数理解析研究所講究録
第1716巻 2010年 84-98

HIGHER-ORDER ALEXANDER INVARIANTS FOR HOMOLOGICALLY FIBERED KNOTS

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1. INTRODUCTION

This note is adapted from the talk at the 2010 Intelligence of Low-dimensional Topology at Research Institute for Mathematical Sciences, Kyoto University. For the detail, see the original papers [12], [13].

Let $\Sigma_{g,n}$ be a compact oriented surface of genus g with $n \geq 1$ boundary components, and the triple (M, i_+, i_-) be an oriented homology cobordism between $\Sigma_{g,n}$ and $\Sigma_{g,n}$ with two markings of ∂M : $i_+, i_- : \Sigma_{g,1} \hookrightarrow \partial M$. We call (M, i_+, i_-) a *homology cylinder* over $\Sigma_{g,n}$. This object was introduced by Goussarov [14] and Habiro [16] since it is suitable for applying the theory of clovers and claspers, and then has been studied together with finite type invariants of 3-manifolds. The following have been known as methods for constructing homology cylinders:

- connected sums of the trivial cobordism with homology 3-spheres;
- Levine's method [19] using string links in the 3-ball;
- Habegger's method [15] giving homology cylinders as results of surgeries along string links in homology 3-balls; and
- clasper surgeries (see [14] and [16]).

In [12], the authors gave an explicit construction of homology cylinders, i.e. we introduced a notion of a *homologically fibered knot* and construct a homology cylinder using it. The family of the homologically fibered knots include that of the fibered knots. So, roughly speaking, the following relationships exist:

$$\begin{tikzcd}
\text{Pure Braid} \arrow[r, leftrightarrow] & \text{Mapping cylinder} \arrow[r, leftrightarrow] & \text{Fibered knot} \\
\cap & \cap & \cap \\
\text{Pure String link} \arrow[r, leftrightarrow, "\text{Levine}"] & \text{Homology cylinder} \arrow[r, leftrightarrow] & \text{Homologically fibered knot} \\
\text{(Habegger-Lin)} & \text{(Goussarov, Habiro)} &
\end{tikzcd}$$

In [18], Kirk-Livingston-Wang introduced a Reidemeister torsion for string links, then the second author studied the corresponding Reidemeister torsion for homology cylinders in [23]. Note that this torsion may be regarded as a special case of a decatogorification of sutured Floer homology [8]. In this note, we study the Reidemeister torsion for homologically fibered knots and show a factorization formula. Further, we give a MATHEMATICA program for explicit calculations of the invariants for homologically fibered knots.

2000 *Mathematics Subject Classification.* Primary 57M27, Secondary 57M25.

The authors are partially supported by Grant-in-Aid for Scientific Research, (No. 21540071 and No. 21740044), Ministry of Education, Science, Sports and Technology, Japan.

2. HOMOLOGICALLY FIBERED KNOTS

In this section, we introduce two main objects in this note: homology cylinders and sutured manifolds. First, we define homology cylinders over surfaces, which have their origin in Goussarov [14], Habiro [16], Garoufalidis-Levine [11] and Levine [19]. Let $\Sigma_{g,n}$ be a compact connected oriented surface of genus $g \geq 0$ with $n \geq 1$ boundary components.

Definition 2.1. A *homology cylinder* (M, i_+, i_-) over $\Sigma_{g,n}$ consists of a compact oriented 3-manifold M with two embeddings $i_+, i_- : \Sigma_{g,n} \hookrightarrow \partial M$ such that:

(i) i_+ is orientation-preserving and i_- is orientation-reversing;
(ii) $\partial M = i_+(\Sigma_{g,n}) \cup i_-(\Sigma_{g,n})$ and $i_+(\Sigma_{g,n}) \cap i_-(\Sigma_{g,n}) = i_+(\partial\Sigma_{g,n}) = i_-(\partial\Sigma_{g,n})$;
(iii) $i_+|_{\partial\Sigma_{g,n}} = i_-|_{\partial\Sigma_{g,n}}$; and
(iv) $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Z}) \to H_*(M; \mathbb{Z})$ are isomorphisms.

If we replace (iv) with the condition that $i_+, i_- : H_*(\Sigma_{g,n}; \mathbb{Q}) \to H_*(M; \mathbb{Q})$ are isomorphisms, then (M, i_+, i_-) is called a *rational homology cylinder*.

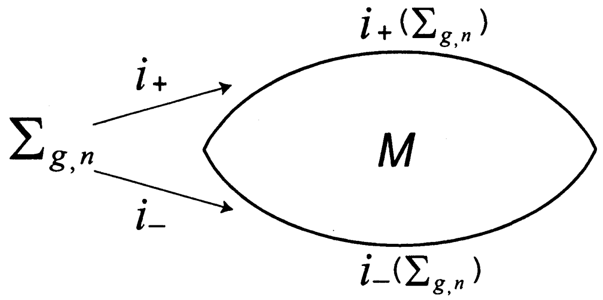

FIGURE 1. Homology cylinder

We often write a (rational) homology cylinder (M, i_+, i_-) briefly by M. Note that our definition is the same as that in [11] and [19] except that we may consider homology cylinders over surfaces with multiple boundaries.

Two (rational) homology cylinders (M, i_+, i_-) and (N, j_+, j_-) over $\Sigma_{g,n}$ are said to be *isomorphic* if there exists an orientation-preserving diffeomorphism $f : M \xrightarrow{\cong} N$ satisfying $j_+ = f \circ i_+$ and $j_- = f \circ i_-$. We denote the set of isomorphism classes of homology cylinders (resp. rational homology cylinders) over $\Sigma_{g,n}$ by $\mathcal{C}_{g,n}$ (resp. $\mathcal{C}_{g,n}^{\mathbb{Q}}$).

