数理解析研究所講究録
第 1715 巻 2010 年 135-142

THE ASYMPTOTIC GROWTH OF THE L^2-NORM OF GEODESIC CYCLES IN CONGRUENCE COVER OF ARITHMETIC HYPERBOLIC MANIFOLDS

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ABSTRACT. Let M be a compact (congruence) arithmetic hyperbolic manifold and $C \subset M$ be a totally geodesic cycle of dimension greater that $\frac{1}{2}\dim M$. We compute the explicit growth rate of the L^2-norm of the lifts of C in the congruence tower above M. In particular, these lifts end up representing non-zero homology classes.

CONTENTS

1. INTRODUCTION

This paper is based on my talk at the annual RIMS Symposium on *Automorphic forms, automorphic representations and related topics* in January 2010. In the talk I gave a survey of my joint work with L. Clozel [5]. There we prove conjectures stated in [2, 6, 4] which relate the cohomology of a (congruence) arithmetic hyperbolic manifold to the cohomology of its totally geodesic cycles. As a particular case we prove that if M is a compact (congruence) arithmetic manifold and $C \subset M$ is an (immersed) totally geodesic cycle of dimension greater that $\frac{1}{2}\dim M$ then there exists a finite cover $\widehat{M}$ of M and a (connected) lift $\widehat{C}$ of C such that the class of $\widehat{C}$ in $H_\bullet(M)$ is non-trivial.

In this note I want to describe a quantified version of this result. The precise theorem is stated in the next section. As in the work with Clozel the proof relies on a "Selberg type" spectral gap for the eigenvalues of the Laplace operator on *differential forms*. We were able to deduce such a result from recent works of J. Arthur on the classification of automorphic representations of classical groups, see [1, §30]. In section 3, I give a survey of (a small part of) Arthur's theory with a view toward direct use of it in the classification of archimedean components of automorphic representations. In section 4, I deduce from Arthur's theory the

"Selberg type" theorem mentioned above. This is joint work with Clozel. Finally in the last section I sketch a proof of the main theorem along the lines of [3].

I would like to thank Professors T. Oda and M. Tsuzuki for organizing this very interesting conference this year and make my participation possible.

2. Main result

2.1. **Geodesic cycles.** Let $\mathbb{H}^n$ be the n-dimensional hyperbolic space. Here k-dimensional subspaces of $\mathbb{H}^n$ will always be *totally geodesic*; note that such a subspace is isometric to $\mathbb{H}^k$. Let $M = \Gamma\backslash\mathbb{H}^n$ be a compact hyperbolic manifold. A k-dimensional subspace $H \subset \mathbb{H}^k$ is a Γ-*subspace* if $\mathrm{Stab}_\Gamma(H)\backslash H$ is compact. A Γ-subspace thus projects onto a compact immersed totally geodesic submanifold in M.

Assume that H projects onto an *embedded* submanifold C in M. It gives a class $[C] \in H^{n-k}(M, \mathbb{C})$. Hodge theory provides a natural inner product on $H^{n-k}(M, \mathbb{C})$. We denote by $|[C]|_2$ the norm of the class $[C]$. In this paper we study the asymptotic growth of $|[C]|_2$ in congruence cover of M.

2.2. **Congruence hyperbolic manifolds.** Recall the general definition of a congruence hyperbolic manifold. Let $\mathbf{G}$ be a $\mathbb{Q}$-algebraic group such that its group of real points, $\mathbf{G}(\mathbb{R})$, is the product (with finite intersection) of a compact group by $G^{\mathrm{nc}} = \mathrm{SO}(n,1)$.

A *congruence subgroup* Γ of $\mathbf{G}(\mathbb{Q})$ is the intersection $\mathbf{G}(\mathbb{Q}) \cap K$, where K is a compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$ the group of finite adèlic points of $\mathbf{G}$. According to a classical theorem of Borel and Harish-Chandra, it is a lattice in $G^{\mathrm{nc}} = \mathrm{SO}(n,1)$. It is a cocompact lattice if and only if $\mathbf{G}$ is anisotropic over $\mathbb{Q}$. For simplicity we will always assume that it is indeed the case. If Γ is *sufficiently deep*, i.e. K is a sufficiently small compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$, then Γ is moreover torsion-free.

