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INVARIANTS OF CONJUGACY CLASSES OF SURFACE BRAIDS DERIVED FROM ALEXANDER QUANDLES OR CORE QUANDLES

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ABSTRACT. In this paper, we introduce new invariants of conjugacy classes of surface braids via colorings by Alexander quandles or core quandles of groups and explain some applications.

1. INTRODUCTION

In 2-dimensional knot theory, it is known that any surface-link in $\mathbf{R}^4$ is represented as the closure of a surface braid ([15, 9]). A surface braid and the closure of it are often studied by charts in a 2-disk and a 2-sphere U_0, respectively (cf. [10]). In this paper, we define invariants of conjugacy classes of surface braids in terms of charts. We also explain some applications for the following topics by the invariants (and quandle cocycle invariants [1]):

- Existence of infinite sequences of mutually non-conjugate surface braids representing same surface-links
- Characterizations of charts representing a given surface-link
- Braid index

This paper consists of seven sections: In §2, we review surface-links, surface braids, charts and their relations. In §3, we review quandle colorings for charts and define invariants K_X related to colorings by Alexander quandles or core quandles of groups. In §4, we give examples of infinite sequences of mutually non-conjugate surface braids representing same surface-links. In §5, we give a simple classification of 4-charts by K_X and the number of X-colorings when X is a dihedral quandle. In §6, we study examples of a pair of non-conjugate surface braids representing same nonribbon surface-links by dihedral quandle cocycle invariants and and the classification given in §5. In §7, we study the braid index of a surface-link by the dihedral quandle cocycle invariant.

2. PRELIMINARIES

A *surface-link* S is a closed oriented surface embedded in Euclidean 4-space $\mathbf{R}^4$ locally flatly. If S is connected, then it is called a *surface-knot.* A surface-knot S is *trivial* if S bounds a handlebody in $\mathbf{R}^4$ and a surface-link S is a *trivial* if S is a split union of trivial surface-knots. Two surface-links F and F' are *equivalent* if there is an orientation preserving homeomorphism $f : \mathbf{R}^4 \longrightarrow \mathbf{R}^4$ such that $f(F) = F'$.

A *surface braid* S of degree m is an oriented surface embedded in $D_1 \times D_2 (\subset \mathbf{R}^4)$ locally flatly and properly such that the restriction map $\pi|_S$ of the projection map

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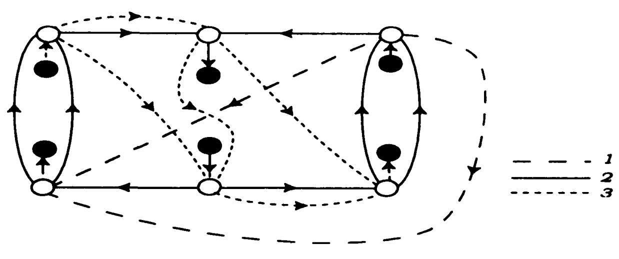

FIGURE 1. Example of a 4-chart

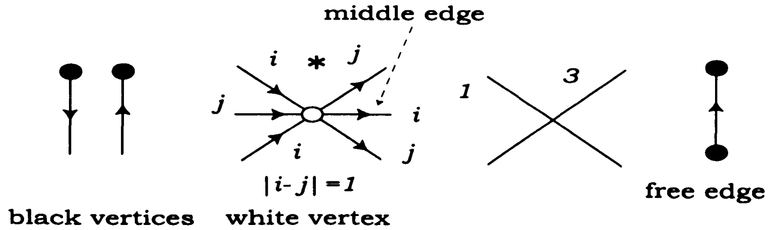

FIGURE 2

$\pi : D_1 \times D_2 \longrightarrow D_2$ is an m-fold branched covering map and $\partial S = X_m \times \partial D_2$, where D_1 and D_2 are 2-disks, X_m is a fixed set of m interior points of D_1. If the branched covering map is simple, then S is called *simple.*

Two surface braids S and S' with same degree are *equivalent* if they are ambient isotopic by a fiber-preserving isotopy $\{h_u\}_{0 \le u \le 1}$ of $D_1 \times D_2$, as a D_1-bundle over D_2, rel $D_1 \times \partial D_2$. For a surface braid S of degree m, we have a surface-link obtained from S by attaching m parallel 2-disks onto the boundary of S in $\overline{\mathbf{R}^4 \backslash D_1 \times D_2}$. We call the surface the *closure* of S.