Example 2.2 (Mapping cylinder). For each diffeomorphism φ of $\Sigma_{g,n}$ which fixes $\partial\Sigma_{g,n}$ pointwise (hence, φ preserves the orientation of $\Sigma_{g,n}$), we can construct a homology cylinder by setting

$$(\Sigma_{g,n} \times [0,1], \mathrm{id} \times 1, \varphi \times 0),$$

where collars of $i_+(\Sigma_{g,n})$ and $i_-(\Sigma_{g,n})$ are stretched half-way along $(\partial\Sigma_{g,n}) \times [0,1]$. It is easily checked that the isomorphism class of $(\Sigma_{g,n} \times [0,1], \mathrm{id} \times 1, \varphi \times 0)$ depends only on the (boundary fixing) isotopy class of φ. Therefore, this construction gives a map from the mapping class group $\mathcal{M}_{g,n}$ of $\Sigma_{g,n}$ to $\mathcal{C}_{g,n}$.

Next, we recall the definition of sutured manifolds given by Gabai [10]. We here use a special case of them.

A *sutured manifold* (M, γ) is a compact oriented 3-manifold M together with a subset $\gamma \subset \partial M$ which is a union of finitely many mutually disjoint annuli. For each component of γ, an oriented core circle called a *suture* is fixed, and we denote the set of sutures by $s(\gamma)$. Every component of $R(\gamma) = \partial M - \operatorname{Int} \gamma$ is oriented so that the orientations on $R(\gamma)$ are coherent with respect to $s(\gamma)$, that is, the orientation of each component of $\partial R(\gamma)$ induced from that of $R(\gamma)$ is parallel to the orientation of the corresponding component of $s(\gamma)$. We denote by $R_+(\gamma)$ (resp. $R_-(\gamma)$) the union of those components of $R(\gamma)$ whose normal vectors point out of (resp. into) M.

Example 2.3. For a knot K in S^3 and a Seifert surface $\bar{R}$ of K, we set $R := \bar{R} \cap E(K)$, called also a Seifert surface, where $E(K) = \overline{S^3 - N(K)}$ is the complement of a regular neighborhood $N(K)$ of K. Then $(M_R, \gamma) := (\overline{E(K) - N(R)}, \overline{\partial E(K) - N(\partial R)})$ defines a sutured manifold. We call it the *complementary sutured manifold* for R. In this paper, we simply call it the sutured manifold for R.

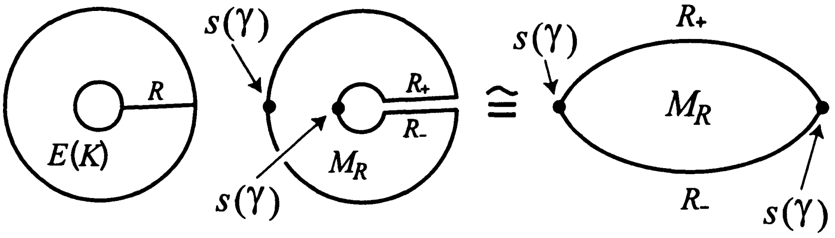

FIGURE 2. Complementary sutured manifold

Let L be an oriented link in the 3-sphere S^3, and $\Delta_L(t)$ the normalized (one variable) Alexander polynomial of L, i.e. the lowest degree of $\Delta_L(t)$ is 0.

Definition 2.4. An n-component link L in S^3 is said to be *homologically fibered* if L satisfies the following two conditions:

(i) The degree of $\Delta_L(t)$ is $2g + n - 1$, where g is the genus of a connected Seifert surface of L; and

(ii) $\Delta_L(0) = \pm 1$.

If an n-component link L satisfies (i), then L is said to be *rationally homologically fibered*.

The Alexander polynomial that satisfies the condition (ii) is said to be *monic* in this paper.

Remark 2.5. In general, if L bounds a connected Seifert surface of genus g, then

$$2g + n - 1 \geq (\text{the degree of } \Delta_L(t)).$$

It is known ([5], [21]) that if L has an alternating diagram that gives, by the Seifert algorithm, a connected Seifert surface of genus g, then the degree of $\Delta_L(t)$ is equal to $2g + n - 1$.

Remark 2.6. Suppose L is an alternating link. Then, L is fibered if and only if $\Delta_L(t)$ is monic, by Murasugi [22] (see also 13.26 (c) in [1]). Therefore, if a homologically fibered link L is not fibered, then L is non-alternating.

Let L be an n-component link and $\Sigma_{g,n}$ the compact oriented surface that is diffeomorphic to a Seifert surface R of L. We fix a diffeomorphism $\vartheta : \Sigma_{g,n} \xrightarrow{\cong} R$ and denote by (M_R, γ) the complementary sutured manifold for R. Then we may see that there are an orientation-preserving embedding $i_+ : \Sigma_{g,n} \to M_R$ and an orientation-reversing embedding $i_- : \Sigma_{g,n} \to M_R$ with $i_+(\Sigma_{g,n}) = R_+(\gamma)$ and $i_-(\Sigma_{g,n}) = R_-(\gamma)$, where two embeddings $i_\pm$ are the composite mappings of ϑ and embeddings $\iota_\pm : R \hookrightarrow M_R$ such that $i_\pm = \iota_\pm \circ \vartheta : \Sigma_{g,n} \to R_\pm(\gamma) \subset M_R$:

$$\begin{tikzcd}
\Sigma_{g,n} \arrow[r, "\vartheta"] \arrow[dr, "i_\pm"'] & R \arrow[d, "\iota_\pm"] \\
& M_R
\end{tikzcd}$$

If i_+, $i_- : H_1(\Sigma_{g,n}) \to H_1(M_R)$ are isomorphisms, we may regard (M_R, γ) as a homology cylinder. The next proposition was essentially mentioned in [6]. A proof is given in [12].

Proposition 2.7. *Let R be a Seifert surface of a link L. If the complementary sutured manifold for R is a homology cylinder, then L is homologically fibered. Conversely, if L is homologically fibered, then the complementary sutured manifold for each minimal genus Seifert surface of L is a homology cylinder.*

It is known that all homologically fibered knots are fibered among prime knots with at most 11 crossings. On the other hand, Friedl-Kim [9] (see also [2]) showed that there are 13 non-fibered homologically fibered knots with 12-crossings. See Figure 7.

