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What are o-minimal sheaves

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Abstract

In this small note we present an introduction to o-minimal sheaves and their connection to semi-algebraic and sub-analytic sheaves.

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1 Introduction

O-minimal structures are a class of ordered structures which are a model theoretic (logic) generalization of interesting classical structures such as:

- the field of real numbers;
- the field of real numbers expanded by restricted globally analytic functions ([7]).

More precisely, an ordered structure

$$\mathcal{M} = (M, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}, <)$$

is o-minimal if every definable subset of M in the structure is already definable in the ordered set $(M, <)$.

The development of o-minimality has been strongly influenced by real analytic geometry and it is based on: (i) adaptation of methods of real analytic geometry to the o-minimal setting; (ii) construction of new and mathematically interesting examples of o-minimal structures; (iii) new insights originated from model-theoretic methods into the real analytic setting. O-minimal structures provide: a generalization, a uniform treatment and new tools.

Good references on o-minimality are, for example, the book [8] by van den Dries and the notes [3] by Coste. For semialgebraic geometry relevant to this paper the reader should consult the work by Delfs [5], Delfs and Knebusch [6] and the book [2] by Bochnak, Coste and Roy. For subanalytic geometry we refer to the work [1] by Bierstone and Milmann.

Given an o-minimal structure

$$\mathcal{M} = (M, (c)_{c\in\mathcal{C}}, (f)_{f\in\mathcal{F}}, (R)_{R\in\mathcal{R}}, <)$$

we have:

- the category Def of definable spaces with continuous definable maps.
- the geometry of Def is called o-minimal geometry.

Examples 1.1 (Special cases of o-minimal geometry)

- $\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, <)$ - *semi-algebraic geometry (includes real algebraic geometry);*

- $\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, (f)_{f \in \mathrm{an}}, <)$ - *restricted globally sub-analytic geometry;*

The model theoretic language allows a uniform development of o-minimal geometry in non-standard o-minimal structures. Concrete non-standard o-minimal structures are:

- $\mathbb{R}((t^{\mathbb{Q}})) = (\mathbb{R}((t^{\mathbb{Q}})), 0, 1, +, \cdot, <)$ (or any ordered real closed field),
- $\mathbb{R}((t^{\mathbb{Q}}))_{\mathrm{an}} = (\mathbb{R}((t^{\mathbb{Q}})), 0, 1, +, \cdot, (f)_{f \in \mathrm{an}}, <)$

where $\mathbb{R}((t^{\mathbb{Q}}))$ is the field of power series with well ordered supports on which every restricted globally analytic function $f \in$ an can be interpreted in a canonical way ([9]). There are many important o-minimal expansions

$$\mathcal{M} = (\mathbb{R}, 0, 1, +, \cdot, (f)_{f \in \mathcal{F}}, <)$$

of the ordered field of real numbers. For example $\mathbb{R}_{\mathrm{an}}$, $\mathbb{R}_{\mathrm{exp}}$, $\mathbb{R}_{\mathrm{an,exp}}$, $\mathbb{R}_{\mathrm{an}^*}$, $\mathbb{R}_{\mathrm{an}^*,\mathrm{exp}}$ see resp., [7, 29, 10, 12, 13]. For each such we have 2^{κ} many non-isomorphic non standard o-minimal models for each $\kappa >$ cardinality of the language! There is however a non-standard o-minimal structure

$$\mathcal{M} = (\bigcup_{n \in \mathbb{N}} \mathbb{R}((t^{\frac{1}{n}})), 0, 1, +, \cdot, (f_p)_{p \in \mathbb{R}[[\zeta_1, \ldots, \zeta_n]]}, <)$$

which does not came from a standard one ([23, 17]). O-minimal geometry includes the geometry of all those (standard) tame analytic structures but it goes beyond and includes also a generalization of PL-geometry: any ordered vector space over an ordered division ring

$$\mathcal{M} = (M, 0, +, (\lambda_d)_{d \in D}, <)$$

is an o-minimal structure ([8]).

