数理解析研究所講究録
第 1717 巻 2010 年 8-18

Applications of Hankel determinant for p-valently starlike and convex functions of order α

Toshio Hayami and Shigeyoshi Owa

Abstract

For p-valently starlike and convex functions $f(z)$ in the open unit disk $\mathbb{U}$, the upper bounds of the functional $|a_{p+1}a_{p+3}-\mu a_{p+2}^{2}|$, defined by using the second Hankel determinant $H_{2}(n)$ due to J. W. Noonan and D. K. Thomas (Trans. Amer. Math. Soc. **223**(2). (1976), 337-346), are discussed.

1 Introduction

Let $\mathcal{A}_{p}$ denote the class of functions $f(z)$ of the form

$$f(z)=z^{p}+\sum_{n=p+1}^{\infty}a_{n}z^{n}\quad(p\in\mathbb{N}=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$.

Furthermore, let $\mathcal{P}(p,\alpha)$ denote the class of functions $p(z)$ of the form

$$p(z)=p+\sum_{k=1}^{\infty}c_{k}z^{k} \tag{1.2}$$

which are analytic in $\mathbb{U}$ and satisfy

$$\operatorname{Re}p(z)>\alpha\qquad(z\in\mathbb{U})$$

for some α $(0\leqq\alpha<p)$. In particular, we say that $p(z)\in\mathcal{P}\equiv\mathcal{P}(1,0)$ is the Carathéodory function (cf. [1]).

If $f(z)\in\mathcal{A}_{p}$ satisfies the following condition

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha\quad(z\in\mathbb{U}) \tag{1.3}$$

for some α $(0\leqq\alpha<p)$, then $f(z)$ is said to be p-valently starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}_{p}^{*}(\alpha)$ the subclass of $\mathcal{A}_{p}$ consisting of functions $f(z)$ which are p-valently starlike of order α in $\mathbb{U}$.

2000 Mathematics Subject Classification: Primary 30C45.
Keywords and Phrases: Hankel determinant, analytic function, p-valently starlike function, p-valently convex function.

Similarly, we say that $f(z)$ belongs to the class $\mathcal{K}_p(\alpha)$ of p-valently convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}_p$ satisfies the following inequality

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{U}) \tag{1.4}$$

for some α $(0 \leqq \alpha < p)$.

As usual, in the present investigation, we write

$$\mathcal{S}_p^* = \mathcal{S}_p^*(0), \quad \mathcal{K}_p = \mathcal{K}_p(0), \quad \mathcal{S}^*(\alpha) = \mathcal{S}_1^*(\alpha) \quad \text{and} \quad \mathcal{K}(\alpha) = \mathcal{K}_1(\alpha).$$

Remark 1.1 For a function $f(z) \in \mathcal{A}_p$, it follows that

$$f(z) \in \mathcal{K}_p(\alpha) \quad \text{if and only if} \quad \frac{zf'(z)}{p} \in \mathcal{S}_p^*(\alpha)$$

and

$$f(z) \in \mathcal{S}_p^*(\alpha) \quad \text{if and only if} \quad \int_0^z \frac{pf(\zeta)}{\zeta} d\zeta \in \mathcal{K}_p(\alpha).$$

Example 1.2

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}} \in \mathcal{S}_p^*(\alpha)$$

and

$$f(z) = z^p {}_2F_1\left(2(p-\alpha), p; p+1; z\right) \in \mathcal{K}_p(\alpha)$$

where ${}_2F_1(a, b; c; z)$ represents the hypergeometric function.

In [5], Noonan and Thomas stated the q–th Hankel determinant as

$$H_q(n) = \det \begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{pmatrix} \qquad (n, q \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

This determinant is discussed by several authors. For example, we know that the Fekete and Szegö functional $|a_3 - a_2^2| = |H_2(1)|$ and Fekete and Szegö [2] have considered the further generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Moreover, we also know that the functional $|a_2a_4 - a_3^2|$ is equivalent to $|H_2(2)|$.

Janteng, Halim and Darus [4] have shown the following theorems.