An *m-chart* Γ is a (possibly empty) finite graph in an oriented 2-disk D_2, which may have *hoops* (that are closed edges without vertices), satisfying the following conditions:

(i) Every vertex has degree one, four or six.
(ii) Every edge is directed and labeled by an integer in $\{1, 2, \ldots, m-1\}$.
(iii) For each vertex of degree six, three consecutive edges are directed inward and the other three are directed outward; these six edges are labeled by i and $i+1$ alternately for some i.
(iv) For each vertex of degree four, two consecutive edges are directed inward and the other two are directed outward; these four edges are labeled by i and j alternately with $|i-j| > 1$.

An example of a 4-chart is given in Fig. 1. A vertex of degree one or six is called a *black vertex* or a *white vertex*, respectively. An edge attached to a white vertex is called a *middle edge* if it is the middle of the three consecutive edges which are oriented in the same directions; otherwise a *non-middle edge.* A *free edge* is an edge in a chart whose endpoints are black vertices. See Fig. 2.

Operations listed below (and their inverses) are called a C_I-, C_{II}- and C_{III}-*move*, respectively. See Fig. 3. These moves are called *C-moves.* Two m-charts are *C-move equivalent* if they are related by a finite sequence of such C-moves and ambient isotopies.

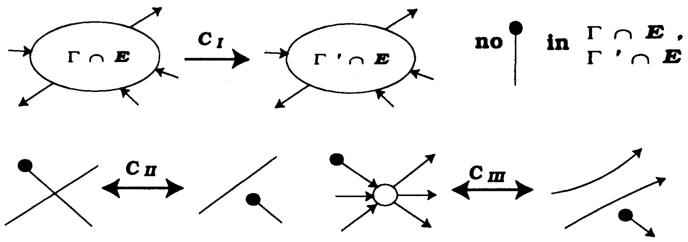

FIGURE 3. C-moves

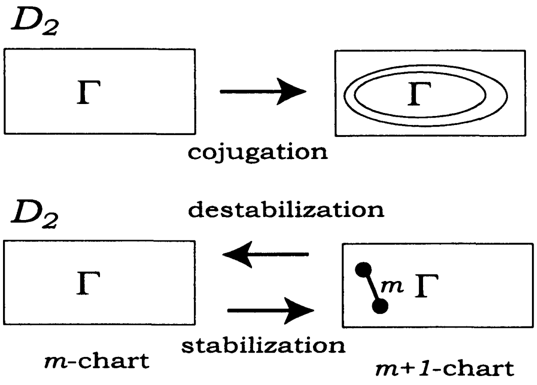

FIGURE 4. Conjugations, stabilizations and destabilizations

(C_I) For a 2-disk E on D_2 such that $\Gamma \cap E$ has neither black vertices nor nodes, replace $\Gamma \cap E$ with an arbitrary chart that has neither black vertices nor nodes.

(C_{II}) Suppose that there is an edge α attached to a black vertex B and a 4-valent vertex v. Remove α and v, attach B to the diagonal edge of α and connect other two edges in a natural way.

(C_{III}) Let a black vertex B and a white vertex W be connected by a non-middle edge α of W. Remove α and W, attach B to the edge of W opposite to α, and connect other four edges in a natural way.

In [10], S. Kamada proved that there is one-to-one correspondence between equivalent classes of simple surface braids of degree m and C-move equivalent classes of m-charts in D_2. For a chart Γ, we denote by $S(\Gamma)$ the closure of a simple surface braid corresponded to Γ. A 4-chart depicted in Fig. 1 represents a 2-twist spun trefoil.