Following or inspired by the work of:

- Verdier (locally compact topological spaces) - [16, 18, 19].
- Delfs (real algebraic geometry) - [5].
- Kashiwara-Schapira, L. Prelli et al. (sub-analytic geometry) - [22, 20, 21, 25, 26].
- Grothendieck (étale framework) - [28].

we would like to develop sheaf theory in the category Def in a fixed but arbitrary o-minimal structures $\mathcal{M}$.

2 What are o-minimal sheaves

Recall that our goal is to develop sheaf theory in the category Def in a fixed but arbitrary o-minimal structures $\mathcal{M}$. Every object of Def is a topological space with topology defined from the ordering of $\mathcal{M}$. So why not topological sheaf theory? Topological sheaf theory is not suitable, since it gives:

- no information in the non standard setting;
- no new information in the standard setting.

In fact we have to use sites (Grothedienck topologies). Usually the problem is having too many or too few open subsets.

So what are o-minimal sheaves? Let X be an object of Def and k a field. An o-minimal sheaf of k-vector spaces on X, called also an o-minimal k-sheaf on X, is a contravariant functor:

$$\begin{aligned} F : \mathrm{Op}(X_{\mathrm{def}}) &\to \mathrm{Mod}(k) \\ U &\mapsto F(U) \\ (V \subset U) &\mapsto (F(U) \to F(V)) \\ &\qquad\qquad s \mapsto s_{|V} \end{aligned}$$

where X_{def} is the o-minimal site on X. Satisfying the following gluing conditions: for $U \in \mathrm{Op}(X_{\mathrm{def}})$ and $\{U_j\}_{j\in J} \in \mathrm{Cov}(U)$ we have the exact sequence

$$0 \to F(U) \to \Pi_{j\in J} F(U_j) \to \Pi_{j,k\in J} F(U_j \cap U_k).$$

What is the o-minimal site on X? The o-minimal site X_{def} on X is the data consisting of:

- The category
$$\mathrm{Op}(X_{\mathrm{def}})$$
of open definable subsets of X with inclusions;
- The collection of admissible coverings
$$\mathrm{Cov}(U),\ \ U \in \mathrm{Op}(X_{\mathrm{def}})$$
such that $\{U_j\}_{j\in J} \in \mathrm{Cov}(U)$ if $\{U_j\}_{j\in J}$ covers U, its elements are in $\mathrm{Op}(X_{\mathrm{def}})$ and has a finite sub-cover.

This includes semi-algebraic and restricted globally sub-analytic sites and sheaves. What about sub-analytic site and sheaves? If we work in the slightly more general category of locally definable spaces with continuous locally definable maps, then the o-minimal site includes also the sub-analytic site on real analytic manifolds.

The gluing condition

$$0 \to F(U) \to \Pi_{j\in J}F(U_j) \to \Pi_{j,k\in J}F(U_j \cap U_k)$$

means:

- if $s \in F(U)$ and $s_{|U_j} = 0$ for each j, then $s = 0$;
- if $s_j \in F(U_j)$ are such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in F(U)$ (i.e. $s_{|U_j} = s_j$).

For X an object of Def and k a field, we use the following notation: $\mathrm{Mod}(k_{X_{\mathrm{def}}}) :=$ k-sheaves in the o-minimal site X_{def} and $\mathrm{Mod}(k_X) :=$ topological k-sheaves on X.

Examples 2.1 (Simple examples) *Let X be an object of* Def. *The following pre-sheaves are in* $\mathrm{Mod}(\mathbb{R}_{X_{\mathrm{def}}})$*:*

- $U \mapsto \mathbb{R}_X(U) := \{f : U \to \mathbb{R} | f \textit{ locally constant}\}$;
- $U \mapsto \{f : U \to \mathbb{R} | f \textit{ bounded}\}$;
- $U \mapsto \mathcal{C}_X(U) := \{f : U \to \mathbb{R} | f \textit{ continuous}\}$;
- $U \mapsto \{f : U \to \mathbb{R} | f \textit{ definable}\}$;

The second and the fourth examples above are not in $\mathrm{Mod}(\mathbb{R}_X)$.