Theorem 1.3 *Let $f(z) \in \mathcal{S}^*$. Then*

$$|a_2a_4 - a_3^2| \leqq 1.$$

Equality is attained for functions

$$f(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots$$

and

$$f(z) = \frac{z}{1-z^2} = z + z^3 + z^5 + z^7 + \cdots .$$

Theorem 1.4 *Let $f(z) \in \mathcal{K}$. Then*

$$|a_2 a_4 - a_3^2| \leqq \frac{1}{8}.$$

The present paper is motivated by these results and the purpose of this investigation is to generalize Theorem 1.3 and Theorem 1.4 by finding the upper bounds of the generalized functional $|a_{p+1}a_{p+3} - \mu a_{p+2}^2|$, defined by the second Hankel determinant, for functions $f(z)$ in the class $\mathcal{S}_p^*(\alpha)$ and $\mathcal{K}_p(\alpha)$, respectively.

2 Preliminary results

To establish our results, we need some lemmas. The following lemmas can be found in [3].

Lemma 2.1 *If a function $p(z) \in \mathcal{P}(p, \alpha)$, then*

$$|c_k| \leqq 2(p-\alpha) \quad (k = 1, 2, 3, \cdots). \tag{2.1}$$

The result is sharp for

$$p(z) = \frac{p + (p-2\alpha)z}{1-z} = p + \sum_{k=1}^{\infty} 2(p-\alpha)z^k.$$

Lemma 2.2 *If a function $p(z) \in \mathcal{P}(p, \alpha)$, then*

$$\left\{ \begin{aligned} &2(p-\alpha)c_2 = c_1^2 + \{4(p-\alpha)^2 - c_1^2\}\zeta \\ &4(p-\alpha)^2 c_3 = c_1^3 + 2\{4(p-\alpha)^2 - c_1^2\}c_1\zeta - \{4(p-\alpha)^2 - c_1^2\}c_1\zeta^2 \\ &\qquad\qquad\qquad +2(p-\alpha)\{4(p-\alpha)^2 - c_1^2\}(1-|\zeta|^2)\eta \end{aligned} \right. \tag{2.2}$$

for some complex numbers ζ and η ($|\zeta| \leqq 1, |\eta| \leqq 1$).

We also need the next lemma concerning with the upper bounds of the coefficients $|a_n|$ for p-valently starlike or convex functions of order α, respectively.

Lemma 2.3 *If a function $f(z) \in \mathcal{S}_p^*(\alpha)$, then*

$$|a_n| \leqq \frac{\prod_{j=1}^{n-p}\Big(2(p-\alpha)+j-1\Big)}{(n-p)!} \qquad (n \geqq p+1) \tag{2.3}$$

with equality for

$$f(z) = \frac{z^p}{(1-z)^{2(p-\alpha)}}.$$

Similarly, if a function $f(z) \in \mathcal{K}_p(\alpha)$, then

$$|a_n| \leqq \frac{p\prod_{j=1}^{n-p}\Big(2(p-\alpha)+j-1\Big)}{n(n-p)!} \qquad (n \geqq p+1) \tag{2.4}$$

with equality for

$$f(z) = z^p{}_2F_1(2(p-\alpha), p; p+1; z).$$

3 Main results

Next, we discuss the upper bound of the functional $|a_{p+1}a_{p+3} - \mu a_{p+2}^2|$ for p-valently starlike and convex functions of order α. For the sake of convenience, we define the following values.

$$s_1(p,\alpha) = (p-\alpha)^2(2(p-\alpha)+1)\left\{\frac{4}{3}(p-\alpha+1) - (2(p-\alpha)+1)\mu\right\},$$

$$s_2(p,\alpha) = \frac{(p-\alpha)^2\left\{6(p-\alpha+1)^2 - 8(p-\alpha+1)(2(p-\alpha)+1)\mu + 3(2(p-\alpha)+1)^2\mu^2\right\}}{3(2(p-\alpha)^2-1)\mu - 4((p-\alpha)^2-1)},$$

$$s_3(p,\alpha) = \frac{(p-\alpha)^2\left\{3(p-\alpha+1) - 2(2(p-\alpha)+1)\mu\right\}}{2(p-\alpha+2) - 3(p-\alpha+1)\mu},$$

$$s_4(p,\alpha) = (p-\alpha)^2\mu$$

and

$$s_5(p,\alpha) = (p-\alpha)^2(2(p-\alpha)+1)\left\{(2(p-\alpha)+1)\mu - \frac{4}{3}(p-\alpha+1)\right\}.$$