If K_∞ is a maximal compact subgroup of $\mathbf{G}(\mathbb{R})$, then $\mathbf{G}(\mathbb{R})/K_\infty$ – the associated symmetric space – is isometric to the n-dimensional hyperbolic space $\mathbb{H}^n$. If $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a torsion-free congruence subgroup, $\Gamma\backslash\mathbb{H}^n$ is a n-dimensional *congruence* hyperbolic manifold.

Let $\Gamma\backslash\mathbb{H}^n$ be a congruence hyperbolic manifold. We consider a decreasing sequence of congruence normal subgroups $\Gamma_N \subset \Gamma_{N-1} \subset \ldots \subset \Gamma$ with the property that $\cap_N \Gamma_N = \{1\}$.

2.3. **Theorem.** *Let H be a k-dimensional Γ-subspace in $\mathbb{H}^n$ with $k \geq n/2$. The projections C_N of H into $\Gamma_N\backslash\mathbb{H}^n$ become embedded when N is large enough and*

$$\frac{|[C_N]|_2}{\sqrt{\mathrm{vol}(C_N)}} \longrightarrow \sqrt{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{n-k}{2}}\Gamma\left(\frac{2k+1-n}{2}\right)}},$$

as $N \to +\infty$.

As a corollary we get – as announced in the introduction – that $[C_N]$ is non-zero for N sufficiently big.

Theorem 2.3 was first conjectured in [3] where the case $k = n-1$ is proved. In [10] Masao Tsuzuki put these questions in a more representation theoretic framework. It follows from the "Selberg type" we prove here that Tsuzuki's Theorem 3 is now unconditional.

3. Arthur's classification of automorphic representations of orthogonal groups

3.1. **Orthogonal groups.** Let F be a totally real numberfield and $\mathbf{G}$ be a special orthogonal group over F. We assume $\mathbf{G}$ is *classical* (not a twisted form ${}^{3,6}D_4$ of $\mathrm{SO}(8)$).

The group $G = \mathbf{G}(F \otimes \mathbb{R})$ is the product (with finite intersection) of a compact group by a semisimple Lie group G^{nc}. We assume that G^{nc} is isomorphic to $\mathrm{SO}(p,q)$ $(p \geq q)$. Set $m = p+q$ and let ℓ the integral part of $m/2$ and $N = 2\ell$. The group $\mathbf{G}$ is an interior form of a *quasi-split* group $\mathbf{G}^*$.

If $m = N+1$ is odd, then $\mathbf{G}^* = \mathrm{SO}(m)$ is *split* and the (complex) dual group $\mathbf{G}^\vee = \mathrm{Sp}(N, \mathbb{C})$.

If $m = N$ is even, then $\mathbf{G}^*$ is a quasi-split form of $\mathrm{SO}(N)$ and $\mathbf{G}^\vee = \mathrm{SO}(N, \mathbb{C})$.

3.2. **Automorphic dual.** We let $\widehat{G}$ be the unitary dual of G endowed with the Fell topology.

Let $\Gamma \subset G$ be a lattice. Call *spectrum* of $\Gamma\backslash G$ the set $\sigma(\Gamma\backslash G)$ of all $\pi \in \widehat{G}$ occuring weakly in the regular representation of G in $L^2(\Gamma\backslash G)$. We recall from [7] the definition of the *automorphic dual* $\widehat{\mathbf{G}}_{\mathrm{Aut}}$:

$$\widehat{\mathbf{G}}_{\mathrm{Aut}} = \overline{\bigcup_{\Gamma \text{ cong}} \sigma(\Gamma\backslash G)}. \tag{3.2.1}$$

The union in (3.2.1) is over all *congruence* subgroup $\Gamma \subset \mathbf{G}(F)$ so that $\widehat{\mathbf{G}}_{\mathrm{Aut}}$ really depends on the rational structure of $\mathbf{G}$; we write $\widehat{\mathrm{SO}(p,q)}_{\mathrm{Aut}}$ if we don't want to refer to a particular rational structure. The closure in (3.2.1) is taken with respect to the Fell topology in $\widehat{G}$.

3.3. **Arthur parameters.** Recall the *Weil group* of $\mathbb{R}$ – denoted $W_{\mathbb{R}}$ – is the nonsplit extension of $\mathbb{C}^*$ by $\mathbb{Z}/2\mathbb{Z}$ given by

$$W_{\mathbb{R}} = \mathbb{C}^* \cup j\mathbb{C}^*,$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$. The Langlands' dual group ${}^L\mathbf{G}$ is an extension of $\mathbf{G}^\vee$ by $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$.

