Uniformly definable subrings of some infinite algebraic extensions of the rationals

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Abstract

We consider the formulas used by Julia Robinson in her proof that number fields are first order undecidable. We extend the result of [1]. We prove that it defines subrings in some infinite algebraic extensions of the rationals. As an application we discuss undecidabilities of those infinite algebraic extensions.

1 Introduction

In 1959 Julia Robinson [8] proved that any number field, as well as the corresponding ring of algebraic integers, is undecidable, by showing that $\mathbb{N}$ is $\emptyset$-definable (in the ring language) in the ring, and the ring is $\emptyset$-definable in its number field.

She first considered the formula

$$\varphi_{m}(s, u, t) : \exists x, y, z(1 - sut^{2m} = x^{2} - sy^{2} - uz^{2}),$$

where $m$ is a positive integer such that $p^{m} \not| 2$ for all prime ideals $p$ of a given number field $F$, that is, $m$ is an integer greater than all the ramification indices of prime ideals of $F$ which divide 2. Then she proved that for a given prime $p_{1}$ of $F$ there are $a, b \in F$ such that $\varphi_{m}(a, b, t)$ defines a finite intersection of valuation rings $\bigcap_{p \in \Delta} \mathcal{O}_{p}$ where $\Delta$ is a finite set of primes of $F$ containing $p_{1}$. (We actually can define the valuation ring of $p_{0}$ using two $\varphi_{m}(s, t, u)$ with some choice of those parameters.) We denote by $\varphi_{m}(a, b, F)$ the solution set of $\varphi_{m}(a, b, t)$ in $F$, that is, $\varphi_{m}(a, b, F) = \{ \alpha \in F : F \models \varphi_{m}(a, b, \alpha) \}$. It is easy to see that $\bigcap_{a, b \in F} \varphi_{m}(a, b, F) = 0$. Therefore in order to define the ring of algebraic integers $\mathcal{O}_{F}$ in a given number field $F$, J. Robinson considered the intersection of all $\varphi_{m}(a, b, F)$ containing $\mathbb{Z}$, which is defined by $\psi_{m}(t)$:

$$\forall s, u(\forall c(\varphi_{m}(s, u, c) \rightarrow \varphi_{m}(s, u, c + 1)) \rightarrow \varphi_{m}(s, u, t)).$$

Note that $\varphi_{m}(s, u, t) \leftrightarrow \varphi_{m}(s, u, -t)$. We denote by $\psi_{m}(F)$ the solution set of $\psi_{m}(t)$ in $F$ as before. It is possible to define $\mathcal{O}_{F}$ since $\mathbb{Z} \subseteq \psi_{m}(F) \subseteq \mathcal{O}_{F}$ and $F$ has an integral basis over the rationals $\mathbb{Q}$. (The defining formula of $\mathcal{O}_{F}$ depends on $F$.)

In this paper we calculate the solution set of $\psi_{2}(t)$ in some infinite algebraic extensions of $\mathbb{Q}$.
2 Construction of $\psi(t)$

Let $F$ be a number field (a finite algebraic extension of the rationals $\mathbb{Q}$) and let $\mathfrak{o}_F$ be the ring of algebraic integers of $F$. $F^*$ will denote the set of non-zero elements of $F$. By $\mathfrak{p}$ we denote a place of $F$ and by $F_{\mathfrak{p}}$ the completion of $F$ with respect to $\mathfrak{p}$. Since non-archimedean places of $F$ are $\mathfrak{p}$-adic valuations for some prime ideal $\mathfrak{p}$ of $F$, we use the same letter $\mathfrak{p}$ for both the place and the prime ideal. The ring of integers of $F_{\mathfrak{p}}$ is denoted by $(\mathfrak{o}_F)_{\mathfrak{p}}$, its maximal ideal is also denoted by $\mathfrak{p}$. Let $\mathfrak{p}$ be a prime ideal of $F$ and $a \in F$. By $\nu_{\mathfrak{p}}(a)$ we denote the order of $a$ at $\mathfrak{p}$. Given $a, b \in F^*$, we use Hilbert symbol $(a, b)_{\mathfrak{p}}$, which is defined to be $+1$ if $ax^2 + by^2 = 1$ is solvable in $F_{\mathfrak{p}}$, otherwise defined to be $-1$. For $a, b \in F^*$ we denote by $S_F(a, b)$ the set of places $\mathfrak{p}$ of $F$ such that $(a, b)_{\mathfrak{p}} = -1$. We know that it contains even number of places of $F$.