Furthermore,

$$k_1(p,\alpha) = \frac{p^2(p-\alpha)^2(2(p-\alpha)+1)}{(p+1)(p+3)(p+2)^2}\left\{\frac{4}{3}(p-\alpha+1)(p+2)^2 - (2(p-\alpha)+1)(p+1)(p+3)\mu\right\},$$

$$k_2(p,\alpha) = \frac{K}{(p+1)(p+3)(p+2)^2\left\{3(2(p-\alpha)^2-1)(p+1)(p+3)\mu - 4((p-\alpha)^2-1)(p+2)^2\right\}},$$

$$k_3(p,\alpha) = \frac{p^2(p-\alpha)^2\left\{3(p-\alpha+1)(p+2)^2 - 2(2(p-\alpha)+1)(p+1)(p+3)\mu\right\}}{(p+1)(p+3)\left\{2(p-\alpha+2)(p+2)^2 - 3(p-\alpha+1)(p+1)(p+3)\mu\right\}},$$

$$k_4(p,\alpha) = \left\{\frac{p(p-\alpha)}{p+2}\right\}^2 \mu$$

and

$$k_5(p,\alpha) = \frac{p^2(p-\alpha)^2(2(p-\alpha)+1)}{(p+1)(p+3)(p+2)^2}\left\{(2(p-\alpha)+1)(p+1)(p+3)\mu - \frac{4}{3}(p-\alpha+1)(p+2)^2\right\},$$

where

$$\begin{aligned} K = {} & p^2(p-\alpha)^2\{6(p-\alpha+1)^2(p+2)^4 \\ & -8(p-\alpha+1)(2(p-\alpha)+1)(p+1)(p+3)(p+2)^2\mu + 3(2(p-\alpha)+1)^2(p+1)^2(p+3)^2\mu^2\}. \end{aligned}$$

The next result is separated into three parts by the region of α below.

Theorem 3.1(1) *If a function $f(z) \in S_p^*(\alpha)$ for $0 \leqq \alpha \leqq p-1$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq \begin{cases} s_1(p,\alpha) & \left(\mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)}\right) \\[2ex] s_2(p,\alpha) & \left(\dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)} \leqq \mu \leqq \dfrac{4(p-\alpha+1)}{3(2(p-\alpha)+1)}\right) \\[2ex] s_3(p,\alpha) & \left(\dfrac{4(p-\alpha+1)}{3(2(p-\alpha)+1)} \leqq \mu \leqq \dfrac{4(p-\alpha)+5}{3(2(p-\alpha)+1)}\right) \\[2ex] s_5(p,\alpha) & \left(\mu \geqq \dfrac{4(p-\alpha)+5}{3(2(p-\alpha)+1)}\right). \end{cases}$$

Theorem 3.1(2) *If a function $f(z) \in S_p^*(\alpha)$ for $p-1 \leqq \alpha \leqq p-\frac{1}{2}$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq \begin{cases} s_1(p,\alpha) & \left(\mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)}\right) \\[2ex] s_2(p,\alpha) & \left(\dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)} \leqq \mu \leqq \dfrac{4(p-\alpha+1)}{3(2(p-\alpha)+1)}\right) \\[2ex] s_3(p,\alpha) & \left(\dfrac{4(p-\alpha+1)}{3(2(p-\alpha)+1)} \leqq \mu \leqq 1\right) \\[2ex] s_4(p,\alpha) & \left(1 \leqq \mu \leqq \dfrac{2(p-\alpha)+1}{3(p-\alpha)}\right) \\[2ex] s_5(p,\alpha) & \left(\mu \geqq \dfrac{2(p-\alpha)+1}{3(p-\alpha)}\right). \end{cases}$$