3. FACTORIZATION FORMULAS OF ALEXANDER INVARIANTS

Let R be a minimal genus Seifert surface of a rationally homologically fibered knot K in S^3, and M_R be the sutured manifold for $R \cong \Sigma_{g,1}$. We fix a basis of $H_1(R;\mathbb{Q})$, which yields an isomorphism $H_1(R;\mathbb{Q}) \cong \mathbb{Q}^{2g}$. Then we can rewrite the definition $\Delta_K(t) = \det(S - tS^T)$ of the Alexander polynomial of K by using the invertibility (over $\mathbb{Q}$) of the Seifert matrix S, and obtain a factorization

$$\Delta_K(t) = \det(S)\det(I_{2g} - t\sigma(M_R)) \tag{3.1}$$

of $\Delta_K(t)$. Note that $\sigma(M_R) := S^{-1}S^T$ represents the composite of isomorphisms

$$\mathbb{Q}^{2g} \cong H_1(R;\mathbb{Q}) \underset{i_-}{\xrightarrow{\cong}} H_1(M_R;\mathbb{Q}) \underset{i_+^{-1}}{\xrightarrow{\cong}} H_1(R;\mathbb{Q}) \cong \mathbb{Q}^{2g}.$$

The matrix $\sigma(M_R)$ can be interpreted as a monodromy of M_R from a view point of the rational homology. Regarding the formula (3.1) as a basic case, we constructed in [12] its generalization under the framework of *higher-order Alexander invariants* due to Cochran [3], Harvey [17] and Friedl [7]. In this procedure, the Seifert matrix S, the monodromy $\sigma(M_R)$ and $\Delta_K(t)$ are generalized to a certain Reidemeister torsion $\tau_\rho^+(M_R)$, the Magnus matrix $r_\rho(M_R)$ and some higher-order (non-commutative) Reidemeister torsion $\tau_\rho(E(K))$ associated with a representation ρ of the fundamental group of M_R.

Here, we review higher-order Alexander invariants quickly. For a matrix A with entries in a group ring $\mathbb{Z}G$ (or its quotient field) for a group G, we denote by $\overline{A}$ the matrix obtained from A by applying the involution induced from ($x \mapsto x^{-1}$, $x \in G$) to each entry. For a module M, we write M^n for the module of column vectors with n entries. For a finite cell complex X, we denote by $\widetilde{X}$ its universal covering. We take a base point p of X and a lift $\widetilde{p}$ of p as a base point of $\widetilde{X}$. $\pi := \pi_1(X, p)$ acts on $\widetilde{X}$ from the *right* through its deck transformation group, so that the lift of a loop $l \in \pi$ starting from $\widetilde{p}$ reaches $\widetilde{p}l^{-1}$. Then the cellular chain complex $C_*(\widetilde{X})$ of $\widetilde{X}$ becomes a right $\mathbb{Z}\pi$-module. For each left $\mathbb{Z}\pi$-algebra $\mathcal{R}$, the twisted chain complex $C_*(X;\mathcal{R})$ is given by the tensor product of the right $\mathbb{Z}\pi$-module $C_*(\widetilde{X})$ and the left $\mathbb{Z}\pi$-module $\mathcal{R}$, so that $C_*(X;\mathcal{R})$ and $H_*(X;\mathcal{R})$ are right $\mathcal{R}$-modules.

In the definition of higher-order Alexander invariants, PTFA groups play important roles, where a group Γ is said to be *poly-torsion-free abelian* (*PTFA*) if it has a sequence

$$\Gamma = \Gamma_0 \rhd \Gamma_1 \rhd \cdots \rhd \Gamma_n = \{1\}$$

whose successive quotients Γ_i/Γ_{i+1} ($i \geq 0$) are all torsion-free abelian. An advantage of using PTFA groups is that the group ring $\mathbb{Z}\Gamma$ (or $\mathbb{Q}\Gamma$) of Γ is known to be an *Ore domain* so that it can be embed into the field (skew field in general)

$$\mathcal{K}_\Gamma := \mathbb{Z}\Gamma(\mathbb{Z}\Gamma - \{0\})^{-1} = \mathbb{Q}\Gamma(\mathbb{Q}\Gamma - \{0\})^{-1}$$

called the *right field of fractions*. A typical example of PTFA groups is $\mathbb{Z}^n$, where $\mathcal{K}_{\mathbb{Z}^n}$ is isomorphic to the field of rational functions with n variables.

For a rationally homologically fibered knot K, we take a homomorphism $\rho : G(K) := \pi_1(E(K)) \to \Gamma$ whose target Γ is PTFA. We suppose that ρ is non-trivial. We regard $\mathcal{K}_\Gamma$ as a local coefficient system on $E(K)$ through ρ.

Lemma 3.1 (Cochran [3, Lemma 3.9]). *For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group Γ, we have $H_*(E(K);\mathcal{K}_\Gamma) = 0$.*

By this lemma, we can define the Reidemeister torsion

$$\tau_\rho(E(K)) := \tau(C_*(E(K);\mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/\pm\rho(G(K))$$

for the acyclic complex $C_*(E(K);\mathcal{K}_\Gamma)$. We refer to Milnor [20] for generalities of torsions. By higher-order Alexander invariants for K, we here mean this torsion $\tau_\rho(E(K))$.

We now describe a factorization of $\tau_\rho(E(K))$ generalizing (3.1). Let $(M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$ be the rational homology cylinder obtained as the sutured manifold for a minimal genus Seifert surface R of K. We use the same notation $\rho : \pi_1(M_R) \to \Gamma$ for the composition $\pi_1(M_R) \to G(K) \xrightarrow{\rho} \Gamma$. Applying Cochran-Orr-Teichner [4, Proposition 2.10], we have the following:

Lemma 3.2. *$i_+, i_- : H_*(\Sigma_{g,1}, p; i_\pm^*\mathcal{K}_\Gamma) \to H_*(M_R, p; \mathcal{K}_\Gamma)$ are isomorphisms as right $\mathcal{K}_\Gamma$-vector spaces. Equivalently, $H_*(M_R, i_\pm(\Sigma_{g,1}); \mathcal{K}_\Gamma) = 0$.*

This lemma provides the following two kinds of invariants for M_R.