The following lemma is well-known.

**Lemma 1** A nonzero element $h$ of $F$ can be represented by the ternary quadratic form $x^2 - ay^2 - bz^2$ in $F$ if and only if $h/(-ab) \notin F_{\mathfrak{p}}^{*2}$ for any place $\mathfrak{p}$ such that $(a, b)_{\mathfrak{p}} = -1$.

This follows from the properties of quaternary quadratic forms and the Hasse-Minkowski theorem on quadratic forms. See [7, p. 187].

**Lemma 2** Given even number of distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_{2k}$ of $F$ there are $a$ and $b$ in $F^*$ such that $S_F(a, b) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{2k}\}$ and $\nu_{\mathfrak{p}_i}(a) = 1$, $\nu_{\mathfrak{p}_i}(b) = 0$ for $i = 1, \ldots, 2k$.

**Proof.** By weak approximation, we get an element $a$ of $F^*$ with $\nu_{\mathfrak{p}_i}(a) = 1$ for all $i$. We know by [7, 71:19, Theorem p. 203] that there is $b \in F^*$ such that $S_F(a, b) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{2k}\}$.

J. Robinson actually proved in [8, Lemma 9] that given a prime ideal $\mathfrak{p}_1$ of $F$ there are relatively prime elements $a$ and $b$ in $\mathfrak{o}_F$ such that $(a) = \mathfrak{p}_1 \cdots \mathfrak{p}_{2k}$, where $\mathfrak{p}_1, \ldots, \mathfrak{p}_{2k}$ are distinct prime ideals that include every prime ideal dividing 2, and $b$ is a totally positive prime element such that $(a, b)_{\mathfrak{p}} = -1$ iff $\mathfrak{p}|a$.

**Lemma 3** Let $a, b, c \in F$. If $a$ and $b$ satisfy Lemma 2 and $m$ be a positive integer such that $\mathfrak{p}^m \not| 2$ for every prime ideal $\mathfrak{p}$. Then

$$1 - abc^{2m} = x^2 - ay^2 - bz^2$$

is solvable for $x, y$ and $z$ in $F$ iff $\nu_{\mathfrak{p}_i}(c) \geq 0$ for each $i$.

**Proof.** By Lemma 1, $h = 1 - abc^{2m}$ can be represented by $x^2 - ay^2 - bz^2$ iff $h/(-ab) \notin F_{\mathfrak{p}_i}^{*2}$ for $1 \leq i \leq 2k$.

If $\nu_{\mathfrak{p}_i}(c) \geq 0$ for each $i$, then we have $\nu_{\mathfrak{p}_i}(h/(-ab)) = -1$, hence $h/(-ab)$ is not a square of $F_{\mathfrak{p}_i}$ for each $i$. 


Suppose \( \nu_p(c) < 0 \) for some \( i \). We know in \( F_p \) that \( (1 + p^r)^2 = 1 + 2p^r \) if \( p^r \subseteq 2p \) by [7, p. 163]. Noting \( h/(-ab) = c^{2m}(1 - 1/(abc^{2m})) \), we see that \( h/(-ab) \) is a square of \( F_p \) since \( \nu_p((1/(abc^{2m})) \geq 2m - 1 \) and \( p^{2m-1} \subseteq 2p \). □

Thus we have that if \( a \) and \( b \) satisfy Lemma 2, \( \varphi_m(a, b, F) = \bigcap_{1 \leq i \leq 2k} \mathfrak{O}_p \), and \( \forall c(\varphi_m(a, b, c) \rightarrow \varphi_m(a, b, c + 1) \) holds in \( F \) since \( \varphi_m(a, b, F) \) is a ring containing \( \mathbb{Z} \).