A (local) *Arthur parameter* for $\mathbf{G}$ is a homomorphism

$$\widetilde{\psi} : W_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathbf{G} \tag{3.3.1}$$

such that

1. the diagram
$$\begin{tikzcd} W_{\mathbb{R}} \arrow[rr] \arrow[dr] & & {}^L\mathbf{G} \arrow[dl] \\ & \mathrm{Gal}(\mathbb{C}/\mathbb{R}) & \end{tikzcd}$$
is commutative;
2. the homomorphisme $\psi_{|\mathrm{SL}_2(\mathbb{C})}$ is algebraic;
3. the image of $\psi_{|W_{\mathbb{R}}}$ has compact closure.

The conjugacy class of an Arthur parameter should be thought to be associated to a formal direct sum of formal tensor products:

$$\mu_1 \boxtimes R_1 \boxplus \ldots \boxplus \mu_r \boxtimes R_r$$

where each μ_j is a cuspidal element of the automorphic dual of $\mathrm{GL}(m_i)/\mathbb{Q}$, R_j is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_j and $N = n_1m_1 + \ldots + n_rm_r$.

Now consider the restriction

$$\psi : \mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{G}^\vee.$$

It follows from the definition (3.3.1) (3) that ψ is semisimple. It may thus be written:

$$\psi = \varphi_1 \otimes R_1 \oplus \ldots \oplus \varphi_r \otimes R_r : \mathbb{C}^* \times \mathrm{SL}_2(\mathbb{C}) \to \mathbf{G}^\vee, \tag{3.3.2}$$

where each φ_j is a semisimple representation of $\mathbb{C}^*$ of rank m_j, R_j is an irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ of dimension n_j and $N = n_1m_1 + \ldots + n_rm_r$. Here each φ_j should be thought as the restriction to $\mathbb{C}^*$ of the L-parameter of μ_j above.

If $\chi = z^p\overline{z}^q$ $(p - q \in \mathbb{Z})$ is a character, we denote by χ^σ the character $z \mapsto \chi(\overline{z})$. As each μ_j is a *real* representation we note that if $\chi \otimes R_j$ appears in (3.3.2) then $\chi^\sigma \otimes R_j$ also appears. Moreover: as μ_j is *cuspidal* and *automorphic*, the generalized Ramanujan conjecture implies that χ should be unitary, i.e. $\mathrm{Re}(p+q) = 0$. This motivates (3) above. Instead of appealing to the (still unproved) Ramanujan conjecture we may use the Luo-Rudnick-Sarnak theorem [9] – as extended in [6, Chap. 7] – which implies that $|\mathrm{Re}(p+q)| < 1 - \frac{2}{m_j^2+1}$. We finally note that, the parameter ψ being self-contragredient, if $\chi \otimes R_j$ appears in (3.3.2) then $\chi^{-1} \otimes R_j$ also appears.

The last paragraph motivates the following definition. We call ψ as in (3.3.2) a *weak Arthur parameter* if for each character $\chi = z^p\overline{z}^q$ such that $\chi \otimes R_j$ appears in (3.3.2), $\chi^\sigma \otimes R_j$ and $\chi^{-1} \otimes R_j$ also appear and

$$p - q \in \mathbb{Z} \text{ and } |\mathrm{Re}(p+q)| < 1 - \frac{2}{m_j^2+1}.$$

3.4. **Infinitesimal characters.** To any weak Arthur parameter ψ we associate a parameter

$$\begin{array}{rccl} \varphi_\psi : & \mathbb{C}^\times & \to & \mathbf{G}^\vee \subset \mathrm{GL}(N,\mathbb{C}) \\ & z & \mapsto & \psi\left(z, \begin{pmatrix} (z\overline{z})^{1/2} & 0 \\ 0 & (z\overline{z})^{-1/2} \end{pmatrix}\right). \end{array}$$

Being semisimple, the image of φ_ψ is conjugate into the maximal torus

$$T^\vee = \{\mathrm{diag}(x_1, \ldots, x_\ell, x_\ell^{-1}, \ldots, x_1^{-1})\}$$

of $\mathbf{G}^\vee$. We may thus write

$$\varphi_\psi = (\eta_1, \ldots, \eta_\ell, \eta_\ell^{-1}, \ldots, \eta_1^{-1})$$

where η_i is a character $z^{P_i}\overline{z}^{Q_i}$. The vector

$$\nu_\psi = (P_1, \ldots, P_\ell) \in \mathbb{C}^\ell,$$

where $\mathbb{C}^\ell$ is seen as a torus in $\mathbf{G}^*$, is uniquely defined modulo the Weyl group W of $\mathbf{G}^*$.