A *conjugation* for a chart is an operation inserting some boundary parallel hoops. A *conjugation* for a surface braid is an operation corresponded to a conjugation for a chart. For an m-chart Γ, an $m+1$-chart is obtained from Γ by inserting a free edge labeled by m. This operation is called a *stabilization*, and the inverse operation is called a *destabilization*. See Fig. 4.

For charts in D_2, we define charts in a 2-sphere U_0 by identifying ∂D_2. We also define C-moves, stabilizations and destabilizations in U_0 naturally.

Theorem 2.1 ([10]). *There is one-to-one correspondence between conjugacy and equivalent classes of simple surface braids of degree m and C-move equivalent classes of m-charts in U_0.*

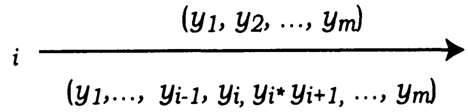

FIGURE 5. Coloring condition

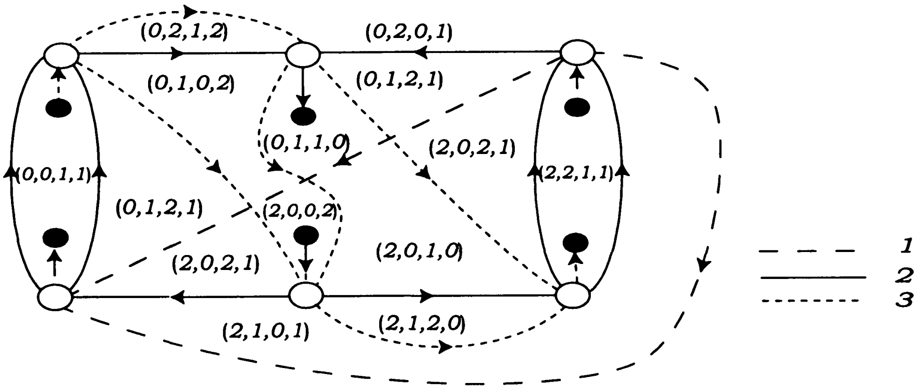

FIGURE 6. Example of an R_3-coloring

From now on, we assumed that that any chart is in U_0.

3. INVARIANTS

In this section, we review quandle colorings of a chart [2, 4] and introduce invariants of conjugacy classes of surface braids.

A *quandle* is a set X with a binary operation $* : X \times X \longrightarrow X$ satisfying the following properties:

(a) For any $x \in X$, $x * x = x$.
(b) For any $x_1, x_2 \in X$, there is a unique $x_3 \in X$ such that $x_1 = x_3 * x_2$.
(c) For any $x_1, x_2, x_3 \in X$, $(x_1 * x_2) * x_3 = (x_1 * x_3) * (x_2 * x_3)$

Example 3.1. (i) The set $\mathbf{Z}_n(\cong \mathbf{Z}/n\mathbf{Z})$ becomes a quandle under the binary operation $a * b = 2b - a \pmod n$, which is called the *dihedral quandle* R_n of order n.
(ii) Set $\Lambda := \mathbf{Z}[t, t^{-1}]$. A Λ-module M becomes a quandle under the binary operation $a * b = ta + (1-t)b$, which is called an *Alexander quandle.* If $M = \Lambda/(n, t+1)$, then M is isomorphic to R_n.
(iii) A group G becomes a quandle under the binary operation $a * b = ba^{-1}b$, which is called the *core quandle* of G. The core quandle of $\mathbf{Z}_n$ is isomorphic to R_n.