The Magnus matrix Let $X \subset \Sigma_{g,1}$ be the bouquet of $2g$ circles $\gamma_1, \ldots, \gamma_{2g}$ tied at p (see Figure 3). X is a deformation retract of $\Sigma_{g,1}$ relative to p. Therefore, for $\pm \in \{+, -\}$, we have

$$H_1(\Sigma_{g,1}, p; i_{\pm}^{*}\mathcal{K}_\Gamma) \cong H_1(X, p; i_{\pm}^{*}\mathcal{K}_\Gamma) = C_1(\widetilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_{\pm}^{*}\mathcal{K}_\Gamma \cong \mathcal{K}_\Gamma^{2g}$$

with a basis

$$\{\widetilde{\gamma}_1 \otimes 1, \ldots, \widetilde{\gamma}_{2g} \otimes 1\} \subset C_1(\widetilde{X}) \otimes_{\pi_1(\Sigma_{g,1})} i_{\pm}^{*}\mathcal{K}_\Gamma$$

as a right $\mathcal{K}_\Gamma$-vector space. Here we fix a lift $\widetilde{p}$ of p as a base point of $\widetilde{X}$, and denote by $\widetilde{\gamma}_i$ the lift of the oriented loop γ_i starting from $\widetilde{p}$.

Definition 3.3. For $M_R = (M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$, the *Magnus matrix*

$$r_\rho(M_R) \in GL(2g, \mathcal{K}_\Gamma)$$

of M_R is defined as the representation matrix of the right $\mathcal{K}_\Gamma$-isomorphism

$$\mathcal{K}_\Gamma^{2g} \cong H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \xrightarrow[i_-]{\cong} H_1(M_R, p; \mathcal{K}_\Gamma) \xrightarrow[i_+^{-1}]{\cong} H_1(\Sigma_{g,1}, p; \mathcal{K}_\Gamma) \cong \mathcal{K}_\Gamma^{2g},$$

where the first and the last isomorphisms use the bases mentioned above.

The matrix $r_\rho(M_R)$ can be interpreted as a monodromy of M_R from a view point of the twisted homology with coefficients in $\mathcal{K}_\Gamma$.

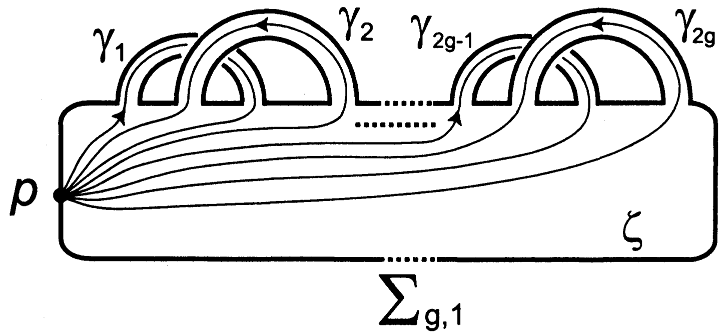

FIGURE 3. Cell decomposition of $\Sigma_{g,1}$

Γ-torsion Since the relative complex $C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)$ obtained from any cell decomposition of $(M_R, i_+(\Sigma_{g,1}))$ is acyclic by Lemma 3.2, we can define the following:

Definition 3.4. For $M_R = (M_R, i_+, i_-) \in \mathcal{C}_{g,1}^{\mathbb{Q}}$, the *Γ-torsion* $\tau_\rho^+(M_R)$ of M_R is defined by

$$\tau_\rho^+(M_R) := \tau(C_*(M_R, i_+(\Sigma_{g,1}); \mathcal{K}_\Gamma)) \in K_1(\mathcal{K}_\Gamma)/ \pm \rho(\pi_1(M_R)).$$

A method for computing $r_\rho(M_R)$ and $\tau_\rho^+(M_R)$ is given in [12, Section 4], which is based on Kirk-Livingston-Wang's method [18] for invariants of string links, and we now recall it briefly. An *admissible presentation* of $\pi_1(M_R)$ is defined to be the one of the form

$$\langle i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), z_1, \ldots, z_l, i_+(\gamma_1), \ldots, i_+(\gamma_{2g}) \mid r_1, \ldots, r_{2g+l} \rangle \tag{3.2}$$

for some integer l. That is, it is a finite presentation with deficiency $2g$ whose generating set contains $i_-(\gamma_1), \ldots, i_-(\gamma_{2g}), i_+(\gamma_1), \ldots, i_+(\gamma_{2g})$ and is ordered as above. Such a presentation always exists. For any admissible presentation, define $2g \times (2g+l)$, $l \times (2g+l)$ and $2g \times (2g+l)$ matrices A, B, C over $\mathbb{Z}\Gamma$ by

$$A = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial i_-(\gamma_i)}\right)}_{\substack{1\le i\le 2g \\ 1\le j\le 2g+l}}, \quad B = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial z_i}\right)}_{\substack{1\le i\le l \\ 1\le j\le 2g+l}}, \quad C = \overline{{}^{\rho}\left(\frac{\partial r_j}{\partial i_+(\gamma_i)}\right)}_{\substack{1\le i\le 2g \\ 1\le j\le 2g+l}}$$

Proposition 3.5 ([12, Propositions 4.5, 4.6]). *As matrices with entries in $\mathcal{K}_\Gamma$, we have:*

(1) *The square matrix* $\begin{pmatrix} A \\ B \end{pmatrix}$ *is invertible and* $\tau_\rho^+(M_R) = \begin{pmatrix} A \\ B \end{pmatrix}$; *and*

(2) $r_\rho(M_R) = -C \begin{pmatrix} A \\ B \end{pmatrix}^{-1} \begin{pmatrix} I_{2g} \\ 0_{(l,2g)} \end{pmatrix}$

Using the above invariants, the factorization formula for $\tau_\rho(E(K))$ is given as follows:

Theorem 3.6. *Let K be a rationally homologically fibered knot of genus g. For any non-trivial homomorphism $\rho : G(K) \to \Gamma$ to a PTFA group Γ, a loop μ representing the meridian of K satisfies $\rho(\mu) \neq 1 \in \Gamma \subset \mathcal{K}_\Gamma$ and we have a factorization*

$$\tau_\rho(E(K)) = \frac{\tau_\rho^+(M_R) \cdot (I_{2g} - \rho(\mu) r_\rho(M_R))}{1 - \rho(\mu)} \in K_1(\mathcal{K}_\Gamma)/\pm\rho(G(K)) \tag{3.3}$$

of the torsion $\tau_\rho(E(K))$.

