_+$.

To see that π is a representation of $\mathcal{AD}$, we need to show that $\pi_T \circ (\pi_{T_1} \otimes \cdots \otimes \pi_{T_d}) = \pi_{T\circ(T_1\times\cdots\times T_d)}$.

When the output object of T has even parity,

$$\pi_T \circ (\pi_{T_1} \otimes \cdots \otimes \pi_{T_d}) = \frac{1}{d^e}\pi^+_{T\circ(C_1^*\times\cdots\times C_d^*)} \circ (\pi_{T_1} \otimes \cdots \otimes \pi_{T_d})$$

and we look into the plugging at the box such that $C_j^* = C$. Then the output parity of T_j is odd and we have $\pi_{T_j} = d^{-e_j}\pi^+_{(1\otimes T_j)\circ(C^*\times\cdots\times C^*)}$, which is used in the above plugging (Figure 9) to see that it results in

$$\frac{1}{d^e}\pi^+_{T\circ(C_1^*\times\cdots\times C_d^*)} \circ (\pi_{T_1} \otimes \cdots \otimes \pi_{T_d}) = \frac{1}{d^f}\pi_{T\circ(T_1\times\cdots\times T_d)\circ(C^*\times\cdots\times C^*)},$$

where $f = \sum_j e_j$ denotes the number of inner boxes of odd parity inside $T_1, \ldots, T_d$.

A similar argument works for T having the outer box of odd parity, proving the associativity of π for plugging. $\square$

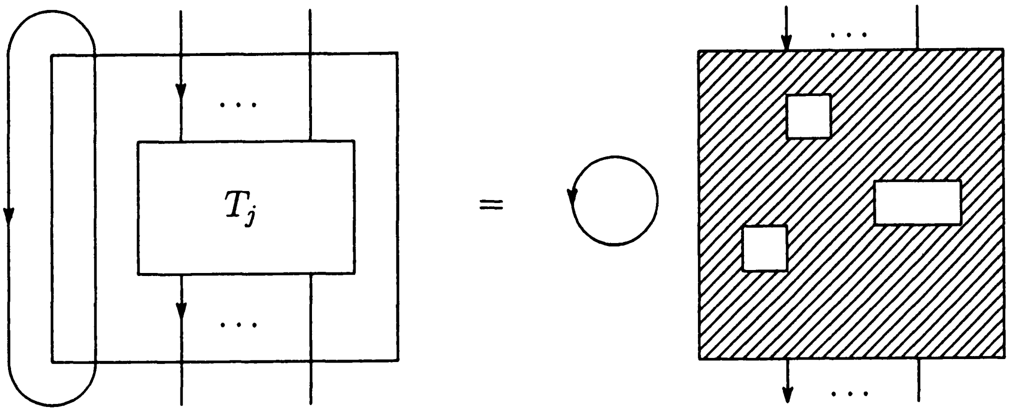


FIGURE 9

Theorem 6.2. *Representations of $\mathcal{AD}$ are in one-to-one correspondence with singly generated pivotal linear bicategories.*

Corollary 6.3. *Planar algebras are in one-to-one correspondence with singly generated pivotal linear bicategories with simple unit objects and satisfying l-$\dim(X) \neq 0$. (l-$\dim$ refers to the left dimension.)*

Corollary 6.4. *Planar C^*-algebras are in one-to-one correspondence with singly generated rigid C^*-bicategories with simple unit object.*

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