Let Γ be an m-chart and the set of regions of $U_0 \backslash \Gamma$ is denoted by $\Sigma(\Gamma)$. A map $C : \Sigma(\Gamma) \longrightarrow X^m$ is an X*-coloring* of Γ if it is such that $C(\lambda_1) = (y_1, \cdots, y_m)$ and $C(\lambda_2) = (y_1, \cdots, y_{i-1}, y_{i+1}, y_i * y_{i+1}, y_{i+2}, \cdots, y_m)$ for each edge e with label i where λ_1 and λ_2 are regions separated by e and λ_1 is on the left-side of e. See Fig. 5. The set of X-colorings of Γ is denoted by $Col_X(\Gamma)$. An example of an R_3-coloring of a 4-chart depicted in Fig. 1 is given in Fig. 6. If $C(\lambda) = (y, \ldots, y)$ for $\lambda \in \Sigma(\Gamma)$ and for some $y \in X$, then we call C a *trivial X-coloring.*

Let Γ be an m-chart and X be an Alexander quandle or the core quandle of a group. We define a map $\kappa : Col_X(\Gamma) \longrightarrow X$ by

$$\kappa(C, \lambda) = \sum_{i=1}^{m} t^{m-i} y_i \tag{1}$$

when X is an Alexander quandle, and

$$\kappa(C, \lambda) = \prod_{i=1}^{m} y_i^{(-1)^{m-i}} \tag{2}$$

when X is the core quandle of a group, where $C(\lambda) = (y_1, y_2, \dots, y_m)$ for $\lambda \in \Sigma(\Gamma)$. In particular, when $X = R_n$, by Equation 1 or 2, $\kappa(C, \lambda)$ is defined by

$$\kappa(C, \lambda) := \sum_{i=1}^{m} (-1)^i y_i. \quad (\text{mod } n) \tag{3}$$

If X is an Alexander quandle and the core quandle of a group, then X is a dihedral quandle. Thus, $\kappa(C, \lambda)$ is well-defined.

Lemma 3.2. *The map $\kappa(C, \lambda)$ is independent of a choice of λ.*

By Lemma 3.2, we denote $\kappa(C, \lambda)$ by $\kappa(C)$. We define a multiset

$$K_X(\Gamma) := \{\kappa(C) \mid C \in Col_X(\Gamma)\}.$$

Theorem 3.3 ([5]). *A multiset $K_X(\Gamma)$ is an invariant of C-move equivalent classes of charts in U_0, and hence $K_X(\Gamma)$ is also an invariant of conjugacy classes of surface braids.*

Set $X = R_n$. Then we also regard $K_{R_n}(\Gamma)$ as an element of $\mathbf{Z}[t, t^{-1}]/(t^n - 1)$ by

$$K_{R_n}(\Gamma) := \sum_{C \in Col_{R_n}(\Gamma)} t^{\kappa(C)}.$$

If Γ is a 4-chart depicted in Fig. 1, then $K_{R_3}(\Gamma) = 3^2$ (see Fig. 6).

An *oval nest* is a free edge together with some concentric hoops. A chart is *ribbon* if it is C-move equivalent to a chart consists of some oval nests. If a surface-link is represented by a ribbon chart, then we call it a *ribbon*.

Remark 3.4. In [2], I. Hasegawa defined another invariant of conjugacy classes of surface braids. Hasegawa's invariant is required that any surface braid corresponded to a ribbon chart has specific value. By the invariant, we have a first example of non-ribbon chart representing a ribbon surface-link and a pair of non-conjugate surface braids. Our invariant K_X do not help us to study whether a chart is ribbon or not, but are useful to study whether two ribbon charts are conjugate or not as in §4.

4. Examples

Let D^n and E^n be 4-charts depicted in Fig. 7 for any n. By a destabilization and C-moves, we see that D^n and E^n represents same surface-knot, the surface-knot is a spun $(2, n)$-torus knot and

