数理解析研究所講究録
第 1718 巻 2010 年 70-74

Morley's theorem on Omitting types

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Abstract

We think about Morley's omitting types theorem for countable first-order theory. Then I introduce the result of having been related to Morley's theorem shown by [4].

1 Introduction

Definition.($\beth$-number) $\beth_0 = \omega$, $\beth_{\alpha+1} = 2^{\beth_\alpha}$, $\beth_\delta = \sup_{\alpha<\kappa} \beth_\alpha$.
Fact.(Erdös-Rado) Let α be infinite cardinal and $n < \omega$. Then $\beth_n^+ \to (\omega^+)^{n+1}_\omega$, $\beth^+_{\alpha+n} \to (\beth^+_\alpha)^{n+1}_{\beth_\alpha}$.
Note. $\alpha \to (\beta)^n_\gamma$ means whenever $|X| = \alpha$ and given any function f from $[X]^n$ into γ, there exists a subset Y of X with $|Y| = \beta$ and an $i < \gamma$ such that for all $\overline{y} \in [Y]^n$, $f(\overline{y}) = i$.
Theorem.(Stretching) Let $\mathcal{L}$ be countable language, M be a model of theory T of $\mathcal{L}$, $\langle A, <\rangle$ be an infinite set of indiscernibles in M, and $\langle B, <\rangle$ be an arbitrary infinite lineary ordered set. Then there exist a model N of T such that $\langle B, <\rangle$ is a set of indiscernibles in N, and for any $a_1 < \cdots < a_n \in A$ and $b_1 < \cdots < b_n \in B$, $tp(a_1, \ldots, a_n) = tp(b_1, \ldots, b_n)$.
Proof. Put $\Sigma := \{t(x_1, \ldots, x_n) : t \text{ is term in } \mathcal{L}\}$. We define an equivalence relation $\sim$ on Σ as follows. If $t(x_1, \ldots, x_n), t'(x_1, \ldots, x_n) \in \Sigma$, define $t \sim t'$ iff for any $a_1 < \cdots < a_n \in A, M \models t(a_1, \ldots, a_n) = t'(a_1, \ldots, a_n)$. Put $\overline{N} := \{t(b_1, \ldots, b_n) : t(x_1, \ldots, x_n) \in \Sigma, b_1 < \cdots < b_n \in B\}$. We define an equivalence relation $\approx$ on $\overline{N}$ as follows. If $t(b_1, \ldots, b_n), t'(b'_1, \ldots, b'_m) \in \overline{N}$, define $t \approx t'$ iff $t_0(z_1, \ldots, z_s) \sim t'_0(z_1, \ldots, z_s)$, where $\{z_1, \ldots, z_s\} := \{x_1, \ldots, x_n\} \cup \{x'_1, \ldots, x'_m\}$ and $t_0(z_1, \ldots, z_s) := t(x_1, \ldots, x_n)$, $t'_0(z_1, \ldots, z_s) := t'(x'_1, \ldots, x'_m)$. Put $N := \{t(\overline{b})^{\approx} : t(\overline{b}) \in \overline{N}\}$. Note that for any $t_1(\overline{b}_1)^{\approx}, \ldots, t_n(\overline{b}_n)^{\approx} \in N$ there exists, for some $\overline{b}' \in B$ and $t'_i \in \Sigma$ such that $t_i(\overline{b}_i)^{\approx} = t'_i(\overline{b}')^{\approx}$. We treat B as a subset of N by identifying each $b \in B$ with $b^{\approx}$.

N can be made into a $\mathcal{L}$-structure by defining constants, functions and relations as follows:

(Constants) $N \models c_N = c^{\approx}$.
(Functions) $N \models F(t_1(\overline{b})^{\approx}, \ldots, t_n(\overline{b})^{\approx}) = (F(t_1(\overline{b}), \ldots, t_n(\overline{b})))^{\approx}$.
(Relations) $N \models R(t_1(\overline{b})^{\approx}, \ldots, t_n(\overline{b})^{\approx})$
$:\overset{\text{def}}{\Leftrightarrow}$ for all $a_1 < \cdots < a_m \in A$, $M \models R(t_1(\overline{a}), \ldots, t_n(\overline{a}))$.