For a given \( c \in F^* \) there are \( a, b \in F^* \) such that \( c \not\in \varphi_m(a, b, F) \) since we can construct \( a, b \in F^* \) such that \( 1 - 1/(abc^{2m}) \) is a square of \( F_p \) for some \( p \) with \( (a, b)_p = -1 \). Noting \( 0 \notin \varphi_m(a, b, F) \) for all \( a, b \) we have \( \bigcap_{a, b \in F} \varphi_m(a, b, F) = 0 \).

Nevertheless we have that \( \psi_m(F) \) is a subset of \( \mathfrak{o}_F \) containing \( \mathbb{Z} \) since \( \psi_m(F) \) is the intersection of all the solution set of

\[
\forall c(\varphi_m(a, b, c) \rightarrow \varphi_m(a, b, c + 1)) \rightarrow \varphi_m(a, b, t).
\]

If the premise of the above formula fails, the solution set is \( F \).

We don't know what \( \psi_m(F) \) is. But we can show what \( \psi_2(K) \) is, if \( K \) is a certain infinite algebraic extension of \( \mathbb{Q} \).

**Remark 4** For a given prime ideal \( p_1 \) we can define the valuation ring of \( p_1 \). Take three prime ideal \( p_1, p_2, p_3 \) of \( F \) and \( a, b, c, d \in \mathfrak{o}_F \) such that \( S_F(a, b) = \{p_1, p_2\} \) and \( S_F(c, d) = \{p_1, p_3\} \), then we easily see that \( \varphi_m(a, b, F) + \varphi_m(c, d, F) \) defines \( \mathfrak{o}_{p_1} \).

### 3 The solution set of \( \psi(t) \) in some infinite algebraic extensions

Let \( F \) be a number field and let \( \mathcal{F} \) be an infinite set of finite Galois extensions \( M \) of \( F \) such that \( [M : F] \) is odd and every prime ideal of \( M \) dividing 2 is unramified in \( M/\mathbb{Q} \). (We say that 2 is unramified in \( M/\mathbb{Q} \). Note \( p^2 \nmid 2 \) for all prime ideals \( p \) of \( M \).) Let \( K \) be the composite field of all fields in \( \mathcal{F} \). Then \( K \) is an infinite Galois extension of \( F \) and every finite Galois subextension \( M \) has odd extension degree over \( \mathbb{Q} \). We denote by \( \mathfrak{O}_K \) the ring of algebraic integers of \( K \).

In this section we will prove that the solution set \( \psi_2(K) \) of \( \psi_2(t) \) in \( K \) is a subset of \( \mathfrak{O}_K \) containing \( \mathbb{Z} \).

We need the following lemma, which is proved in [2, pp. 272,337].

**Lemma 5** Let \( M, L \) be number fields with \( L \supset M \) and let \( \mathfrak{P} \supset \mathfrak{p} \) be primes of \( L \) and \( M \) respectively. For \( \alpha \in L_{\mathfrak{p}}, \) let \( a = N_{L_{\mathfrak{p}}/M_{\mathfrak{p}}}(\alpha) \) and \( b \in M_{\mathfrak{p}} \). Then we have \( (\alpha, b)_{\mathfrak{p}} = (a, b)_\mathfrak{p} \).

The next lemma follows from Lemma 5.
Lemma 6 Let $L$ be a finite Galois extension of a number field $M$ with $[L : M]$ odd. Let $\mathfrak{p}$ be a prime ideal of $M$ and let $\mathfrak{P}$ be a prime of $L$ lying over $\mathfrak{p}$. Then for $a, b \in M^*$, we have $(a, b)_{\mathfrak{p}} = 1$ iff $(a, b)_{\mathfrak{P}} = 1$.