To compare (3.3) with (3.1), recall Milnor's formula [20] that $\dfrac{\Delta_K(t)}{1-t}$ represents the Reidemeister torsion associated with the abelianization map $\rho_1 : G(K) \to \langle t \rangle \subset \mathbb{Q}(t)$. Taking ρ_1 as ρ, we recover the formula (3.1).

4. COMPUTATIONS

Although all the ingredients in the formula (3.3) are theoretically determined by information on fundamental groups, it is difficult to compute them explicitly because of the non-commutativity of $\mathcal{K}_\Gamma$ except in some special cases including the following.

Let K be a homologically fibered knot with a minimal genus Seifert surface R and let M_R be the sutured manifold for R. Consider the group extension

$$1 \longrightarrow G(K)'/G(K)'' \longrightarrow D_2(K) \longrightarrow G(K)/G(K)' = H_1(E(K)) \cong \mathbb{Z} \longrightarrow 1 \tag{4.1}$$

relating to the metabelian quotient $D_2(K) := G(K)/G(K)''$ of $G(K)$. We have

$$G(K)'/G(K)'' \cong H_1(R) \cong H_1(M_R)$$

since it coincides with the first homology of the infinite cyclic covering of $E(K)$, which can be seen as the product of infinitely many copies of M_R. In particular, we may regard $H_1(M_R)$ as a natural (namely, independent of choices of minimal genus Seifert surfaces) subgroup of $D_2(K)$. We take ρ to be the natural projection

$$\rho_2 : G(K) \longrightarrow D_2(K).$$

It is known that $D_2(K)$ is PTFA, so that $\mathcal{K}_{D_2(K)}$ is defined. Then, Proposition 3.5 shows that $\tau^+_{\rho_2}(M_R)$ and $r_{\rho_2}(M_R)$ can be computed by calculations on a commutative subfield $\mathcal{K}_{H_1(M_R)}$ of $\mathcal{K}_{D_2(K)}$.

Let us see an example of calculations of our invariants. Let K be the knot as the boundary of the Seifert surface R illustrated in Figure 4. This is the knot 0057 in Figure 7. We can easily compute that $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and the genus of R is 2. Hence K is a homologically fibered knot and R is of minimal genus. The graph G in the right hand side of Figure 4 is obtained from R by a deformation retract. Thus $\pi_1(M_R) \cong \pi_1(S^3 - \overset{\circ}{N}(G))$. Then $\pi_1(M_R)$ has a presentation:

$$\left\langle\ z_1, z_2, \ldots, z_{10}\ \middle|\ z_1z_5z_6^{-1},\ z_2z_3z_4z_1,\ z_3z_9^{-1}z_5^{-1},\ z_7z_4z_8^{-1},\ z_8z_{10}z_6,\ z_2z_5z_7^{-1}z_5^{-1},\ z_9z_4z_{10}^{-1}z_4^{-1}\ \right\rangle.$$

The first 5 relations come from the vertices of G and the last 2 relations come from the crossings of G. We can drop the last relation $z_9z_4z_{10}^{-1}z_4^{-1}$ because it is derived from the others.

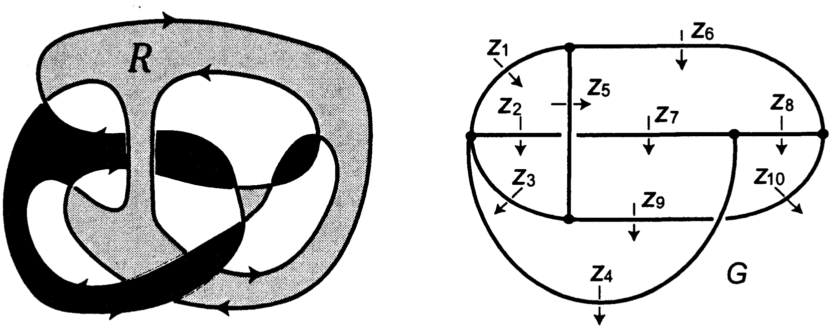

FIGURE 4

We take a spine of R as in Figure 5, by which we can fix an identification of $\Sigma_{g,1}$ and R. A direct computation shows that

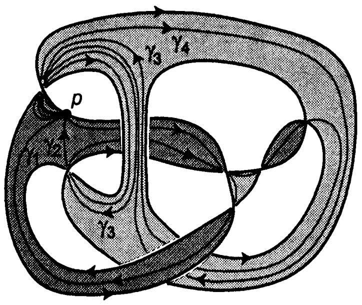

FIGURE 5

$$\begin{array}{llll} i_-(\gamma_1) = z_5z_1 & i_-(\gamma_2) = z_2^{-1} & i_-(\gamma_3) = z_5z_7^{-1}z_8^{-1}z_4^{-1} & i_-(\gamma_4) = z_4^{-1} \\ i_+(\gamma_1) = z_5 & i_+(\gamma_2) = z_6z_9 & i_+(\gamma_3) = z_6z_5^{-1}z_3z_5z_7^{-1}z_4^{-1}z_6^{-1} & i_+(\gamma_4) = z_6z_7z_6^{-1}. \end{array}$$

Here the darker color in R is the +-side. Then, we obtain an admissible presentation of $\pi_1(M_R)$:

Generators $i_-(\gamma_1), \dots, i_-(\gamma_4),\ z_1, \dots, z_{10},\ i_+(\gamma_1), \dots, i_+(\gamma_4)$

Relations $z_1 z_5 z_6^{-1},\ z_2 z_3 z_4 z_1,\ z_3 z_9^{-1} z_5^{-1},\ z_7 z_4 z_8^{-1},\ z_8 z_{10} z_6,\ z_2 z_5 z_7^{-1} z_5^{-1},$
$i_-(\gamma_1) z_1^{-1} z_5^{-1},\ i_-(\gamma_2) z_2,\ i_-(\gamma_3) z_4 z_8 z_7 z_5^{-1},\ i_-(\gamma_4) z_4,$
$i_+(\gamma_1) z_5^{-1},\ i_+(\gamma_2) z_9^{-1} z_6^{-1},\ i_+(\gamma_3) z_6 z_4 z_7 z_5^{-1} z_3^{-1} z_5 z_6^{-1},\ i_+(\gamma_4) z_6 z_7^{-1} z_6^{-1}$

If we have an admissible presentation, we can use the program shown in Section 5. However, we here demonstrate a calculation by hand.