This definition dose not depend on the choice of representatives of the equivalence classes under $\approx$.

By induction on the complexity of formulas and use Skolem function it can be shown that for any $b_1 < \cdots < b_n \in B$ and $\phi(x_1, \ldots, x_n) \in \mathcal{L}$,

$$N \models \phi(b_1, \ldots, b_n) \quad \text{iff} \quad \text{for all } a_1 < \cdots < a_n \in A, \\ M \models \phi(a_1, \ldots, a_n).$$

By indiscernibility of $\langle A, < \rangle$, $\langle B, < \rangle$ is a set of indiscrnibles in N and for any $a_1 < \cdots < a_n \in A$ and $b_1 < \cdots < b_n \in B$, $tp(a_1, \ldots, a_n) = tp(b_1, \ldots, b_n)$. In particular, $N \equiv M$, hence N is a model of T.

2 Morley's Theorem

Theorem. (Morley's omitting types theorem) Let T be a theory of countable language $\mathcal{L}$, Γ a set of partial types in finitely many variables over $\emptyset$, $\mu = (2^\omega)^+$. Suppose $\{M_\alpha : \alpha < \mu\}$ is a sequence of models of T such that

1. $|M_\alpha| > \beth_\alpha$,
2. M_α omits each member of Γ.

Then for every $\lambda \geq \omega$, there is a model N with $|N| = \lambda$ of T such that N omits each member of Γ.

Proof. Assume to simplify an argument T has built-in Skolem functions and the set of formulas Γ in the unary. Let $C = \langle c_i : i < \omega \rangle$ be a sequence of new constant symbols, $\mathcal{L}^* = \mathcal{L} \cup C$.

Now we construct the consistent $\mathcal{L}^*$-theory Φ as following properties:

1. $T \cup \{c_i \neq c_j : i < j < \omega\} \subset \Phi$;
2. for each term $t(x_1, \ldots, x_n)$ and $p \in \Gamma$, there is a $\phi_p \in p$ such that for all $i_1 < \cdots < i_n < \omega$,
$$\neg\phi_p(t(c_{i_1}, \ldots, c_{i_n})) \in \Phi;$$
3. for any $\psi(x_1, \ldots, x_n) \in \mathcal{L}$ if $i_1 < \cdots < i_n < \omega$ and $j_1 < \cdots < j_n < \omega$,
$$\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) \in \Phi.$$

Notation. $F := \{(M_\alpha, A_\alpha) : \alpha < \mu\}$ is a sequence such that M_α is satisfied the hypotheses of the theorem and A_α is subset of M_α with $|A_\alpha| > \beth_\alpha$.

We say that $F' = \{(M'_\alpha, B_\alpha) : \alpha < \mu\}$ is subsequence of F if for each M'_α there is $\beta \geq \alpha$ such that $M'_\alpha = M_\beta$ and $B_\alpha \subset A_\beta$ with $|B_\alpha| > \beth_\alpha$.

Fix a linear ordering of each M_α in an arbitrary fashion denoting them all by $<$.

Claim 1. Fix a term $t(x_1, \ldots, x_n)$. There is subsequence F' of F as following property: for each $p \in \Gamma$ there is a $\phi_p \in p$ such that for any $(M'_\alpha, B'_\alpha) \in F'$, if $i_1 < \cdots < i_n < \omega$ and $b_{i_j} \in B'_\alpha$ then $M'_\alpha \models \neg\phi_p(t(b_{i_1}, \ldots, b_{i_n}))$.