Proof. Since $L/M$ is a Galois extension, the local degree at $\mathfrak{P}$ divides the degree of $L/M$, that is, $[(L)_{\mathfrak{P}} : (M)_{\mathfrak{p}}][[L : M]$ (see [7, p. 32]). Let $u$ be the local degree at $\mathfrak{P}$. Then $N_{(L)_{\mathfrak{P}}/(M)_{\mathfrak{p}}}(a) = a^u$ and $(a, b)_{\mathfrak{P}} = (a^u, b)^u = (a, b)_{\mathfrak{p}}^u$. Since $u$ is odd, it follows that $(a, b)_{\mathfrak{p}} = 1$ iff $(a, b)_{\mathfrak{P}} = 1$. \hfill \qed

We recall that $\varphi_2(s, u, t)$ is

$$ \exists x, y, z(1 - sut^4 = x^2 - sy^2 - uz^2) $$

and $\psi_2(t)$ is

$$ \forall s, u(\forall c(\varphi(s, u, c) \rightarrow \varphi(s, u, c + 1)) \rightarrow \varphi_2(s, u, t)). $$

Lemma 7 Let $M$ be a subfield of $K$ with $M/F$ finite and Galois. Let $a, b, \alpha \in M$ with $ab \neq 0$. Then

$$ M \models \varphi(a, b, \alpha) \iff K \models \varphi(a, b, \alpha). $$

Proof. If $M \models \varphi(a, b, \alpha)$, then we have trivially $K \models \varphi(a, b, \alpha)$.

If $M \models \neg \varphi(a, b, \alpha)$, then $(1 - ab \alpha^4)/(ab) \in M^*_p$ for some $p$ a place of $M$ such that $(a, b)_p = -1$. Let $L$ be any subfield of $K$ with $L/M$ finite and Galois and let $\mathfrak{P}$ be a place of $M$ lying above $\mathfrak{p}$. Since $[L : M]$ is odd we have $(a, b)_{\mathfrak{P}} = -1$ and $(1 - ab \alpha^4)/(ab) \in L^*_\mathfrak{P}$. Hence $L \models \neg \varphi(a, b, \alpha)$ and $K \models \neg \varphi(a, b, \alpha)$. Note that for archimedean places $\mathfrak{p} \subset \mathfrak{P}$, it is also true that $(a, b)_\mathfrak{p} = 1$ iff $(a, b)_{\mathfrak{P}} = 1$. \hfill \qed

Theorem 8 The solution set $\psi_2(K)$ of $\psi_2(t)$ in $K$ is a subset of $\mathcal{D}_K$ containing $\mathbb{Z}$ ($\mathbb{Z} \subseteq \psi_2(K) \subseteq \mathcal{D}_K$).

Proof. We have trivially $\mathbb{Z} \subseteq \psi_2(K)$. Let $t \in K \setminus \mathcal{D}_K$. We show that there are $a, b \in K$ such that

$$ K \models \neg \varphi_2(a, b, t) \land \forall c(\varphi_2(a, b, c) \rightarrow \varphi_2(a, b, c + 1)). $$

We fix a subfield $M$ of $K$ such that $[M : F]$ is finite and $t \in M$. Then we have $\nu_{\mathfrak{p}_i}(t) < 0$ for some prime $\mathfrak{p}_i$ of $M$. Take a prime $\mathfrak{p}_2 \neq \mathfrak{p}_1$ of $M$. By Lemma 2, there are $a$ and $b$ in $M^*$ such that $\nu_{\mathfrak{p}_1}(a) = 1$, $\nu_{\mathfrak{p}_1}(b) = 0$ and $(a, b)_{\mathfrak{p}_i} = -1$ for $i = 1, 2$, and $t \not\in \varphi_2(a, b, M)$. By Lemma 7, $1 - abt^4 = x^2 - ay^2 - bz^2$ is not solvable for $x, y, z$ in $K$.

Let $c$ in $K$ and suppose $K \models \varphi_2(a, b, c)$. Take a subfield $L$ of $K$ such that $L$ contains $c$ and $L/M$ is a finite Galois extension, then we have $L \models \varphi_2(a, b, c)$ by
Lemma 7. Let $h = 1 - abc^4$ and $h' = 1 - ab(c+1)^4$. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ be all the primes of $L$ lying above $\mathfrak{p}_1$ and $\mathfrak{P}_{k+1}, \ldots, \mathfrak{P}_{k+s}$ be all the primes of $L$ lying above $\mathfrak{p}_2$. By Lemma 5, we have $S_L(a, b) = \{\mathfrak{P}_1, \ldots, \mathfrak{P}_{k+s}\}$, that is, $\mathfrak{P}_i$ are all the primes $\mathfrak{P}$ of $L$ such that $(a, b)_{\mathfrak{P}} = -1$. $k$ and $s$ are odd since $L/M$ is Galois with odd extension degree. We will show that for all $\mathfrak{P}_i$, $h'(-ab)$ is not a square of $L^\mathfrak{P}_i$, assuming $h/(-ab)$ is not. Take one $\mathfrak{P} = \mathfrak{P}_i$. We will break into cases according to whether or not $\mathfrak{P}$ divides 2.