Theorem 3.1(3) *If a function $f(z) \in \mathcal{S}_p^*(\alpha)$ for $p - \dfrac{1}{2} \leqq \alpha < p$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq \begin{cases} s_1(p,\alpha) & \begin{pmatrix} p - \frac{1}{2} \leqq \alpha < p - \frac{1}{4};\ \mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)} \\ p - \frac{1}{4} \leqq \alpha < p;\ \mu \leqq \frac{2}{3} \end{pmatrix} \\ s_2(p,\alpha) & \begin{pmatrix} p - \frac{1}{2} \leqq \alpha < p - \frac{1}{4};\ \dfrac{(p-\alpha+1)(4(p-\alpha)-1)}{3(p-\alpha)(2(p-\alpha)+1)} \leqq \mu \leqq 1 \\ p - \frac{1}{4} \leqq \alpha < p;\ \frac{2}{3} \leqq \mu \leqq 1 \end{pmatrix} \\ s_4(p,\alpha) & \left(1 \leqq \mu \leqq \dfrac{2(p-\alpha)+1}{3(p-\alpha)}\right) \\ s_5(p,\alpha) & \left(\mu \geqq \dfrac{2(p-\alpha)+1}{3(p-\alpha)}\right). \end{cases}$$

For each α and μ, we see that the following equalities

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| = s_1(p,\alpha) \quad \text{and} \quad |a_{p+1}a_{p+3} - \mu a_{p+2}^2| = s_5(p,\alpha)$$

are attained for function $f(z) = \dfrac{z^p}{(1-z)^{2(p-\alpha)}}$. Similarly, the equality

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| = s_4(p,\alpha)$$

is attained for function $f(z) = \dfrac{z^p}{(1-z^2)^{p-\alpha}}$.

Taking $\alpha = 0$ or $p = 1$ in Theorem 3.1(1)–(3), we derive the following corollaries.

Corollary 3.2 *If a function $f(z) \in \mathcal{S}_p^*$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq$$

$$\begin{cases} p^2(2p+1)\left\{\frac{4}{3}(p+1) - (2p+1)\mu\right\} & \left(\mu \leqq \dfrac{(p+1)(4p-1)}{3p(2p+1)}\right) \\ \dfrac{p^2\{6(p+1)^2 - 8(p+1)(2p+1)\mu + 3(2p+1)^2\mu^2\}}{3(2p^2-1)\mu - 4(p^2-1)} & \left(\dfrac{(p+1)(4p-1)}{3p(2p+1)} \leqq \mu \leqq \dfrac{4(p+1)}{3(2p+1)}\right) \\ \dfrac{p^2\{3(p+1) - 2(2p+1)\mu\}}{2(p+2) - 3(p+1)\mu} & \left(\dfrac{4(p+1)}{3(2p+1)} \leqq \mu \leqq \dfrac{4p+5}{3(2p+1)}\right) \\ p^2(2p+1)\left\{(2p+1)\mu - \frac{4}{3}(p+1)\right\} & \left(\mu \geqq \dfrac{4p+5}{3(2p+1)}\right). \end{cases}$$

Corollary 3.3(1)–(2) *If a function $f(z) \in \mathcal{S}^*(\alpha)$ with $0 \leqq \alpha \leqq \dfrac{1}{2}$, then*

$$|a_2a_4 - \mu a_3^2| \leqq \begin{cases} s_1(1,\alpha) & \left(\mu \leqq \dfrac{(2-\alpha)(3-4\alpha)}{3(1-\alpha)(3-2\alpha)}\right) \\ s_2(1,\alpha) & \left(\dfrac{(2-\alpha)(3-4\alpha)}{3(1-\alpha)(3-2\alpha)} \leqq \mu \leqq \dfrac{4(2-\alpha)}{3(3-2\alpha)}\right) \\ s_3(1,\alpha) & \left(\dfrac{4(2-\alpha)}{3(3-2\alpha)} \leqq \mu \leqq 1\right) \\ s_4(1,\alpha) & \left(1 \leqq \mu \leqq \dfrac{3-2\alpha}{3(1-\alpha)}\right) \\ s_5(1,\alpha) & \left(\mu \geqq \dfrac{3-2\alpha}{3(1-\alpha)}\right). \end{cases}$$