$$K_{R_n}(D^n) = n(1 + t + \cdots + t^{n-1}),$$
$$K_{R_n}(E^n) = n^2.$$

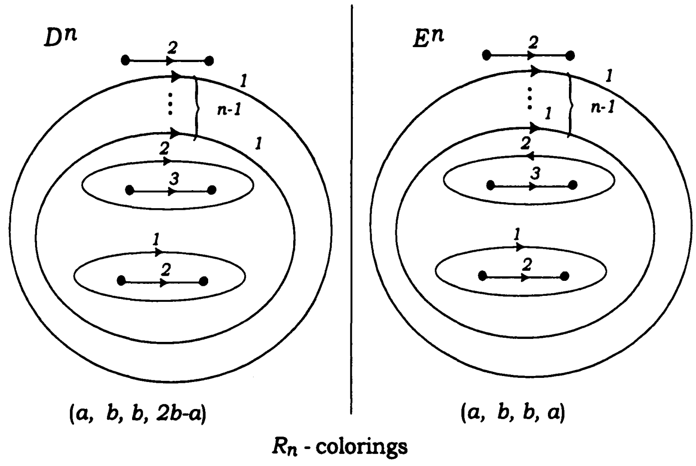

FIGURE 7. $D^n|E^n$

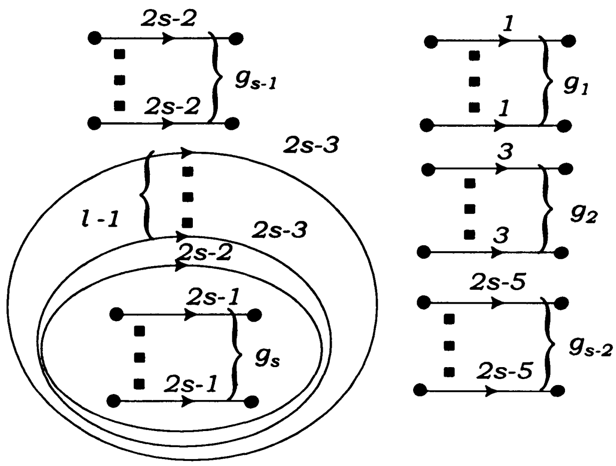

FIGURE 8. $B^l_{s,g_1,\dots,g_s}$

Thus, we have Theorem 4.1.

Theorem 4.1 ([5]). *There is a pair of non-conjugate surface braids with degree* 4 *representing a spun* $(2, n)$*-torus knot for* $n \geq 3$.

Let $s, g_1, \dots, g_s, l$ be integers with $s \geq 2$, $g_1, \dots, g_s \geq 0$ and $l \geq 2$. Let $B^l_{s,g_1,\dots,g_s}$ be a $2s$-chart depicted in Fig. 8. By a destabilization and C-moves, we see that $B^l_{s,g_1,\dots,g_s}$ represents same surface-link for any l and the surface-link is an s component trivial surface-link whose components have genera $g_1, \dots, g_s$. Let $\mathbb{P}$ be the set of prime integers. Then we also see that $\{B^p_{s,g_1,\dots,g_s}\}_{p \in \mathbb{P}}$ are the set of $2s$-charts representing mutually non-conjugate surface braids by the set of invariants $\{K_{R_p}\}_{p \in \mathbb{P}}$. Thus, we have Theorem 4.2.

Theorem 4.2 ([5]). *There is an infinite sequence of mutually non-conjugate surface braids with degree* $2s$ *representing the trivial* s *component surface-link for any* $s \geq 2$ *and any genus.*

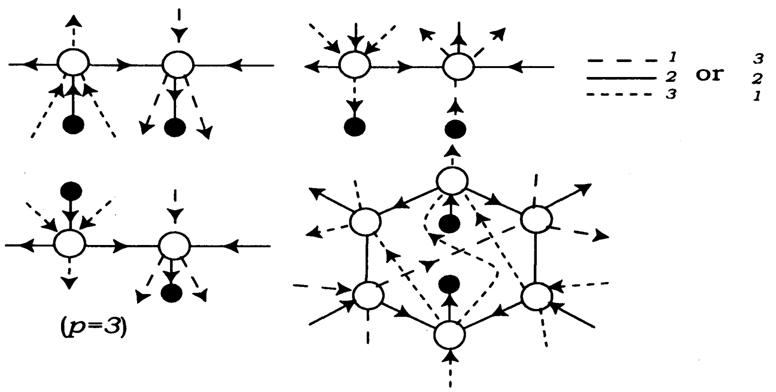

FIGURE 9. $\mathfrak{A}$

5. Simple classification of 4-charts

In §4, we give examples of pairs of non-conjugate surface braids representing a nontrivial ribbon surface-knot. We would like to find an example of a pair of non-conjugate surface braids representing a nonribbon surface-knot. In [8], Kamada proved that any m-chart represents a ribbon surface-link if $m \leq 3$. Thus, we study 4-charts.