In our context the gluing condition gives rise to the following gluing criteria. Let X be an object of Def (resp. a real analytic manifold) and F a presheaf on X_{def} (resp. on X_{sa} - the sub-analytic site of X). Assume that

- $F(\varnothing) = 0$;
- for all $U, V \in \mathrm{Op}(X_{\mathrm{def}})$ (resp. in $\mathrm{Op}(X_{sa})$) the sequence

  $$0 \to F(U \cup V) \to F(U) \oplus F(V) \to F(U \cap V)$$

  is exact.

Then F is a sheaf on X_{def} (resp. X_{sa}).

Examples 2.2 ([20] - Deep examples) *M. Kashiwara and P. Schapira combined classical analytical results of S. Lojasiewicz and the gluing criteria to show that the following pre-sheaves*

- *tempered distributions* $\mathcal{D}b_X^t$;
- *tempered* C^∞ *functions;*
- *Whitney* C^∞ *functions;*
- *tempered holomorphic* $\mathcal{O}_X^t$ *functions;*

are sheaves on X_{sa}. *This is very deep and has applications to the theory of* D*-modules.*

3 Some results

Of course all the classical homological results for sheaves on sites hold in the category $\mathrm{Mod}(k_{X_{\mathrm{def}}})$. So if we want to obtain specific results on the geometry of objects of Def we have to introduce something more. For this it will be convenient to replace the o-minimal site X_{def} by the o-minimal spectrum $\widetilde{X}$ of X. See [14]. This method was also used in the semi-algebraic context but never in the sub-analytic case where everything is standard - [2, 4, 5].

The o-minimal spectrum $\widetilde{X}$ of X is the set of ultrafilters of definable subsets of X equipped with the topology generated by the open subsets of the form $\widetilde{U}$ where $U \in \mathrm{Op}(X_{\mathrm{def}})$. This is a spectral topological space - [3, 14, 24].

Example 3.1 (The connection to real algebraic geometry) *If R is a real closed field and X an affine real algebraic variety over R with coordinate ring $R[X]$, then $\widetilde{X} \simeq \mathrm{Specr}R[X]$ (the real spectrum of the commutative ring $R[X]$).*

The tilde operation determines the tilde functor $\mathrm{Def} \longrightarrow \widetilde{\mathrm{Def}}$ which determines morphisms of sites

$$\nu_X : \widetilde{X} \longrightarrow X_{\mathrm{def}}$$

given by the functor $\nu_X^t : \mathrm{Op}(X_{\mathrm{def}}) \longrightarrow \mathrm{Op}(\widetilde{X}) : U \mapsto \widetilde{U}$.

Theorem 3.2 ([14]) *The functor* Def $\longrightarrow \widetilde{\mathrm{Def}}$ *induces an isomorphism of categories*

$$\mathrm{Mod}(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{Mod}(k_{\widetilde{X}}) : F \mapsto \widetilde{F},$$

where $\mathrm{Mod}(k_{\widetilde{X}})$ *is the category of sheaves of* k*-modules on the topological space* $\widetilde{X}$.

The isomorphism is the inverse image ν_X^{-1} and its inverse is the direct image ν_{X*}. The canonical isomorphism extends to the derived categories

$$\mathrm{D}^*(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^*(k_{\widetilde{X}}) : I \mapsto \widetilde{I}$$

where $\mathrm{D}^*(k_{\widetilde{X}}) = \mathrm{D}^*(\mathrm{Mod}(k_{\widetilde{X}}))$ and $(* = b, +, -)$.