By sliding the edges v_1 and v_2 of G as in Figure 6, we obtain a graph whose complement is clearly a genus 4 handlebody. This means that the complement of G (and hence M_R) is homeomorphic to a genus 4 handlebody. Let $D_1, \dots, D_4$ be the meridian disks of the handlebody as illustrated in the figure.

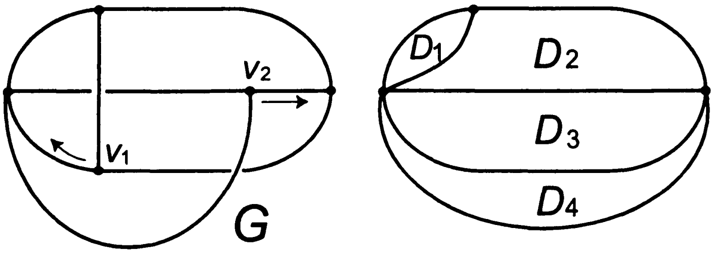

FIGURE 6

Then, $H_1(M_R)$ is the free abelian group generated by t_i $(i = 1, \dots, 4)$ where t_i corresponding to an oriented loop which intersects D_i transversely in one point from the above to the down side in Figure 6 and is disjoint from D_j $(i \neq j)$.

We have the natural homomorphism $\pi_1(M_R) \to H_1(M_R)$ which maps

$$\begin{array}{lllll}
z_1 \mapsto t_1^{-1} & z_2 \mapsto t_2 t_3^{-1} & z_3 \mapsto t_1 t_2^{-1} t_3 t_4^{-1} & z_4 \mapsto t_4 & z_5 \mapsto t_1 t_2^{-1} \\
z_6 \mapsto t_2^{-1} & z_7 \mapsto t_2 t_3^{-1} & z_8 \mapsto t_2 t_3^{-1} t_4 & z_9 \mapsto t_3 t_4^{-1} & z_{10} \mapsto t_3 t_4^{-1}
\end{array}$$

$$\begin{array}{llll}
i_-(\gamma_1) \mapsto t_2^{-1} & i_-(\gamma_2) \mapsto t_2^{-1} t_3 & i_-(\gamma_3) \mapsto t_1 t_2^{-3} t_3^2 t_4^{-2} & i_-(\gamma_4) \mapsto t_4^{-1} \\
i_+(\gamma_1) \mapsto t_1 t_2^{-1} & i_+(\gamma_2) \mapsto t_2^{-1} t_3 t_4^{-1} & i_+(\gamma_3) \mapsto t_1 t_2^{-2} t_3^2 t_4^{-2} & i_+(\gamma_4) \mapsto t_2 t_3^{-1}
\end{array}$$

Under the bases $\langle [\gamma_1], [\gamma_2], [\gamma_3], [\gamma_4] \rangle$ of $H_1(\Sigma_{2,1})$ and $\langle t_1, t_2, t_3, t_4 \rangle$ of $H_1(M_R)$, the induced maps i_-, i_+ are represented by

$$S_- = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & -1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -2 & -1 \end{pmatrix}, \quad S_+ = \begin{pmatrix} 1 & 0 & 1 & 0 \\ -1 & -1 & -2 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -1 & -2 & 0 \end{pmatrix}$$

respectively. Note that $\det(I - t\,(S_+^{-1} S_-)) = 1 - 2t + 3t^2 - 2t^3 + t^4$ is the Alexander polynomial of K.

Since M_R is homeomorphic to a handlebody, we have the following admissible presentation of $\pi_1(M_R)$ by setting $x_1 := z_1^{-1}, x_2 = z_6^{-1}, x_3 := (z_6 z_7)^{-1}$ and $x_4 := z_4$, which are mapped to t_1, t_2, t_3 and t_4 by the homomorphism $\pi_1(M_R) \to H_1(M_R)$.

Generators $i_-(\gamma_1), \dots, i_-(\gamma_4),\ x_1, x_2, x_3, x_4,\ i_+(\gamma_1), \dots, i_+(\gamma_4)$

Relations $i_-(\gamma_1) x_1 x_2 x_1^{-1},\ i_-(\gamma_2) x_1 x_3^{-1} x_2 x_1^{-1},\ i_-(\gamma_3) x_4 x_2 x_3^{-1} x_4 x_2 x_3^{-1} x_2 x_1^{-1},\ i_-(\gamma_4) x_4,$
$i_+(\gamma_1) x_2 x_1^{-1},\ i_+(\gamma_2) x_4 x_3^{-1} x_2,\ i_+(\gamma_3) x_2^{-1} x_4 x_2 x_3^{-1} x_2 x_1^{-1} x_4 x_3^{-1} x_2,\ i_+(\gamma_4) x_2^{-1} x_3$

We write $r_1, \dots, r_8$ for these relations in order. Note that $\mathcal{K}_{H_1(M_R)}$ is isomorphic to the field of rational functions with variables $x_1, \dots, x_4$. Then we have:

$$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{array}{c} \\ i_-(\gamma_1) \\ i_-(\gamma_2) \\ i_-(\gamma_3) \\ i_-(\gamma_4) \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ i_+(\gamma_1) \\ i_+(\gamma_2) \\ i_+(\gamma_3) \\ i_+(\gamma_4) \end{array} \!\!\begin{array}{c} \begin{array}{cccccccc} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array},$$

where $g_{ij} = \overline{\dfrac{\partial r_j}{\partial x_i}}$. Thus $\tau^+_{\rho_2}(M_R) = \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ g_{11} & g_{12} & g_{13} & g_{14} & g_{15} & g_{16} & g_{17} & g_{18} \\ g_{21} & g_{22} & g_{23} & g_{24} & g_{25} & g_{26} & g_{27} & g_{28} \\ g_{31} & g_{32} & g_{33} & g_{34} & g_{35} & g_{36} & g_{37} & g_{38} \\ g_{41} & g_{42} & g_{43} & g_{44} & g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix}$. As a torsion, it is equivalent to $\begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\ g_{25} & g_{26} & g_{27} & g_{28} \\ g_{35} & g_{36} & g_{37} & g_{38} \\ g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix}$, where

