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ON THE ARTHUR-SELBERG TRACE FORMULA OVER FUNCTION FIELDS

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The Arthur-Selberg trace formula for a connected reductive group G over number fields is a very powerful tool in the theory of automorphic representations. The trace formula was first introduced by Selberg and then it has been studied extensively in a very impressive series of papers of Arthur [1, 2, 3]. It has applications in the study of L-functions associated to certain Shimura varieties and in the study of transfers between automorphic representations of classical groups. We refer the reader to the excellent surveys of Arthur [4] and Labesse [12] or the very well-written book of Gelbart [9] for more details on the subject.

It is expected by experts that a similar trace formula would hold for function fields. When the reductive group in question is either the general linear group $\mathrm{GL}(n)$ or some inner forms of $\mathrm{GL}(n)$, several non-invariant versions of the trace formula over function fields have been introduced and studied in the works of Drinfeld [7, 8], Laumon [15] and Lafforgue [13, 14]. However, it seems that the trace formula for a general connected reductive group over function fields has not been written in the literature. We will report our progress on the project aiming to fill this missing part. Details will be appeared elsewhere [19, 20].

1. Notations

Let us fix the notations. Throughout this paper, let $\mathbb{F}_q$ be a finite field of q elements and F be a function field over $\mathbb{F}_q$. We will denote by X the projective smooth geometrically connected curve over $\mathbb{F}_q$ with the function field F. It is well known that the set of closed points x of X is in bijection with the set of places of F. Let x be a place of F, then it is non-archimedian. Denote by F_x the completion of F with respect to x, $x : F_x^\times \longrightarrow \mathbb{Z}$ the normalized valuation, $\mathcal{O}_x$ the ring of integers. Denote by κ_x the residue field of $\mathcal{O}_x$. It is a finite extension

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of $\mathbb{F}_q$ and the degree of the extension will be denoted by $\deg(x)$ and called the degree of x.

We will denote by $\mathbb{A}$ the ring of adèles of F. It is defined as the restricted product $\prod'(F_x, \mathcal{O}_x)$. An element a of $\mathbb{A}$ (resp. of $\mathbb{A}^\times$) is given by a collection $a = (a_x)_{x\in X}$ where for every place x of F, we have $a_x \in F_x$, and for almost except a finite number of places x of F, we have $a_x \in \mathcal{O}_x$ (resp. $a_x \in \mathcal{O}_x^\times$). We define the degree map $\deg : \mathbb{A}^\times \longrightarrow \mathbb{Z}$ as follows. Let $a = (a_x)_{x\in X} \in \mathbb{A}^\times$, then

$$\deg(a) = -\sum_{x\in X} x(a_x)\deg(x).$$

The group F (resp. $F^\times$) can be considered as a discrete subgroup of $\mathbb{A}$ (resp. of $\mathbb{A}^\times$). Moreover, $F^\times$ sits inside the subgroup $\mathbb{A}^{\times,0}$ of $\mathbb{A}^\times$ consisting of elements of degree 0 and the quotient $F^\times\backslash\mathbb{A}^{\times,0}$ is compact.

Let G be a connected reductive group over F. We choose a maximal split torus T of G over F and a minimal parabolic subgroup P_0 of G defined over F and containing T. Denote by M_0 its Levi subgroup containing T, it is defined over F. By definition, a parabolic subgroup (resp. a Levi subgroup) of G is a parabolic subgroup P (resp. a Levi subgroup M) of G defined over F and containing M_0 and a standard parabolic subgroup is a parabolic subgroup of G defined over F and containing P_0.

Let Φ (resp. $\Phi^\vee$) be the set of roots of G (resp. that of coroots of G) with respect to the choice of T. It is a root system. Recall that we have fixed a minimal parabolic subgroup P_0 of G containing T, therefore we get a partition of Φ into two subsets: the set of positive roots Φ^+ and that of negative roots Φ^-. Denote by Δ the set of simple roots, says $\Delta = \{\alpha_1, \dots, \alpha_n\}$. Denote by $\widehat{\Delta}$ the set of corresponding simple coroots and by $\alpha_i^\vee$ the simple coroot associated to α_i. There exists a bijection between subsets I of Δ and standard parabolic subgroups P_I of G. Under this bijection, the empty set will correspond to G and the whole set Δ will correspond to P_0.