The following proposition is a (very) weak form of the classification theorem of representations of classical groups announced by Arthur in his Clay lectures, see in particular theorem 30.2 in [1, §30]. The proof should appear soon. Note that it will depend on the recent proofs by Ngô, Waldspurger, Laumon and Chaudouard of the "fundamental lemma" and its avatara (twisted and weighted). It also uses [6, Lemma 6.3.1 & 6.4.1].

3.5. **Proposition.** *The automorphic dual of* $\mathbf{G}^*$ *decomposes as a union of finite packets of representations* $\Pi(\psi) \subset \widehat{G}$ *indexed by weak Arthur parameters. Members of a same packet all share the same infinitesimal character* $\nu_\psi \in \mathbb{C}^\ell/W$.

Arthur's stable trace formula allows to compare the automorphic spectrum of $\mathbf{G}$ with that of his endoscopic subgroups. These consist of $\mathbf{G}^*$ and products of smaller quasi-split special orthogonal groups to which proposition 3.5 applies. This is described in [5] and theorem 6.1 in loc. cit. implies:

3.6. **Proposition.** *If* π *belongs to the automorphic dual* $\widehat{\mathbf{G}}_{\mathrm{Aut}}$*, then there exists a weak Arthur parameter* ψ *such that the infinitesimal character of* π *is* $\nu_\psi \in \mathbb{C}^\ell/W$.

4. Spectra of congruence hyperbolic manifolds

4.1. Proposition 3.6 has interesting consequences on the spectrum of hyperbolic manifolds: We now assume that $p = n$ and $q = 1$. Let $M = \mathrm{S}(\mathrm{O}(n-1) \times \mathrm{O}(1,1)) \subset G^{\mathrm{nc}} = \mathrm{SO}(n,1)$. It is the Levi subgroup of a minimal parabolic subgroup. The connected component of the identity ${}^0M \cong \mathrm{SO}(n-1)$ is compact. And it follows from Langlands classification that any irreducible admissible representation of $\mathrm{SO}(n,1)$ is either a member of the discrete series (this can only occurs if n is even) or an irreducible subquotient $J(\tau, s)$ of an induced representation $I(\tau, s)$ where $\tau \in \widehat{{}^0M}$ and $\mathrm{Re}(s) \geq 0$. See [6, §6.3 & 6.4] for more details.

4.2. **Spectrum of the Laplace operator and unitary representations.** Let $\mathfrak{g}$ be the complexified Lie algebra of $\mathrm{SO}(n,1)$. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and $\{x_i^*\}$ be the dual basis with respect to the invariant bilinear form $\langle , \rangle$ on $\mathfrak{g}$ which induces the Riemannian metric of constant curvature -1 on $\mathbb{H}^n$. Define the Casimir element C by

$$C = \sum_j x_j x_j^*.$$

Then C belongs to the center of the universal enveloping algebra of $\mathfrak{g}$. It acts on the space of smooth functions on $\mathrm{SO}(n,1)$. A differential form on $\mathbb{H}^n$ lifts as a function on $\mathrm{SO}(n,1)$ and the Laplace operator (on forms) is induced by the restriction of $-C$.

Let $\Gamma \subset \mathrm{SO}(n,1)$ be a (cocompact) congruence lattice. The right regular representation of $\mathrm{SO}(n,1)$ on $L^2(\Gamma\backslash\mathrm{SO}(n,1))$ decomposes into a discrete sum of irreducible unitary representations of $\mathrm{SO}(n,1)$. This gives rise to a spectral decomposition for the action of C on $L^2(\Gamma\backslash\mathrm{SO}(n,1))$. Matsushima's formula (see e.g. [6]) then implies that λ belongs to the spectrum of the Laplace operator on k-forms if and only if there exists a representation $J(\tau, s)$ in the support of $L^2(\Gamma\backslash\mathrm{SO}(n,1))$ such that $J(\tau,s)(C)$ acts as the scalar $-\lambda$ and τ occurs as an irreducible summand in the restriction to ${}^0M$ of the standard $\mathrm{SO}(n)$-representation: $\wedge^k\mathbb{C}^n$.