Case 1: $\mathfrak{P} 
/2.$

As mentioned before we have $(1 + \mathfrak{P})^2 = 1 + 2\mathfrak{P}$ if $\mathfrak{P} \subseteq 2\mathfrak{P}$ by [7, p. 163]. Hence we have $(1 + \mathfrak{P})^2 = 1 + \mathfrak{P}$. If $\nu_{\mathfrak{P}}(c) \geq 0$, then $h' = 1 - ab(c+1)^4$ is a square of $L_{\mathfrak{P}}$ since $\nu_{\mathfrak{P}}(-ab(c+1)^4) > 0$. Since $(a, b)_{\mathfrak{P}} = (a, -ab)_{\mathfrak{P}} = -1$ we have $-ab$ is not a square of $L_{\mathfrak{P}}$, hence $h'/(-ab)$ is also not.

We consider the case $\nu_{\mathfrak{P}}(c) < 0$. Since $h/(-ab) = c^4(1 - 1/(abc^4)$ it follows that $\nu_{\mathfrak{P}}(-abc^4) \geq 0$. Let $\mathfrak{P}$ lie above $\mathfrak{p}_i$ and let $e = e(\mathfrak{P}/\mathfrak{p}_i)$ be the ramification index of $\mathfrak{P}$. $e$ must be odd since $L/M$ is Galois with odd extension degree. Hence we have $\nu_{\mathfrak{P}}(-abc^4) > 0$. Then we have $\nu_{\mathfrak{P}}(-ab(c+1)^4) = \nu_{\mathfrak{P}}(-ab) + 4\nu_{\mathfrak{P}}(c) = \nu_{\mathfrak{P}}(-abc^4) > 0$, hence $h' = 1 - ab(c+1)^4$ is a square of $L_{\mathfrak{P}}$ and $h'/(-ab)$ is not.

Case 2: $\mathfrak{P}|2.$

Since 2 is unramified in $L/Q$ we have $\nu_{\mathfrak{P}}(2) = 1$ and $\nu_{\mathfrak{P}}(-ab) = 1$. Furthermore we know $(1 + \mathfrak{P})^2 = 1 + \mathfrak{P}^3$ by [7, p. 163]. If $\nu_{\mathfrak{P}}(c) < 0$ then $h/(-ab) = c^4(1 - 1/(abc^4)$ would be a square of $L_{\mathfrak{P}}$, hence we have $\nu_{\mathfrak{P}}(c) \geq 0$. It follows that $\nu_{\mathfrak{P}}(h'/(-ab)) = -1$ and $h'/(-ab)$ is not a square of $L_{\mathfrak{P}}$.

Example 9 1. Let $F = \mathbb{Q}((\zeta_l))$ and $\mathfrak{F}$ be a set of all $M_n = \mathbb{Q}(\zeta_{(n)}$ ($n > 1$), where $l$ is an odd integer $> 1$ and $\zeta_{(n)}$ is a primitive $l^n$-th root of unity. $K_n = \bigcup_n M_n$.

2. Let $F = \mathbb{Q}$ and $\mathcal{F}$ be a set of all $\mathbb{Q}(\cos(2\pi/l^n))$, where $n \in \mathbb{N}$ and $l$ is an odd prime with $l \equiv -1$ (mod 4). $K = \mathbb{Q}(\{\cos(2\pi/l^n) : n \in \mathbb{N}, l$ a prime, $l \equiv -1$ (mod 4))

Remark 10 In the proof of Theorem 8, we have $\varphi_2(a, b, M) = \mathcal{O}^{M}_{\mathfrak{p}_1} \cap \mathcal{O}^{M}_{\mathfrak{p}_2}$. Here $\mathcal{O}^{M}_{\mathfrak{p}_i}$ denotes the valuation ring of $\mathfrak{p}_i$ in $M$. But it is not necessarily true that $\varphi_2(a, b, L) = \mathcal{O}^{\mathfrak{p}_i}_{\mathfrak{p}_i}$. Actually we have $\varphi_2(a, b, M) \subseteq \mathcal{O}_{\mathfrak{P}_i} \subseteq \varphi_2(a, b, L)$.