We consider a Levi subgroup M of G. Let A_M be the split maximal torus of M over F containing T and A'_M be the maximal split quotient torus of M over F. The natural map $A_M \longrightarrow A'_M$ is an isogeny and we denote by $\mathfrak{a}_M$ (resp. $\mathfrak{a}_M^*$) the real vector space $X_*A_M \otimes_{\mathbb{Z}} \mathbb{R}$ (resp. $X^*A_M \otimes_{\mathbb{Z}} \mathbb{R}$). For any parabolic subgroup P of G admitting M as a Levi subgroup, we define $\mathfrak{a}_P$ (resp. $\mathfrak{a}_P^*$) to be the real vector space $\mathfrak{a}_M$ (resp. $\mathfrak{a}_M^*$) associated to M.

Let P and Q be two parabolic subgroups of G such that $P \subset Q$. The natural maps $A_{M_P} \longrightarrow A_{M_Q} \longrightarrow A'_{M_Q} \longrightarrow A'_{M_P}$ induces an injection $\mathfrak{a}_Q \hookrightarrow \mathfrak{a}_P$ and a surjection $\mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$ such that the composition $\mathfrak{a}_Q \hookrightarrow \mathfrak{a}_P \twoheadrightarrow \mathfrak{a}_Q$ is identity. Hence we obtain a decomposition $\mathfrak{a}_P = \mathfrak{a}_Q \oplus \mathfrak{a}_P^Q$. Similarly, we have a canonical decomposition $\mathfrak{a}_P^* = \mathfrak{a}_Q^* \oplus \mathfrak{a}_P^{Q*}$. In particular, if we take $P = P_0$ and $Q = G$, we get vector spaces $\mathfrak{a}_{P_0}^{G*}$ and $\mathfrak{a}_{P_0}^G$. The set Δ (resp. $\Delta^\vee$) forms a basis of $\mathfrak{a}_{P_0}^{G*}$ (resp. $\mathfrak{a}_{P_0}^G$). The dual basis of $\Delta^\vee$ will be denoted by $\widehat{\Delta}$ and contain fundamental weights of G, says $\{\varpi_1, \ldots, \varpi_n\}$. Similarly, we get the basis of fundamental coweights of $\mathfrak{a}_{P_0}^G$.

Now, we consider a standard parabolic subgroup P of G, then $P_0 \subset P$. Denote by Δ_P the subset of Δ corresponding to P. The previous discussion gives a canonical decomposition $\mathfrak{a}_{P_0}^{G*} = \mathfrak{a}_P^{G*} \oplus \mathfrak{a}_{P_0}^{P*}$. The set of simple roots Δ forms a basis of the real vector space $\mathfrak{a}_{P_0}^{G*}$. By abuse of notations, we still denote by Δ_P the image of Δ_P via the projection $\mathfrak{a}_{P_0}^{G*} \twoheadrightarrow \mathfrak{a}_P^{G*}$, it forms also a basis of $\mathfrak{a}_P^{G*}$. Let α be an element in Δ_P which is the image of a simple root $\beta \in \Delta$ under the above projection. We will denote by $\alpha^\vee$ the projection of $\beta^\vee$ in $\mathfrak{a}_P^G$ and call it the simple coroot of $\alpha \in \Delta_P$. We will denote by $\widehat{\Delta}_P$ (resp. $\widehat{\Delta}_P^\vee$) the set of projections of fundamental weights ϖ_i (resp. fundamental coweights $\varpi_i^\vee$) such that $\alpha_i \in \Delta_P$. They form another basis of $\mathfrak{a}_P^{G*}$ and $\mathfrak{a}_P^G$.

Let P and Q be two parabolic subgroups of G such that $P \subset Q$ and Δ_P, Δ_Q the subsets of Δ corresponding to P and Q. Since $P \subset Q$, we have $\Delta_Q \subset \Delta_P$. We will denote by Δ_P^Q (resp. $\widehat{\Delta}_P^Q$) the image under the canonical projection $\mathfrak{a}_{P_0}^{G*} \twoheadrightarrow \mathfrak{a}_P^{G*} \twoheadrightarrow \mathfrak{a}_P^{Q*}$ of simple roots $\alpha \in \Delta_P - \Delta_Q$ (resp. of fundamental weights ϖ_α with $\alpha \in \Delta_P - \Delta_Q$). They forms different basis for the vector space $\mathfrak{a}_P^{Q*}$. Similarly, we can define two different basis $\Delta_P^{Q\vee}$ and $\widehat{\Delta}_P^{Q\vee}$ of $\mathfrak{a}_P^Q$.