The infinitesimal character of $J(\tau, s)$ is equal to $(\lambda_\tau, s) \in \mathbb{C}^\ell/W$ so that $J(\tau,s)(C)$ acts as the scalar

$$\langle \lambda_\tau + s, \lambda_\tau + s\rangle - \langle \rho, \rho\rangle.$$

Here λ_τ – the infinitesimal character of τ – is a strictly increasing sequence of elements in $\frac{n-1}{2} + \mathbb{Z}$ and $\rho = ((n-1)/2, (n-3)/2, \ldots)$.

Now proposition 3.6 and a simple combinatorial game yield:

4.3. **Proposition.** *Assume* $J(\tau, s) \in \widehat{\mathrm{SO}(n,1)}_{\mathrm{Aut}}$. *Then: either* $s \in \frac{1}{2}\mathbb{Z}$ *or* $|\mathrm{Re}(s)| < \frac{1}{2} - \frac{1}{(n+1)^2+1}$.

Proof. Indeed according to proposition 4.3, the infinitesimal character (λ_τ, s) is associated to a weak Arthur parameter $\psi = \oplus_j \varphi_j \otimes R_j$. Suppose $s \notin \frac{1}{2}\mathbb{Z}$. As $\nu_\psi = (\lambda_\tau, s)$ there exists some index j, an integer $k \in [1, n_j]$ and a character $\chi = z^p \overline{z}^q$ appearing in φ_j such that

$$s = p + \frac{n_j + 1 - 2k}{2}.$$

In particular $p \notin \frac{1}{2}\mathbb{Z}$. Moreover: as $\chi \otimes R_j$ appears in ψ, $\chi^{-1} \otimes R_j$ also appears (and $\chi \neq \chi^{-1}$ as $p \neq 0$). As ν_ψ has only one coordinate $\notin \frac{1}{2}\mathbb{Z}$, we conclude that $n_j = 1$ and $s = p$.

Similarly we conclude that $\chi^\sigma = \chi$ or χ^{-1}. In otherwords: $\chi = (z\overline{z})^p$ or $(z/\overline{z})^p$. The second case cannot occur as $p = s \notin \frac{1}{2}\mathbb{Z}$. The only remaining possibility is that $s = p = q$ so that

$$|\mathrm{Re}(s)| = \frac{1}{2}|\mathrm{Re}(p+q)| < \frac{1}{2}\left(1 - \frac{2}{m_j^2 + 1}\right) \leq \frac{1}{2} - \frac{1}{(n+1)^2 + 1}.$$

□

One can be a little more precise; see [5, Prop. 6.2]. Anyway, we easily conclude from §4.2 that proposition 4.3 already implies the following proposition that was conjectured in [2] and proved in our joint work with Clozel [5].

4.4. **Proposition.** *Let $\mathbb{H}^n$ be the real hyperbolic n-space and let k be a non-negative integer strictly less that the "middle dimension" (i.e. $k < [n/2]$). There exists a positive real constant $\varepsilon = \varepsilon(n,k)$ such that if $\Gamma \subset \mathrm{SO}(n,1)$ is a congruence arithmetic subgroup, the non-zero eigenvalues λ of the Laplace operator acting on the space $\Omega^k(\Gamma\backslash\mathbb{H}^n)$ of differential forms of degree k satisfy:*

$$\lambda \geq \varepsilon.$$

5. Proof of theorem 2.3

It follows from [6, Prop. 15.2.2] that there exists an algebraic $\mathbb{Q}$-subgroup $\mathbf{H} \subset \mathbf{G}$ such that if $K^H = K \cap \mathbf{H}(\mathbb{A}_f)$ and $\Lambda = K^H \cap \mathbf{H}(\mathbb{Q})$ then C coincides with the image of the map

$$\Lambda\backslash\mathbf{H}(\mathbb{R})/K_\infty^H \to \Gamma\backslash\mathbf{G}(\mathbb{R})/K_\infty. \tag{5.0.1}$$

Moreover: choosing K sufficiently small we may assume the map (5.0.1) to be injective. Let $\Lambda_N = \Lambda \cap \Gamma_N$ (so that $C_N = \Lambda_N\backslash H$) and $M_N = \Gamma_N\backslash\mathbb{H}^n$.