Proof of claim 1. Note $|\Gamma| \leq 2^{\omega}$. Let $N_\alpha = M_{\alpha+n}$. Define, for all $\alpha < \mu$, $f_\alpha : [A_{\alpha+n}]^n \to \mathcal{L}^{\Gamma}$ $(\bar{a} \mapsto f_\alpha(\bar{a}))$ where $f_\alpha(\bar{a}) : \Gamma \to \mathcal{L}$ $(p \mapsto (f_\alpha(\bar{a}))(p) := \phi_{\bar{a},p} \in p)$ such that $N_\alpha \models \neg\phi_{\bar{a},p}(t(\bar{a}))$ such a $\phi_{\bar{a},p}$ exists since N_α omits p.

Now $|A_{\alpha+n}| > \beth_{\alpha+n}$ and for $\alpha \geq 3$, $\beth_\alpha \geq |\mathcal{L}^{\Gamma}|$.
By Erdös-Rado Theorem, $(\beth_{\alpha+n})^+ \to (\beth_\alpha^+)^n_{|\mathcal{L}^{\Gamma}|}$. Thus we obtain $B_\alpha \subset A_{\alpha+n}$ and $\phi_{\alpha,p} \in \mathcal{L}$ such that

1. $|B_\alpha| > \beth_\alpha$,
2. for all $\bar{b} \in [B_\alpha]^n$, $N_\alpha \models \neg\phi_{\alpha,p}(t(\bar{b}))$.

Namely, for all $\bar{b} \in [B_\alpha]^n$, $f_\alpha(\bar{b}) =$ constant.

As $\mu = (2^{\omega})^+$, by Erdös-Rado, there is subsequence $\{M'_\alpha : \alpha < \mu\}$ of $\{N_\alpha : \alpha < \mu\}$ such that for all $\bar{b} \in [B'_\alpha]^n$ and $p \in \Gamma$, $(f_\alpha(\bar{b}))(p) =$ constant. Thus $\{(M'_\alpha, B'_\alpha) : \alpha < \mu\}$ and $\phi_p := (f_\alpha(\bar{b}))(p)$ are required.

Claim 2. Fix a $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n)$. There is subsequence F' of F as following property: for any $(M'_\alpha, B_\alpha) \in F'$ if $i_1 <, \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{i_k}, b_{j_k} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$

Proof of Claim 2. Define, for all $\alpha < \mu$, $h_\alpha : [A_\alpha]^n \to 2$ as follows:

$$h_\alpha(\bar{a}) = \begin{cases} 0 & \text{if } M_\alpha \models \psi(\bar{a}), \\ 1 & \text{otherwise.} \end{cases}$$

By Erdös-Rado theorem, there is $B_\alpha \subset A_\alpha$ such that $|B_\alpha| > \beth_\alpha$ and for all $\bar{b} \in [B_\alpha]^n$, $h_\alpha(\bar{b}) =$ constant. Thus, $\{(M_\alpha, B_\alpha) : \alpha < \mu\}$ is required.

Let $\{t_i : i < \omega,\ t_i \text{ is a term of } \mathcal{L}\}$ and $\{\psi_i : i < \omega,\ \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$ be enumerations of all the terms of $\mathcal{L}$ and all the $\mathcal{L}$-formula, respectively. Now we construct Φ by induction on $i < \omega$. Suppose $F_0 := \{(M_\alpha, M_\alpha) : \alpha < \mu\}$ and $\Phi_0 := T \cup \{c_i \neq c_j : i < j < \omega\}$. Clearly, for any $(M_\alpha, M_\alpha) \in F_0$, $M_\alpha \models \Phi_0$ and $|M_\alpha| > \beth_\alpha$.

Case 1 ($i < \omega$ is even). Assume we have found F_i and Φ_i. We take new term $t(x_1, \ldots, x_n) \in \{t_i : i < \omega,\ t_i \text{ is a term of } \mathcal{L}\}$, by claim 1,
there is subsequence F_{i+1} of F_i as following property: for each $p \in \Gamma$, there is a $\phi_p \in p$ such that for any $(M'_\alpha, B_\alpha) \in F_{i+1}$, if $i_1 < \cdots < i_n < \omega$ and $b_{i_j} \in B_\alpha$,

$$M'_\alpha \models \neg\phi_p(t(b_{i_1}, \ldots, b_{i_n})).$$

We put $\Phi_{i+1} = \Phi_i \cup \{\neg\phi_p(t(c_{i_1}, \ldots, c_{i_n})) : p \in \Gamma,\ i_1 < \cdots < i_n < \omega\}$.