Following Langlands and Arthur, we can introduce the acute chamber $\mathfrak{a}_P^{Q+}$ and the obtuse chamber ${}^+\mathfrak{a}_P^Q$ of $\mathfrak{a}_P^Q$ as follows:

$$\mathfrak{a}_P^{Q+} = \{p \in \mathfrak{a}_P^{Q+} : \langle \alpha, p \rangle \geq 0 \text{ for all } \alpha \in \Delta_P^Q\}$$
$${}^+\mathfrak{a}_P^Q = \{p \in \mathfrak{a}_P^{Q+} : \langle \varpi, p \rangle \geq 0 \text{ for all } \varpi \in \widehat{\Delta}_P^Q\}$$

Remark that we always have the inclusion $\mathfrak{a}_P^{Q+} \in^+ \mathfrak{a}_P^Q$.

2. Harder-Narasimhan filtration

We choose a maximal compact subgroup K of $G(\mathbb{A})$ as in [3, section 1]. Let M be a Levi subgroup of G. We define the map $H_M : M(\mathbb{A}) \longrightarrow \mathfrak{a}_M$ as follows. Let m be an element of $M(\mathbb{A})$. To each character

$\chi : M \longrightarrow \mathbb{G}_m$ of M defined over F, we get an integer $\deg(\chi(m))$. Hence, we obtain an element $H_M(m) \in \mathfrak{a}_M$ such that

$$\langle H_M(m), \chi \rangle = \deg \chi(m).$$

Remark that the homomorphism H_M takes value in a lattice of the real vector space $\mathfrak{a}_M$.

Next, let P be a parabolic subgroup of G, with Levi subgroup M and radical unipotent subgroup N. We extend the previous homomorphism to get a map $H_P : G(\mathbb{A}) \longrightarrow \mathfrak{a}_M$. Since $G(\mathbb{A}) = P_0(\mathbb{A})K$, we also have $G(\mathbb{A}) = P_0(\mathbb{A})K$. Hence any element $g \in G(\mathbb{A})$ can be written $g = nmk$ with $m \in M(\mathbb{A})$, $n \in N(\mathbb{A})$ and $k \in K$, and we define $H_P(g) = H_M(m)$.

We review the notion of orthogonal family appeared in the work of Arthur. Let M be a Levi subgroup of G. We consider the real vector space $\mathfrak{a}_M^G$ defined in the first section. The roots Φ_M of M defines different hyperplanes in $\mathfrak{a}_M^G$ and divides $\mathfrak{a}_M^G$ into chambers. The open chambers are called Weyl chambers. We will denote by $\mathcal{F}(M)$ the set of parabolic subgroups of G admitting M as a Levi subgroup. It is known that $\mathcal{F}(M)$ is in bijection with the set of Weyl chambers, therefore we say that two elements P and P' in $\mathcal{F}(M)$ are adjacent if the corresponding Weyl chambers are adjacent. Furthermore, there is a unique root $\alpha \in \Phi_M$ such that α is a simple root for P and $-\alpha$ is a simple root for P'.

Let $\underline{\lambda} = \{\lambda_P\}_{P \in \mathcal{F}(M)}$ be a family of points in $\mathfrak{a}_M^G$. We say that this family is (G, M)-*orthogonal* if the following condition is satisfied: for every pair of adjacent parabolic subgroups P and P', we denote by α the unique simple root of P such that $-\alpha$ is a simple root of P', then we require that $\lambda_P - \lambda_{P'} = m_P \alpha^\vee$ for some real number $m_P \in \mathbb{R}$. If we require further that $m_P \geq 0$, then the family $\underline{\lambda}$ is said *positive* (G, M)-*orthogonal.*

We provide some examples to illustrate the previous definition. In the first example, let λ be a point in $\mathfrak{a}_M^G$ (resp. in the Weyl chamber $\mathfrak{a}_M^{G+}$). We take $\underline{\lambda}$ the family of Weyl orbit of λ. Then it is (G, M)-orthogonal (resp. positive (G, M)-orthogonal). In the second example, we consider an element $g \in G(\mathbb{A})$. Then the family $\underline{\lambda} = \{H_P(g)\}_{P \in \mathcal{F}(M)}$ is positive (G, M)-orthogonal.