It is known that $Col_{R_p}(\Gamma)$ is a linear space over $\mathbf{Z}_p$ (cf. [3]) and the dimension $Col_{R_p}(\Gamma)$ is at most 4 ([14]). We classify 4-charts the following five types for odd prime p by the dimension of the set of $Col_{R_p}(\Gamma)$, which is denoted by $\dim Col_{R_p}(\Gamma)$, and K_{R_p}.

(I-p) It is satisfied that $\dim Col_{R_p}(\Gamma) = 1$.
(II-I-p) It is satisfied that $\dim Col_{R_p}(\Gamma) = 2$ and $K_{R_p}(\Gamma) = p^2$.
(II-II-p) It is satisfied that $\dim Col_{R_p}(\Gamma) = 2$ and $K_{R_p}(\Gamma) \neq p^2$.
(III-p) It is satisfied that $\dim Col_{R_p}(\Gamma) = 3$.
(IV-p) It is satisfied that $\dim Col_{R_p}(\Gamma) = 4$.

If Γ is a 4-chart depicted in Fig. 1, then Γ satisfies (II-I-p) (see Fig. 6).

Lemma 5.1. *We have the following.*

(i) *If Γ satisfies* (I-p), *then all R_p-colorings are trivial.*
(ii) *If Γ satisfies* (III-p), *then $K_{R_p}(\Gamma) \neq p^3$.*
(iii) *If Γ satisfies* (IV-p), *then Γ represents the trivial* 4*-component* 2*-link.*

Let $\mathfrak{A}$ be the set of subcharts in U_0 depicted in Fig. 9 and their mirror images. Let $\mathfrak{B}$ be the set of subcharts in U_0 depicted in Fig. 10 and their mirror images. By colorings conditions around each subgraph in $\mathfrak{A}$ and $\mathfrak{B}$, we have the following lemmas.

Lemma 5.2. *Let Γ be a* 4*-chart.*

(i) *If there is a subchart G of Γ with $G \in \mathfrak{A}$, then Γ satisfies* (I-p) *or* (II-I-p).
(ii) *If there is a subchart G of Γ with $G \in \mathfrak{B}$, then Γ satisfies* (I-p) *or* (II-II-p).

Moreover, if there is a subchart G and G' of Γ with $G \in \mathfrak{A}$ and $G' \in \mathfrak{B}$, then all R_p-coloring of Γ is trivial.