Case 2 ($i < \kappa$ is odd). Assume we have found F_i and Φ_i. We take new formula $\psi(x_1, \ldots, x_n) \in \{\psi_i : i < \omega,\ \psi_i \text{ is a } \mathcal{L}\text{-formula}\}$, by claim 2, there is subsequence F_{i+1} of F_i as following property: for any $(M'_\alpha, B_\alpha) \in F_{i+1}$ if $i_1 <, \cdots < i_n < \mu$, $j_1 < \cdots < j_n < \mu$ and $b_{i_k}, b_{j_k} \in B_\alpha$

$$M'_\alpha \models \psi(b_{i_1}, \ldots, b_{i_n}) \leftrightarrow \psi(b_{j_1}, \ldots, b_{j_n}).$$

We put $\Phi_{i+1} := \Phi_i \cup \{\psi(c_{i_1}, \ldots, c_{i_n}) \leftrightarrow \psi(c_{j_1}, \ldots, c_{j_n}) : i_1 < \cdots < i_n < \omega,\ j_1 < \cdots < j_n < \omega\}$.

If put $\Phi := \cup_{i<\omega}\Phi_i$ then it is required $\mathcal{L}^*$-theory. We take any $(M_\alpha, A_\alpha) \in F := \cap_{i<\omega}F_i$. By construction $M_\alpha \models \Phi$.

Let A be the set of all interpretation $C = \{c_i : i < \omega\}$ in M_α, N be Skolem closure of A in M_α. Thus N is model of T, omitting Γ, indicernibles in M_α, and $|N| = \omega$.

Take any $\lambda \geq \omega$. By stretching theorem, there is a model of T which the cardinality of λ such that omitting Γ. Note that if $|\Gamma| \leq \omega$ then it is sufficient $\mu = \omega_1$, see [2].

It is known that Morley's theorem is proved in infinitary logic, and it is effective means to show existence of models in infinitary logic that the compactness theorem is false generally, see [1], [3].

3 Related Result

The following result is related to Morley's omitting type theorem. This theorem says the thing that is stronger than Morley's theorem under a certain condition.

Theorem.(Tsuboi) Let T be a countable complete $\mathcal{L}$-theory and Γ a set of complete types with $|\Gamma| < 2^\omega$. Suppose that for each $\alpha < \omega_1$, there is a model $M_\alpha \models T$ with the following properties:

1. $|M_\alpha| > \beth_\alpha$,
2. M_α omits each member of Γ.

Then for each $\lambda \geq \omega$ there is a model N omitting Γ and with $|N| = \lambda$.

Proof. Let $X = \omega_1$ and $\{I_i : i \in X\}$ be a set of infinite indiscernible sequences and $\{t_n; n < \omega\}$ be an enumeration of all the $\mathcal{L}$-terms. We may assume that t_n has n-variables. We will say that the set $\{I_i : i \in X\}$ is t_n-uniform if the following condition holds: If $i, j \in X$, then $tp(t_n(I_i)) = tp(t_n(I_j))$ where $tp(t_n(I_i)) := tp(t_n(a_0, \ldots, a_{n-1}))$ $(a_0 < \cdots < a_{n-1} \in I_i)$. We will say that $\{I_i : i \in X\}$ is essentially t_n-uniform if there is an uncountable subset Y of X such that $\{I_i : i \in Y\}$ is t_n-uniform. For a formula $\phi(x)$, define $X^{\phi,t_n} := \{i \in X : \phi(x) \in tp(t_n(I_i))\}$. Put $X_\emptyset = \omega_1$, and for each $i \in X_\emptyset$ we fix a sequence $I_\emptyset(i)$ enumerating the universe M_i.