We will recall now a result of Behrend on positive (G, M_0)-orthogonal families. Let $\underline{\lambda} = \{\lambda_P\}_{P \in \mathcal{F}(M_0)}$ be a positive (G, M_0)-orthogonal family. Let P be a parabolic subgroup of G, not necessarily in $\mathcal{F}(M_0)$. We define

$$R(P) = \{\beta \in \Phi : [\beta]_P = \sum_{\alpha \in \Delta_P} m_\alpha \alpha \text{ with } m_\alpha \geq 0\}$$

and

$$\deg_P(\underline{\lambda}) = \sum_{\beta \in R(P)} \langle \beta, \lambda_P \rangle.$$

With these notations, Behrend proved [5]:

Theorem 2.1 (Behrend). *There exists a unique parabolic subgroup P of G verifying the following conditions:*

a) For every parabolic subgroup Q of G such that $Q \in P$ and $\Delta_Q - \Delta_P$ contains a unique simple root, says α, then we require

$$\langle \alpha, \lambda_P \rangle \leq 0.$$

b) The element λ_P lies in the Weyl chamber $\mathfrak{a}_P^{G+}$.

Further, it is characterized by the following property: we consider the set of parabolic subgroups P of G such that

$$\deg_P(\underline{\lambda}) = \max_Q \{\deg_Q(\underline{\lambda})\}$$

where we consider all parabolic subgroups Q of G, then P is the unique maximal element of this set.

We will apply Behrend's theorem to our situation. Our goal is to define the notion of Harder-Narasimhan polygon associated to an element $g \in G(\mathbb{A})$. It could be considered as a generalization of the classical Harder-Narasimhan polygon associated to vector bundles over curves.

We have to introduce some more notations. Recall that for every element s of the relative Weyl group W of G, we have fixed a representative $w_s \in G(F)$ of s. By the choice of the maximal open compact subgroup K, Arthur proved [3, Lemma 1.1] that there exists a unique point $T_0 \in \mathfrak{a}_{M_0}^G$ considered as a point in $\mathfrak{a}_{M_0}$ such that, for every element $s \in W$, we have

$$H_{M_0}(w_s^{-1}) = T_0 - s^{-1}T_0.$$

We remark that if G is split over $\mathbb{F}_q$, then we can choose $K = \prod_x G(\mathcal{O}_x)$. It implies that the elements w_s can be chosen in K and hence $H_{M_0}(w_s^{-1}) = 0$. Consequently, $T_0 = 0$. However, for general connected reductive group G over F, $T_0 \neq 0$.

Now, let g be an element in $G(\mathbb{A})$ and Q be a parabolic subgroup of G defined over F but not necessarily containing M_0. There exists

$\delta \in G(F)$ and $P \in \mathcal{P}(M_0)$ such that $P \subseteq Q' := \delta Q \delta^{-1}$. We define:

$$\deg_Q(g) := (\sum_{\alpha \in R(Q')} \alpha, [H_P(\delta g)]_P^G - T_0).$$

The previous choice of T_0 implies that the degree $\deg_Q(g)$ does not depend on our choice (δ, P).

We introduce the set $\mathcal{A}$ consisting of parabolic subgroups P de G defined over F, but not necessarily in $\mathcal{P}(M_0)$ such that $\deg_P(g) = \max_Q \deg_Q(g)$. It follows from Behrend's previous result that this set contains a unique maximal element. Remark that the set of parabolic subgroups of G defined over F is in bijection with the set of pairs (P, δ) where P is a standard parabolic subgroup of G and δ is a class in the quotient $P(F)\backslash G(F)$. Under this identification, we can reformulate our previous result as follows. Let (P, δ) be a pair as before. We define the polygon

$$p_P^{\delta g} = [H_P(\delta g)]_P^G - [T_0]_P^G.$$

It is a point in $\mathfrak{a}_P^G$. We have proved:

Theorem 2.2. *We keep the previous notations. There exists a unique pair (P, δ) called the canonical Harder-Narasimhan filtration associated to $g \in G(\mathbb{A})$ which satisfies the following conditions:*

a) For every standard parabolic subgroup Q such that $Q \subsetneq P$ and $\Delta_Q - \Delta_P$ contains a unique simple root α, and for every $\delta_Q \in Q(F)\backslash P(F)$, we have the inequality

$$\langle \alpha, p_P^{\delta_Q \delta g} \rangle \leq 0.$$

b) The element $p_P^{\delta g}$ lies in the Weyl chamber $\mathfrak{a}_P^{G+}$.