$$\begin{aligned} g_{15} &= -1, & g_{16} &= 0, & g_{18} &= 0, \\ g_{25} &= x_1^{-1}x_2, & g_{26} &= x_2, & g_{28} &= -x_3, \\ g_{35} &= 0, & g_{36} &= -x_2, & g_{38} &= x_3, \\ g_{45} &= 0, & g_{46} &= x_2x_3^{-1}x_4, & g_{48} &= 0, \end{aligned}$$

$$\begin{aligned} g_{17} &= -x_2x_3^{-1}x_4, \\ g_{27} &= x_2 + x_1^{-1}x_2^2x_3^{-1}x_4 + x_1^{-1}x_2^3x_3^{-2}x_4 - x_1^{-1}x_2^3x_3^{-2}x_4^2, \\ g_{37} &= -x_2 - x_1^{-1}x_2^2x_3^{-1}x_4, \\ g_{47} &= x_2x_3^{-1}x_4 + x_1^{-1}x_2^3x_3^{-2}x_4^2. \end{aligned}$$

Then we have:

$$\det(\tau^+_{\rho_2}(M_R)) = \det \begin{pmatrix} g_{15} & g_{16} & g_{17} & g_{18} \\ g_{25} & g_{26} & g_{27} & g_{28} \\ g_{35} & g_{36} & g_{37} & g_{38} \\ g_{45} & g_{46} & g_{47} & g_{48} \end{pmatrix} = -\frac{x_2^3x_4^2}{x_1x_3^2}(x_2 - x_3 - x_2x_4).$$

The Magnus matrix $r_{\rho_2}(M_R)$ can be computed by the formula in Proposition 3.5 (2). However we omit here.

Remark 4.1. If we change bases of $H_1(\Sigma_{2,1}) \cong H_1(M_R)$ by

$$x_1 = \gamma_2^{-2}\gamma_3, \quad x_2 = \gamma_1^{-1}\gamma_2^{-2}\gamma_3, \quad x_3 = \gamma_1^{-1}\gamma_2^{-2}\gamma_3\gamma_4^{-1}, \quad x_4 = \gamma_2^{-1}\gamma_4^{-1},$$

where γ_j denotes $i_+(\gamma_j)$, we have $\det(\tau_{\rho_2}^{+}(M_R)) = \dfrac{\gamma_3}{\gamma_1^2\gamma_2^5\gamma_4}(1+\gamma_2-\gamma_2\gamma_4)$. This expression is used in the program in Section 5.

5. MATHEMATICA PROGRAM

The following is a MATHEMATICA program which calculates the invariants discussed in the previous section.

```
h1Class = {};
h1Monodromy = {};
torsionMatrix = {};
magnusMatrix = {};

invariants[g_, z_, RELATIONS_] :=
  Module[{reindexedRel, h1Matrix, i, alex},
   GENUS = g;
   Ztotal = z;

   reindexedRel = Map[reindexing, RELATIONS, {2}];

   h1Matrix = -Map[Take[#, -2 GENUS] &, homologyComputation[reindexedRel]];
   h1Class =
    Join[Map[monomialExpression, h1Matrix],
     Table[ToExpression[ToString[SequenceForm["\[Gamma]", i]]], {i, 2 GENUS}]];
   Print["Homology classes of generators = ", h1Class // DisplayForm];

   h1Monodromy = Transpose[Take[h1Matrix, 2 GENUS]];
   Print["Homological monodromy = ", h1Monodromy // MatrixForm];

   alex = Transpose[makeAlexanderMatrix[reindexedRel]];
   torsionMatrix = Take[alex, 2 GENUS + Ztotal];
   Print["torsion matrix = ", torsionMatrix // MatrixForm];
   Print["det(torsion) = ", Expand[Det[torsionMatrix]]];

   magnusMatrix = Simplify[Transpose[
     Take[Transpose[-Drop[alex, 2 GENUS + Ztotal].Inverse[
           torsionMatrix]], 2 GENUS]]];
   Print["Magnus matrix = ", magnusMatrix // MatrixForm]
   ];

reindexing[num_] :=
  Module[{numString, sg},
    If[NumberQ[num], num + 2 GENUS*Sign[num],
     numString = ToString[num];
     sg = If[StringTake[numString, 1] == "-", 1, 0];
     If[StringTake[numString, {1 + sg}] == "m",
      ((-1)^sg)*ToExpression[StringDrop[numString, 1 + sg]],
      ((-1)^sg)*(ToExpression[StringDrop[numString, 1 + sg]] + 2 GENUS + Ztotal)]]
```

```
 ];

homologyComputation[rel_] :=
  Module[{i, j},
   RowReduce[Table[Count[rel[[i]], j] - Count[rel[[i]], -j],
   {i, 1, 2 GENUS + Ztotal}, {j, 1, 4 GENUS + Ztotal}]]];

monomialExpression[list_] :=
  Module[{i, prod = 1},
   For[i = 1, i <= 2 GENUS, i++,
    prod = prod*(ToExpression[ToString[SequenceForm["\[Gamma]", i]]]^list[[i]])];
   prod];

makeAlexanderMatrix[rel_] :=
  Module[{i, j},
   Table[foxDer[rel[[i]], j], {i, 1, Length[rel]}, {j, 1, 4 GENUS + Ztotal}]];

foxDer[word_, var_] :=
  Module[{entry = 0, i},
   For[i = 1, i <= Length[word], i++,
    Which[word[[i]] == var,
     entry = entry + (makeMonomial[Take[word, i - 1]]^(-1)),
     word[[i]] == -var,
     entry = entry - (makeMonomial[Take[word, i]]^(-1))]];
   entry];

makeMonomial[list_] :=
  Module[{prod = 1},
   For[i = 1, i <= Length[list], i++,
    prod = prod*(h1Class[[Abs[list[[i]]]]]^Sign[list[[i]]])];
   prod];
```

A computation by this program goes as follows. Let $(M, i_+, i_-) \in \mathcal{C}_{g,1}$ with an admissible presentation

$$\langle i_-(\gamma_1), \dots, i_-(\gamma_{2g}), z_1, \dots, z_l, i_+(\gamma_1), \dots, i_+(\gamma_{2g}) \mid r_1, \dots, r_{2g+l} \rangle$$

of $\pi_1(M)$. The main function in the program is `invariants` having three slots as the input. These slots correspond to the genus g, the number l of z-generators and the list of relations. For each word in the relations, we make a list by replacing $i_-(\gamma_j)^{\pm 1}$, $z_j^{\pm 1}$ and $i_+(\gamma_j)^{\pm 1}$ by ±`mj`, ±`j` and ±`pj`. By lining up them, we obtain the list of relations.