Using the argument in the paper([4]), for $\eta \in 2^{<\omega}$ and $k < \omega$, we can inductively choose $X_\eta \subset \omega_1$, $\{I_\eta(i) : i \in X_\eta\}$ and formulas $\phi_{\eta,k}$ with the following properties:

1. If $\eta < \nu$, then
   (a) X_ν is an uncountable subset of X_η;
   (b) $I_\nu(i)$ is a subsequence of $I_\eta(i)$ for each $i \in X_\nu$.
2. $i < j \Rightarrow |I_\eta(i)| < |I_\eta(j)|$, and $\sup\{|I_\eta(i)| : i \in X_\eta\} \geq \beth_{\omega_1}$.

3. If $\eta \in 2^n$ then
   (a) each $I_\eta(i)$ is an infinite indiscernible sequence;
   (b) $\{I_\eta(i) : i \in X_\eta\}$: essentially t_n-uniform $\Rightarrow$ it is t_n-uniform.

4. If $\eta \in 2^n$ and $k \leq n$ then
   $\{I_\eta(i) : i \in X_\eta\}$: not t_n-uniform
   $\Rightarrow X_{\eta^\frown 0} \subset (X_\eta)^{(\phi_{\eta,k}),t_k}$ and $X_{\eta^\frown 1} \subset (X_\eta)^{(\neg\phi_{\eta,k}),t_k}$.

For all $\nu \in 2^\omega$, we define the following:

1. K_ν is the set of all $n < \omega$ such that $\{I_{\nu|n}(i) : i \in X_{\nu|n}\}$ is not t_n-uniform;
2. for $n \in K_\nu$, $\Delta_\nu^n(x) := \bigcup_{n \leq m < \omega}\{\phi_{\nu|m,n}(x) : \nu(m) = 0\} \cup \bigcup_{n \leq m < \omega}\{\neg\phi_{\nu|m,n}(x) : \nu(n) = 1\}$;
3. $\Phi_\nu := \{\{x_i\}_{i<\omega} \text{ is indiscernible}\} \cup \bigcup_{n \in K_\nu} \Delta_\nu^n(t_n(\overline{x}_n)) \cup \bigcup_{n \notin K_\nu} p_{\nu|n}(t_n(\overline{x}_n))$;
4. $F_\nu := \{(M_i^\nu, I_\nu(i)) : i \in X_\nu\} \quad (I_\nu(i) \subset M_i^\nu)$.

$$
2^\omega \begin{cases}
F_{\langle 0\cdots\rangle} : \overbrace{(M_0^{\langle 0\cdots\rangle}, I_{\langle 0\cdots\rangle}(0)), \cdots, (M_i^{\langle 0\cdots\rangle}, I_{\langle 0\cdots\rangle}(i)), \cdots}^{\omega_1} & \models \Phi_{\langle 0\cdots\rangle} \\
\vdots \qquad\qquad \vdots & \vdots \\
F_\nu : (M_0^\nu, I_\nu(0)), \cdots, (M_i^\nu, I_\nu(i)), \cdots & \models \Phi_\nu \\
\vdots \qquad\qquad \vdots & \vdots \\
F_{\langle 1\cdots\rangle} : (M_0^{\langle 1\cdots\rangle}, I_{\langle 1\cdots\rangle}(0)), \cdots, (M_i^{\langle 1\cdots\rangle}, I_{\langle 1\cdots\rangle}(i)), \cdots & \models \Phi_{\langle 1\cdots\rangle}
\end{cases}
$$

We can take $\nu \in 2^\omega$ well, see [4], such that if $\{c_i : i < \omega\}$ realizing Φ_ν in M_0^ν, and N is Skolem closure of $\{c_i : i < \omega\}$ in M_0^ν then N omits Γ. The rest of the statement is clear from Stretching Theorem.

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