The element $p_P^{\delta g} \in \mathfrak{a}_P^G$ considered as a point in $\mathfrak{a}_{M_0}^G$ is called the Harder-Narasimhan polygon of $g \in G(\mathbb{A})$.

3. Harder's reduction theory and truncation parameters

Let p be a point in the acute chamber $\mathfrak{a}_{M_0}^{G+}$. Let $g \in G(\mathbb{A})$ and (P, δ) be a pair where P is a standard parabolic subgroup of G and δ is a class in the quotient $P(F)\backslash G(F)$. We have defined

$$p_P^{\delta g} = [H_P(\delta g)]_P^G - [T_0]_P^G.$$

We say that $p_P^{\delta g} >_P p$ if we have $p_P^{\delta g} - [p]_P^G \in {}^+\mathfrak{a}_P^G$.

Definition 3.1. We keep the previous notations. Suppose further that (P, δ) is the canonical Harder-Narasimhan filtration associated to g and that $p(g) := p_P^{\delta g} \in \mathfrak{a}_P^G$ considered as a point in $\mathfrak{a}_{M_0}^G$ is the associated

Harder-Narasimhan polygon. We will say that $p(g)$ is bounded by p, write $p(g) \leq p$ if $p - p(g) \in {}^{+}\mathfrak{a}_{M_0}^{G}$.

We review the reduction theory for function fields entirely due to Harder. For the references, we send the reader to [16, section 4] for an excellent overview and to the original papers of Harder [10, 11] for proofs. For a real number c, we consider the following set

$$P_0(c) = \{g \in P_0(\mathbb{A}) : (\alpha, H_{M_0}(g)) \geq c, \alpha \in \Delta\}.$$

Harder proved two fundamental and deep results of the reduction theory over function fields:

Theorem 3.2 (Harder). *There exists a constant $c > -\infty$ such that $G(\mathbb{A}) = G(F)P_0(c)K$. The set $P_0(c)$ such that we have the previous property is called a Siegel domain.*

Obviously, if $P_0(c)$ is a Siegel domain for some real number c, then $P_0(c')$ will be a Siegel domain for any real number c' with $c' \leq c$.

Theorem 3.3 (Harder). *Let c be a real number such that $G(\mathbb{A}) = G(F)P_0(c)K$. Then there exists a positive constant $c_2 = c_2(c)$ depending on c verifying the following property: If $\gamma \in G(F)$ and $x \in P_0(c)$ such that $\gamma x \in P_0(c)$, then $\gamma \in P(F)$ where the parabolic subgroup P corresponds to the subset I of Δ consisting of all simple roots α with $(\alpha, x) \geq c_2$.*

Harder's reduction theory implies immediately that the subset of $G(F)\backslash G(\mathbb{A})/J$ consisting of elements g such that $p(g) \leq p$ is compact.

4. The non-invariant trace formula: results and discussion

We consider the function space $L^2(G(F)\backslash G(\mathbb{A}))$ which consists of square integrable complex functions on $G(F)\backslash G(\mathbb{A})$. Let $f \in C_c^\infty(G(\mathbb{A}))$ be a function over $G(\mathbb{A})$ which is locally constant and of compact support. We choose a Haar measure on $G(\mathbb{A})$ and define an operator $R(f)$ on $L^2(G(F)\backslash G(\mathbb{A}))$ as follows:

$$(R(f)\varphi)(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy)dy.$$

A simple calculation implies that

$$(R(f)\varphi)(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy)dy$$
$$= \int_{G(F)\backslash G(\mathbb{A})} \sum_{\gamma\in G(F)} f(x^{-1}\gamma y)\varphi(y)dy.$$

Hence, this integral operator $R(f)$ admits a kernel $K(x,y)$ given by the formula:

$$K(x,y) = \sum_{\gamma\in G(F)} f(x^{-1}\gamma y).$$

Since f is locally constant and of compact support, the sum vanishes for all except a finite number of $\gamma \in G(F)$, therefore the kernel is well defined.