For example, the knot 0815 in Figure 7 has a minimal genus Seifert surface giving a sutured manifold whose fundamental group has the following admissible presentation:

$$\begin{array}{cc}
\text{Generators} & i_-(\gamma_1), \dots, i_-(\gamma_4),\ z_1, \dots, z_{11},\ i_+(\gamma_1), \dots, i_+(\gamma_4) \\
\text{Relations} & z_1 z_9 z_6,\ z_1 z_2^{-1} z_4^{-1},\ z_4 z_{11}^{-1} z_5,\ z_{10}^{-1} z_5^{-1} z_6 z_7 z_8,\ z_8^{-1} z_6^{-1} z_9 z_6, \\
 & z_7^{-1} z_6^{-1} z_3 z_6,\ z_4 z_3^{-1} z_4^{-1} z_{10}, \\
 & i_-(\gamma_1) z_4 z_3^{-1} z_4^{-1},\ i_-(\gamma_2) z_4 z_{11},\ i_-(\gamma_3) z_9,\ i_-(\gamma_4) z_2^{-1} z_9^{-1}, \\
 & i_+(\gamma_1) z_2^{-1} z_3^{-1} z_4^{-1},\ i_+(\gamma_2) z_{11} z_1,\ i_+(\gamma_3) z_9 z_3^{-1} z_1,\ i_+(\gamma_4) z_9 z_2^{-1} z_9^{-1}
\end{array}$$

Then, the input is:

```
invariants[2, 11, {{1, 9, 6}, {1, -2,-4}, {4,-11, 5},
  {-10, -5, 6, 7, 8}, {-8, -6, 9, 6}, {-7, -6, 3, 6},
```

```
{4, -3, -4, 10}, {m1, 4, -3, -4}, {m2, 4, 11},
{m3, 9}, {m4, -2, -9}, {p1, -2, -3, -4}, {p2, 11, 1},
{p3, 9, -3, 1}, {p4, 9, -2, -9}}]
```

Then the function returns homology classes of generators in terms of γj $:= i_+(\gamma_j) \in H_1(M_R)$, the homological monodromy matrix $\sigma(M_R)$, the torsion matrix $\tau^+_{\rho_2}(M_R)$ and the Magnus matrix $r_{\rho_2}(M_R)$. These data can be referred as the variables `h1Class`, `h1Monodromy`, `torsionMatrix` and `magnusMatrix`.

Using this program, we can easily check the calculations presented in [13] for 13 non-fibered homologically fibered knots with 12-crossings (Figure 7).

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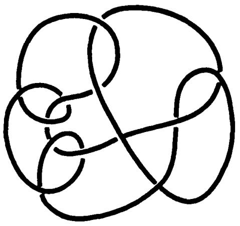
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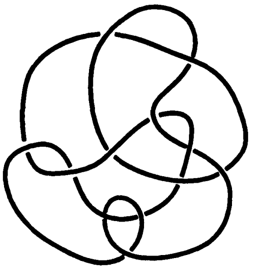
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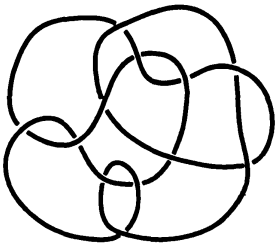
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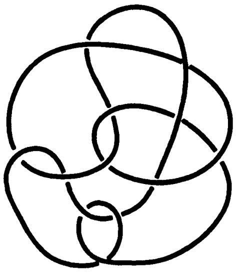
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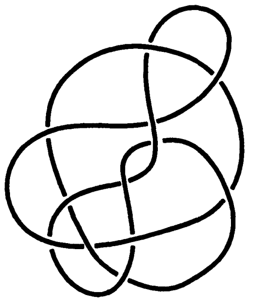
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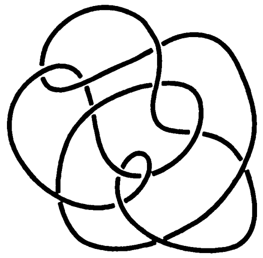
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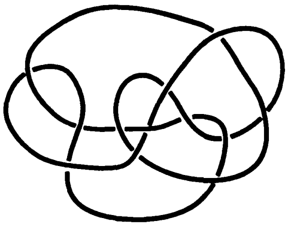
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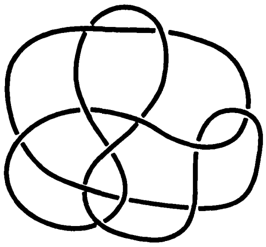
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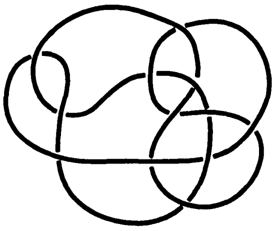
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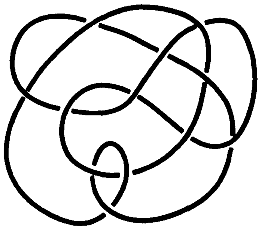
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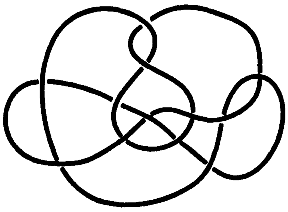
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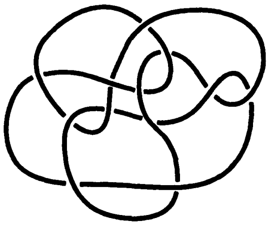
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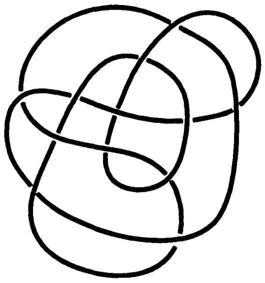
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FIGURE 7. Non-fibered homologically fibered knots with 12-crossings