Corollary 3.3(3) *If a function $f(z) \in \mathcal{S}^*(\alpha)$ with $\dfrac{1}{2} \leqq \alpha < 1$, then*

$$|a_2a_4 - \mu a_3^2| \leqq \begin{cases} s_1(1,\alpha) & \left(\dfrac{1}{2} \leqq \alpha < \dfrac{3}{4};\ \mu \leqq \dfrac{(2-\alpha)(3-4\alpha)}{3(1-\alpha)(3-2\alpha)},\quad \dfrac{3}{4} \leqq \alpha < 1;\ \mu \leqq \dfrac{2}{3}\right) \\ s_2(1,\alpha) & \left(\dfrac{1}{2} \leqq \alpha < \dfrac{3}{4};\ \dfrac{(2-\alpha)(3-4\alpha)}{3(1-\alpha)(3-2\alpha)} \leqq \mu \leqq 1,\quad \dfrac{3}{4} \leqq \alpha < 1;\ \dfrac{2}{3} \leqq \mu \leqq 1\right) \\ s_4(1,\alpha) & \left(1 \leqq \mu \leqq \dfrac{3-2\alpha}{3(1-\alpha)}\right) \\ s_5(1,\alpha) & \left(\mu \geqq \dfrac{3-2\alpha}{3(1-\alpha)}\right). \end{cases}$$

Furthermore, in consideration of Corollary 3.2 and Corollary 3.3, we immediately obtain the following result including Theorem 1.3 by Janteng, Halim and Darus [4].

Corollary 3.4 *If a function $f(z) \in \mathcal{S}^*$, then*

$$|a_2a_4 - \mu a_3^2| \leqq \begin{cases} 8 - 9\mu & \left(\mu \leqq \dfrac{2}{3}\right) \\ \dfrac{8 - 16\mu + 9\mu^2}{\mu} & \left(\dfrac{2}{3} \leqq \mu \leqq \dfrac{8}{9}\right) \\ 1 & \left(\dfrac{8}{9} \leqq \mu \leqq 1\right) \\ 9\mu - 8 & (\mu \geqq 1). \end{cases}$$

By virtue of Remark 1.1, we have

Theorem 3.5(1) *If a function $f(z) \in \mathcal{K}_p(\alpha)$ for $0 \leqq \alpha \leqq p-1$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq$$

$$\begin{cases} k_1(p,\alpha) & \left(\mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)}\right) \\ k_2(p,\alpha) & \left(\dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)} \leqq \mu \leqq \dfrac{4(p-\alpha+1)(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)}\right) \\ k_3(p,\alpha) & \left(\dfrac{4(p-\alpha+1)(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)} \leqq \mu \leqq \dfrac{\{4(p-\alpha)+5\}(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)}\right) \\ k_5(p,\alpha) & \left(\mu \geqq \dfrac{\{4(p-\alpha)+5\}(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)}\right). \end{cases}$$

Theorem 3.5(2) *If a function $f(z) \in \mathcal{K}_p(\alpha)$ for $p-1 \leqq \alpha \leqq p-\dfrac{1}{2}$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq$$

$$\begin{cases} k_1(p,\alpha) & \left(\mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)}\right) \\ k_2(p,\alpha) & \left(\dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)} \leqq \mu \leqq \dfrac{4(p-\alpha+1)(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)}\right) \\ k_3(p,\alpha) & \left(\dfrac{4(p-\alpha+1)(p+2)^2}{3(2(p-\alpha)+1)(p+1)(p+3)} \leqq \mu \leqq \dfrac{(p+2)^2}{(p+1)(p+3)}\right) \\ k_4(p,\alpha) & \left(\dfrac{(p+2)^2}{(p+1)(p+3)} \leqq \mu \leqq \dfrac{(2(p-\alpha)+1)(p+2)^2}{3(p-\alpha)(p+1)(p+3)}\right) \\ k_5(p,\alpha) & \left(\mu \geqq \dfrac{(2(p-\alpha)+1)(p+2)^2}{3(p-\alpha)(p+1)(p+3)}\right). \end{cases}$$

Theorem 3.5(3) *If a function $f(z) \in \mathcal{K}_p(\alpha)$ for $p - \frac{1}{2} \leqq \alpha < p$, then*