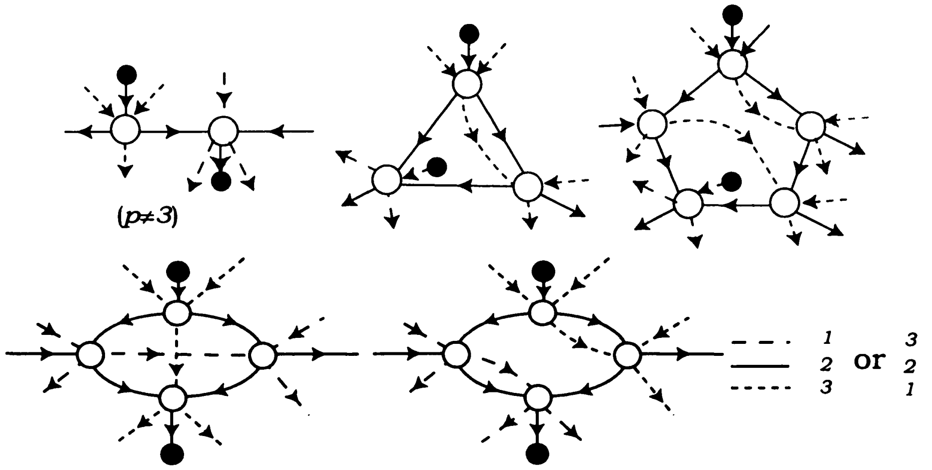

FIGURE 10. $\mathfrak{B}$

6. DIHEDRAL QUANDLE COCYCLES INVARIANTS

In [1], a quandle cocycle invariant $\Phi_f(F)$ for a surface-link F was defined as an element of $\mathbf{Z}[A]$ where f is an A-valued 3-cocycle. By the definition of $\Phi_f(F)$, we see that if $\Phi_f(F) \not\in \mathbf{Z}(\subset \mathbf{Z}[A])$, then F is nonribbon. Thus, we shall consider a chart Γ such that $\Phi_f(S(\Gamma)) \not\in \mathbf{Z}$ for an A-valued 3-cocycle f. First, we consider $\Phi_{\theta_3}(S(\Gamma))$ where θ_3 is the Mochizuki's 3-cocycle of R_3 (cf. [12]). Since $\mathbf{Z}[\mathbf{Z}_p]$ is isomorphic to $\mathbf{Z}[t,t^{-1}]/(t^p-1)$, we also regard $\Phi_{\theta_3}(F)$ as an element of $\mathbf{Z}[t,t^{-1}]/(t^n-1)$. It is known that Γ satisfies (II-I-3) and $\Phi_{\theta_3}(S(\Gamma)) = 3+6t^2$ (or $3+6t$) where Γ is a 4-chart depicted in Fig. 1 (or its mirror image) (cf. [1]).

Theorem 6.1 ([7]). *Let F be a surface-link represented by a 4-chart Γ. If Γ satisfies* (II-II-3), *then $\Phi_{\theta_3}(F) = 3^2$.*

Corollary 6.2. *There is no 4-chart Γ satisfying that $\Phi_{\theta_3}(S(\Gamma)) \not\in \mathbf{Z}$ and* (II-II-3). *In particular, if there is a subchart G of Γ with $G \in \mathfrak{B}$, then Γ does not represent a 2-twist spun trefoil.*

It is implies that we cannot find examples for a pair of non-conjugate surface braids of degree 4 representing a nonribbon surface-link F with $\Phi_{\theta_3}(F) \not\in \mathbf{Z}$ by our approaches.

Next, we consider a chart Γ such that $\Phi_{\theta_5}(S(\Gamma)) \not\in \mathbf{Z}$. Let A_1^{10}, A_2^{10} and A_3^{10} be 4-charts depicted in Fig. 11. Then we see that $K_{R_5}(A_1^{10}) = K_{R_5}(A_2^{10}) = 5(1+t+t^2+t^3+t^4)$ and $K_{R_5}(A_3^{10}) = 5^2$. We also see that $\Phi_{\theta_5}(S(A_i^{10})) = 5(1+2t^2+2t^3)$ and the surface-knot group $G(S(A_i^{10})) \cong \langle a, z \mid a^{-1}za = z^{-1}, z^5 = 1\rangle$ for $i = 1,2,3$. By similar arguments of §21 of [10], we see that A_1^{10} represents a 2-twist spun $(2,5)$-torus knot.

Question 6.3 ([6]). Are $S(A_1^{10})$, $S(A_2^{10})$ and $S(A_3^{10})$ equivalent?

If the answer of Question 6.3 is positive, then we have an example of a pair of non-conjugate surface braids representing a nonribbon surface-link.

7. THE BRAID INDEX OF A SURFACE-LINK

In this section, we study the braid index of a surface-link.