We consider the kernel over the diagonal and define

$$k(x) = K(x,x) = \sum_{\gamma\in G(F)} f(x^{-1}\gamma x)$$

as the function over $G(F)\backslash G(\mathbb{A})$. Remark that this function is invariant by multiplication with $Z(\mathbb{A})$ where Z is the center of G. We will introduce a discrete torsion-free subgroup J of $Z(\mathbb{A})$ such that $JZ(F)\backslash Z(\mathbb{A})$ is compact and $Z(F)\cap J = 0$. In general, the integral $\int_{JG(F)\backslash G(\mathbb{A})} k(x)$ diverges and the goal of the trace formula and its refinements is to regularize this integral.

We follow closely Arthur's procedure to define the modified kernel. Let P be a standard parabolic subgroup of G. Following Arthur, we consider the integral operator $R_P(f)$ acting on the function space $L^2(M(F)N(\mathbb{A})\backslash G(\mathbb{A}))$ defined as follows:

$$(R_P(f)\varphi)(x) = \int_{G(\mathbb{A})} f(y)\varphi(xy)dy.$$

It has a kernel given by the formula

$$K_P(x,y) = \int_{N(\mathbb{A})} \sum_{\gamma\in M(F)} f(x^{-1}\gamma ny)dn$$

for every $x,y \in M(F)N(\mathbb{A})\backslash G(\mathbb{A})$. Denote by $k_P(x) = K_P(x,x)$, in other words, for every $x \in M(F)N(\mathbb{A})\backslash G(\mathbb{A})$,

$$k_P(x,y) = \int_{N(\mathbb{A})} \sum_{\gamma\in M(F)} f(x^{-1}\gamma nx)dn.$$

Let p be a point in the acute chamber $\mathfrak{a}_{M_0}^{G+}$. Suppose that it is sufficiently regular in the sense that $\langle\alpha,p\rangle$ is sufficiently large with respect to the support of f. We define the Arthur's modified kernel

$$k^p(g) = \sum_{P_0 \subseteq P} (-1)^{\dim \mathfrak{a}_P^G} \sum_{\delta \in P(F)\backslash G(F)} 1(p_P^{\delta g} >_P p) k_P(\delta g).$$

It is a function over $G(\mathbb{A})$ which is invariant on the left by the subgroup $JG(F)$. Hence, we will consider the modified kernel as a function over the double quotient $G(F)\backslash G(\mathbb{A})/J$.

In contrast to the number field setting, we will show that, in the function field case, the modified kernel over $G(F)\backslash G(\mathbb{A})/J$ is in fact the product of the original kernel and the characteristic function of a compact set. Consequently, it is automatically integrable over $G(F)\backslash G(\mathbb{A})/J$.

Theorem 4.1. *We keep the previous notations. Then for every $g \in G(\mathbb{A})$, we have the following equality:*

$$k^p(g) = 1(p(g) \leq p) k(g).$$

The modified integral will be defined as follows:

$$J^p(f) = \int_{JG(F)\backslash G(\mathbb{A})} k^p(x).$$

We have proved that our modified integral converges. The modified kernel has an obvious geometric expression. It also has a spectral expression following from deep results of Langlands and Morris [16, 17, 18] called the Langlands decomposition theorem. Hence, we get the first step of our desired non-invariant trace formula, which states an equality between a geometric side and a spectral side.

Next, we need to refine the previous trace formula. On the geometric side, Arthur proposed to decompose the kernel into semisimple conjugacy classes. However, since we are working with the function field F which is not perfect, we cannot apply directly Arthur's procedure and it would be an interesting question how to overcome this difficulty. Let us attract our attention to the spectral side. It turns out that a fine spectral refinement could be obtained using only the Langlands decomposition theorem. It has been proved by Lafforgue for $\mathrm{GL}(n)$ [15, 13] and then generalized by the author for reductive groups over F [19, 20].

Finally, it would be very interesting to investigate the behavior of $J^p(f)$ in function of the truncating parameter p for fixed f. In the number field setting, the parameter takes value in some open chamber of a real vector space and Arthur proved that the function $J^p(f)$ is in fact a polynomial of p. In our function field setting, the parameter p

takes value in a lattice of a real vector space, but we can still prove that the $J^p(f)$ is a quasi-polynomial of p [20].

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