Corollary 3.3 *The functors*

$$\begin{aligned}
&\mathrm{RHom}_{k_{X_{\mathrm{def}}}}(\bullet,\bullet) : \mathrm{D}^-(k_{X_{\mathrm{def}}})^{\mathrm{op}} \times \mathrm{D}^+(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^+(k),\\
&R\mathcal{H}om_{k_{X_{\mathrm{def}}}}(\bullet,\bullet) : \mathrm{D}^-(k_{X_{\mathrm{def}}})^{\mathrm{op}} \times \mathrm{D}^+(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^+(k_{X_{\mathrm{def}}}),\\
&f^{-1} : \mathrm{D}^*(k_{Y_{\mathrm{def}}}) \longrightarrow \mathrm{D}^*(k_{X_{\mathrm{def}}}) \qquad (* = b, +, -),\\
&Rf_* : \mathrm{D}^+(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^+(k_{Y_{\mathrm{def}}}),\\
&\bullet \otimes^L_{k_{X_{\mathrm{def}}}} \bullet : \mathrm{D}^*(k_{X_{\mathrm{def}}}) \times \mathrm{D}^*(k_{X_{\mathrm{def}}}) \longrightarrow \mathrm{D}^*(k_{X_{\mathrm{def}}}) \qquad (* = b, +, -)
\end{aligned}$$

commute with the tilde functor.

In the paper [14] can develop o-minimal sheaf cohomology by setting

$$H^*(X; F) := H^*(\widetilde{X}; \widetilde{F})$$

where X is a definable space and F is a sheaf in $\mathrm{Mod}(k_{X_{\mathrm{def}}})$ and prove the following results:

Theorems 3.4 ([14])

- *Vanishing Theorem.*
- *Vietoris-Begle Theorem.*
- *Eilenberg-Steenrod Axioms.*

The vanishing theorem above has the following application to sub-analytic sheaves:

Theorem 3.5 ([27]) *Let* X *be a real analytic manifold. The homological dimension of* $\mathrm{Mod}(k_{X_{\mathrm{sa}}})$ *is finite.*

After developing the theory of definably compact supports one obtains the following result conjectured by Delfs in the semi-algebraic case:

Theorem 3.6 ([15] - Global Verdier duality) *Let X be definably normal, definably locally compact, definable space. There exists $\mathcal{D}^*$ in $\mathrm{D}^+(k_{X_{\mathrm{def}}})$ and a natural isomorphism*

$$\mathrm{RHom}_{k_{X_{\mathrm{def}}}}(\mathcal{F}^*, \mathcal{D}^*) \simeq \mathrm{RHom}_k(R\Gamma_c(X, \mathcal{F}^*), k)$$

as $\mathcal{F}^*$ varies through $\mathrm{D}^+(k_{X_{\mathrm{def}}})$.

This is a general form of Poincaré duality:

Corollary 3.7 ([15] - Poincaré and Alexander duality) *Let X be definably normal, definably locally compact, definable manifold of dimension n.*

- *If X has an orientation k-sheaf $\mathcal{O}r_X$, then*

$$H^p(X; \mathcal{O}r_X) \simeq H_c^{n-p}(X; \underline{k})^{\vee}.$$

- *If X is k-orientable and Z is a closed definable subset, then*

$$H_Z^p(X; k_X) \simeq H_c^{n-p}(Z; \underline{k})^{\vee}.$$

With L. Prelli we are working on developing the formalism of the six operations on o-minimal sheaves in Def:

$$Rf_*,\ f^{-1},\ \otimes^L,\ R\mathcal{H}om,\ Rf_{!!},\ f^{!!}$$

Such formalism was developed for sub-analytic sheaves by Kashiwara-Schapira using the complicated theory of ind-sheaves and later a direct construction was given by L. Prelli. However, both methods do not generalize to o-minimal sheaves since they rely on the formalism of the six operations on topological sheaves in locally compact topological spaces (Verdier).

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