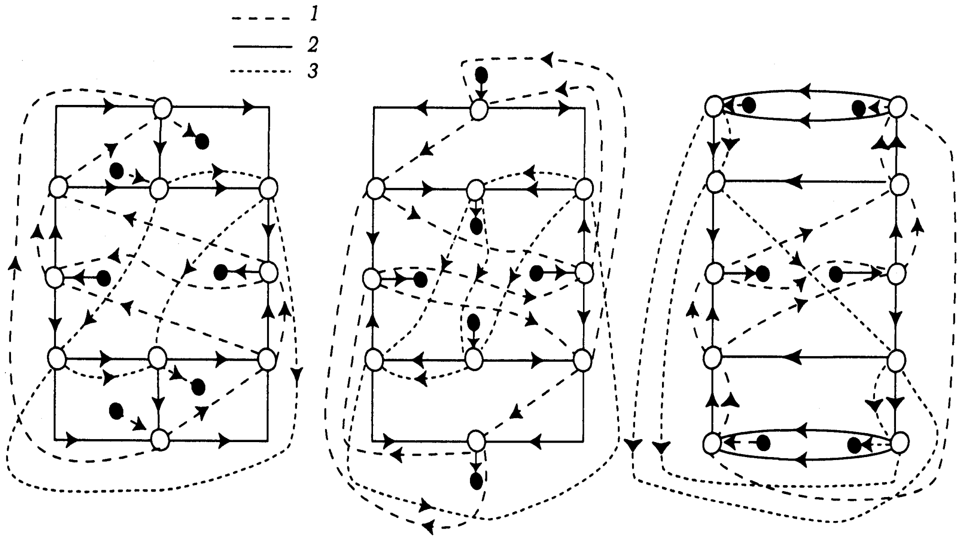

FIGURE 11. A_1^{10}, A_2^{10} and A_3^{10}

For a surface-link F, we define Braid(F) by

$$\begin{aligned}\mathrm{Braid}(F) \ &:= \min\left\{ m \ \middle| \ \begin{array}{l} S \text{ is a surface braid of degree } m \\ \text{whose closure is equivalent to } F \end{array} \right\} \\ &= \min\{m \mid \Gamma \text{ is an } m\text{-chart representing } F\},\end{aligned}$$

and call Braid(F) the *braid index* of F.

Let T_3^r be an r-twist spun trefoil. It is known that $\mathrm{Braid}(T_3^0) = 3$ and $\mathrm{Braid}(T_3^2) = 4$ (cf.[8, 10]). Let F_l and G_l be the connected sums $F_l = \sharp_l T_3^0$ and $G_l = T_3^2 \sharp (\sharp_l T_3^0)$.

Theorem 7.1 ([11]). *Let F and F' be non-trivial surface-links. Then*

$$\mathrm{Braid}(F \sharp F') \leq \mathrm{Braid}(F) + \mathrm{Braid}(F') - 2.$$

Theorem 7.2 ([14]). *Let F be a non-trivial surface-link and X be a finite quandle. If $|Col_X(F)| \geq |X|^l$, then* $\mathrm{Braid}(F) \geq l + 1$.

By Theorems 7.1 and 7.2, Tanaka stated that $\mathrm{Braid}(F_l) = l + 2$ and $\mathrm{Braid}(G_l) = l + 3$ or $l + 4$ for each l, and gave the following problem.

Problem 7.3 ([14]). For each integer $l > 0$, determine the braid index of G_l exactly. Which is the correct value of this index, $l + 3$ or $l + 4$?

By Theorem 6.1, we can prove the following corollary.

Corollary 7.4 ([7]). *Let F be a surface-link represented by a 4-chart Γ. If Γ satisfies* (III-3), *then $\Phi_{\theta_3}(F) = 3^3, 21 + 6t$ or $21 + 6t^2$.*

We see that $\Phi_{\theta_3}(G_1) = 9 + 18t^2$. By Theorem 7.1 and Corollary 7.4, we have Problem 7.3 for $l = 1$.

Corollary 7.5 ([7]). *The braid index of G_1 is equal to* 5.

Acknowledgments

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