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| \leqq$$

$$\begin{cases} k_1(p,\alpha) & \left(\begin{array}{l} p - \frac{1}{2} \leqq \alpha < \frac{1}{4};\ \mu \leqq \dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)} \\ p - \frac{1}{4} \leqq \alpha < p;\ \mu \leqq \dfrac{2(p+2)^2}{3(p+1)(p+3)} \end{array} \right) \\ k_2(p,\alpha) & \left(\begin{array}{l} p - \frac{1}{2} \leqq \alpha < \frac{1}{4};\ \dfrac{(p-\alpha+1)(4(p-\alpha)-1)(p+2)^2}{3(p-\alpha)(2(p-\alpha)+1)(p+1)(p+3)} \leqq \mu \leqq \dfrac{(p+2)^2}{(p+1)(p+3)} \\ p - \frac{1}{4} \leqq \alpha < p;\ \dfrac{2(p+2)^2}{3(p+1)(p+3)} \leqq \mu \leqq \dfrac{(p+2)^2}{(p+1)(p+3)} \end{array} \right) \\ k_4(p,\alpha) & \left(\dfrac{(p+2)^2}{(p+1)(p+3)} \leqq \mu \leqq \dfrac{(2(p-\alpha)+1)(p+2)^2}{3(p-\alpha)(p+1)(p+3)} \right) \\ k_5(p,\alpha) & \left(\mu \geqq \dfrac{(2(p-\alpha)+1)(p+2)^2}{3(p-\alpha)(p+1)(p+3)} \right). \end{cases}$$

For each α and μ, we see that the following equalities

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| = k_1(p,\alpha) \quad \text{and} \quad |a_{p+1}a_{p+3} - \mu a_{p+2}^2| = k_5(p,\alpha)$$

are attained for function $f(z) = z^p {}_2F_1(2(p-\alpha), p; p+1; z)$. Similarly, the equality

$$|a_{p+1}a_{p+3} - \mu a_{p+2}^2| = k_4(p,\alpha)$$

is attained for function $f(z) = z^p {}_2F_1\left(\frac{p}{2}, p-\alpha; 1+\frac{p}{2}; z^2\right)$.

Setting $\alpha = 0$ or $p = 1$ in Theorem 3.5(1)–(3), the following corollaries are obtained.

Corollary 3.6 *If a function $f(z) \in \mathcal{K}_p$, then*

$$|a_{p+1}a_{p+3}-\mu a_{p+2}^2| \leqq$$

$$\begin{cases} \dfrac{p^4(2p+1)}{(p+3)(p+2)^2}\left\{\dfrac{4}{3}(p+2)^2-(2p+1)(p+3)\mu\right\} & \left(\mu \leqq \dfrac{(4p-1)(p+2)^2}{3p(2p+1)(p+3)}\right) \\ \dfrac{p^4\left\{6(p+1)^2(p+2)^4-8(2p+1)(p+1)(p+3)(p+2)^2\mu+3(2p+1)^2(p+1)(p+3)^2\mu^2\right\}}{(p+3)(p+2)^2\left\{3(2p^2-1)(p+1)(p+3)\mu-4(p^2-1)(p+2)^2\right\}} & \\ & \left(\dfrac{(4p-1)(p+2)^2}{3p(2p+1)(p+3)} \leqq \mu \leqq \dfrac{4(p+2)^2}{3(2p+1)(p+3)}\right) \\ \dfrac{p^4\left\{3(p+2)^2-2(2p+1)(p+3)\mu\right\}}{(p+3)\left\{2(p+2)^3-3(p+1)^2(p+3)\mu\right\}} & \left(\dfrac{4(p+2)^2}{3(2p+1)(p+3)} \leqq \mu \leqq \dfrac{(4p+5)(p+2)^2}{3(2p+1)(p+1)(p+3)}\right) \\ \dfrac{p^4(2p+1)}{(p+3)(p+2)^2}\left\{(2p+1)(p+3)\mu-\dfrac{4}{3}(p+2)^2\right\} & \left(\mu \geqq \dfrac{(4p+5)(p+2)^2}{3(2p+1)(p+1)(p+3)}\right). \end{cases}$$