The finite group Λ/Λ_N acts freely on M_N. Let $\overline{M}_N$ denote the quotient manifold. We note that $\overline{M}_N$ is a finite cover of $\overline{\Gamma}_N\backslash\mathbb{H}^n$ where $\overline{\Gamma}_N = K_N K^H \cap \mathbf{G}(\mathbb{Q})$. Here K_N is a compact-open subgroup of $\mathbf{G}(\mathbb{A}_f)$ such that $\Gamma_N = K_N \cap \mathbf{G}(\mathbb{Q})$. In particular: $\cap_N \overline{\Gamma}_N = \Lambda$. Set $\overline{M}_\infty = \Lambda\backslash\mathbb{H}^n$. The tower of finite coverings

$$\ldots \to \overline{M}_N \to \ldots \to M$$

converges (on compact subsets) toward $\overline{M}_\infty$.

5.1. **The dual form.** Kudla and Millson [8] give an explicit construction of the harmonic form dual to C in $\overline{M}_N$. We briefly recall their construction: Note that $C = \Lambda\backslash H$ naturally embeds into $\overline{M}_\infty = \Lambda\backslash\mathbb{H}^n$. We thus get normal coordinates:

$(x,r,\sigma) \in C\times]0,+\infty[\times\mathbb{S}^{n-k+1}$. Define a holomorphic family of forms ψ_s on $\overline{M}_\infty$ for $s \in \mathbb{C}$ by the formula

$$\phi_s = \frac{(\sinh r)^{n-(k+1)}}{(\cosh r)^{k+2s}} d\mathrm{vol}_{\mathbb{S}}^{n-(k+1)} \wedge dr.$$

We may then average ϕ_s and define a meromorphic family of forms on $\overline{M}_N$ by the following series, convergent for $\mathrm{Re}(s) > \frac{n-1-k}{2}$:

$$\omega_s^N = \frac{2}{\mathrm{vol}(\mathbb{S}^{n-(k+1)})} \frac{\Gamma\left(\frac{2s+k+1}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{2s+2k+1-n}{2}\right)} \sum_{\gamma\in\Lambda\backslash\overline{\Gamma}_N} \gamma^*\phi_s. \tag{5.1.1}$$

Kudla and Millson then prove that ω_s^N has a meromorphic extension to all of $\mathbb{C}$ and satisfies a differential functional equation. Moreover: $s = 0$ is a regular value and ω_0 is the harmonic form dual to C in $\overline{M}_N$.

5.2. **Asymptotic growth of** $(|\omega_0^N|_{L^2(\overline{M}_N)})_{N\geq 0}$. Note that for fixed s in the half-plane of absolute convergence, the sequence of sums (5.1.1) converges toward

$$\frac{2}{\mathrm{vol}(\mathbb{S}^{n-(k+1)})} \frac{\Gamma\left(\frac{2s+k+1}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{2s+2k+1-n}{2}\right)} \phi_s.$$

In [3] we prove that under the hypothesis – named (H) in [3] – that the first positive eigenvalue of the Laplace operator on closed $(n-k)$-forms is bigger that a positive uniform (in N) constant, the sequence $(|\omega_0^N|_{L^2(\overline{M}_N)})_{N\geq 0}$ converges toward

$$\frac{2}{\mathrm{vol}(\mathbb{S}^{n-(k+1)})} \frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{n-k}{2}\right)\Gamma\left(\frac{2k+1-n}{2}\right)} |\phi_0|_{L^2(\overline{M}_\infty)} = \sqrt{\frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{n-k}{2}}\Gamma\left(\frac{2k+1-n}{2}\right)}\mathrm{vol}(C)}.$$

This should not be a surprise: only eigenvalues approaching 0 can contribute to harmonic forms at infinity.

5.3. **Conclusion of the proof.** Proposition 4.4 implies that hypothesis (H) is satisfied in our case. Finally note that the harmonic form representing $[C_N] \in H^{n-k}(M_N)$ is invariant under the action of Λ/Λ_N on M_N. It thus coincides with ω_0^N. And we conclude that

$$|[C_N]|_2 = |\omega_0^N|_{L^2(M_N)} = \frac{|\omega_0^N|_{L^2(\overline{M}_N)}}{\sqrt{[\Lambda : \Lambda_N]}}.$$

This finishes the proof.

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