Corollary 3.7(1)–(2) *If a function $f(z) \in \mathcal{K}(\alpha)$ with $0 \leqq \alpha \leqq \frac{1}{2}$, then*

$$|a_2a_4-\mu a_3^2| \leqq \begin{cases} k_1(1,\alpha) & \left(\mu \leqq \dfrac{3(2-\alpha)(3-4\alpha)}{8(1-\alpha)(3-2\alpha)}\right) \\ k_2(1,\alpha) & \left(\dfrac{3(2-\alpha)(3-4\alpha)}{8(1-\alpha)(3-2\alpha)} \leqq \mu \leqq \dfrac{3(2-\alpha)}{2(3-2\alpha)}\right) \\ k_3(1,\alpha) & \left(\dfrac{3(2-\alpha)}{2(3-2\alpha)} \leqq \mu \leqq \dfrac{9}{8}\right) \\ k_4(1,\alpha) & \left(\dfrac{9}{8} \leqq \mu \leqq \dfrac{3(3-2\alpha)}{8(1-\alpha)}\right) \\ k_5(1,\alpha) & \left(\mu \geqq \dfrac{3(3-2\alpha)}{8(1-\alpha)}\right). \end{cases}$$

Corollary 3.7(3) *$f(z) \in \mathcal{K}(\alpha)$ with $\frac{1}{2} \leqq \alpha < 1$, then*

$$|a_2a_4-\mu a_3^2| \leqq \begin{cases} k_1(1,\alpha) & \left(\dfrac{1}{2} \leqq \alpha < \dfrac{3}{4};\ \mu \leqq \dfrac{3(2-\alpha)(3-4\alpha)}{8(1-\alpha)(3-2\alpha)},\quad \dfrac{3}{4} \leqq \alpha < 1;\ \mu \leqq \dfrac{3}{4}\right) \\ k_2(1,\alpha) & \left(\dfrac{1}{2} \leqq \alpha < \dfrac{3}{4};\ \dfrac{3(2-\alpha)(3-4\alpha)}{8(1-\alpha)(3-2\alpha)} \leqq \mu \leqq \dfrac{9}{8},\quad \dfrac{3}{4} \leqq \alpha < 1;\ \dfrac{3}{4} \leqq \mu \leqq \dfrac{9}{8}\right) \\ k_4(1,\alpha) & \left(\dfrac{9}{8} \leqq \mu \leqq \dfrac{3(3-2\alpha)}{8(1-\alpha)}\right) \\ k_5(1,\alpha) & \left(\mu \geqq \dfrac{3(3-2\alpha)}{8(1-\alpha)}\right). \end{cases}$$

Also, by Corollary 3.6 and Corollary 3.7, we can establish the following corollary including Theorem 1.4 due to Janteng, Halim and Darus [4].

Corollary 3.8 *If a function $f(z) \in \mathcal{K}$, then*

$$|a_2a_4 - \mu a_3^2| \leqq \begin{cases} 1-\mu & \left(\mu \leqq \dfrac{3}{4}\right) \\ \dfrac{9-16\mu+8\mu^2}{8\mu} & \left(\dfrac{3}{4} \leqq \mu \leqq 1\right) \\ \dfrac{1}{8} & \left(1 \leqq \mu \leqq \dfrac{9}{8}\right) \\ \mu - 1 & \left(\mu \geqq \dfrac{9}{8}\right). \end{cases}$$

References

[1] P. L. Duren, *Univalent Functions*, Springer-Verlag, New York, Berlin, Heidelberg, Tokyo, 1983.

[2] M. Fekete and G. Szegö, *Eine Bemerkung uber ungerade schlichte Funktionen*, J. London Math. Soc. **8**(1933), 85-89.

[3] T. Hayami and S. Owa, *Hankel determinant for p–valently starlike and convex functions of order α*, General Math. **17**(4) (2009), 29–44.

[4] A. Janteng, S. A. Halim and M. Darus, *Hankel determinant for starlike and convex functions*, Int. J. Math. Anal. **1**(2007), 619-625.

[5] J. W. Noonan and D. K. Thomas, *On the second Hankel determinant of areally mean p-valent functions*, Trans. Amer. Math. Soc. **223**(2) (1976), 337-346.

Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577-8502
Japan
E-mail: ha_ya_to112@hotmail.com
owa@math.kindai.ac.jp