Nevertheless we can prove $\varphi_2(a, b, L) = \bigcap_{i} \mathcal{O}_{\mathfrak{P}_i}$ for $K = \bigcup_n \mathbb{Q}(\zeta_{(n)}$, where $l$ is an odd prime and $\zeta_{in}$ is a primitive $l^n$-th root of unity.

4 The structure of $\psi(K)$

In this section we let $F = \mathbb{Q}$, that is, let $K$ be the composite of all fields in $\mathfrak{F}$ where $\mathfrak{F}$ is a set of infinitely many finite Galois extensions $M$ of $\mathbb{Q}$ such that $[M : \mathbb{Q}]$ is odd
and 2 is unramified in $M/\mathbb{Q}$. We let $\mathcal{F}$ be the family of all finite Galois subextensions of $K$. Then every $M$ also has odd extension degree over $\mathbb{Q}$ and 2 is unramified in $M/\mathbb{Q}$. We write $\varphi$ and $\psi$ instead of $\varphi_2$ and $\psi_2$ respectively.

We shall investigate what $\psi(K)$ is. For $a, b \in K$ we let $T_{a,b}$ be the set of elements $\alpha$ of $K$ such that

$$K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \rightarrow \varphi(a, b, \alpha).$$

Then we have $\psi(\mathcal{O}_K) = \bigcap_{a,b \in K} T_{a,b}$. We easily see $T_{a,b} = K$ for $a,b$ with $ab = 0$. So we shall investigate what $T_{a,b}$ is, for $a, b \in K^*$. We recall that for $a, b \in M^*$, $M \models \neg \varphi(a, b, \alpha)$ iff $\alpha^4 - 1/ab \in M_{p}^{*2}$ for some $p \in S_M(a,b)$. Hence we easily see the following: for $a, b \in K^*$, if $S_M(a,b) = \emptyset$ for some $M \in \mathcal{F}$ with $a, b \in M$, then $\varphi(a, b, K) = T_{a,b} = K$ by Lemma 6. So we shall investigate what $T_{a,b}$ is, for $a, b \in K^*$ such that for some $M \in \mathcal{F}$ with $a, b \in M$, $S_M(a,b) \neq \emptyset$.

From now on we use the following notation. For a number field $M$, the ring of integers of $M_p$ is denoted by $(\mathfrak{o}_M)_p$, its maximal ideal is also denoted by $\mathfrak{p}$, its residue field $(\mathfrak{o}_M)_p/\mathfrak{p}$ by $(\tilde{M})_p$, and the group of units of $(\mathfrak{o}_M)_p$ by $(U_M)_p$. For $\alpha \in \mathcal{M}$, we denote by $\overline{\alpha}$ its residue class in $(\tilde{M})_p$. Furthermore we usually let $\mathfrak{p}$ lie above a rational prime $p$. Note that $(\tilde{M})_p \simeq \mathfrak{o}_M/\mathfrak{p} \simeq \mathbb{F}_p^f$, where $f$ is the residue degree of $M$ at $\mathfrak{p}$.

**Lemma 11** Let $a, b \in K^*$ such that

$$K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$$

holds. Then for every $M \in \mathcal{F}$ with $a, b \in M$, every $\mathfrak{p} \in S_M(a,b)$ is not archimedean.

This is proved similarly as Lemma 14 in [1].

**Lemma 12** Let $M \in \mathcal{F}$. Let $a, b \in M^*$, $\alpha \in \mathfrak{o}_M$ and $\mathfrak{p}_0 \in S_M(a,b)$ with $\mathfrak{p}_0 \not\parallel 2$ such that

1. $K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$ and
2. $\alpha^4 - 1/ab \in M_{p_0}^{*2}$ hold.

Then $\nu_{\mathfrak{p}_0}(-ab) = 0$ and $\nu_{\mathfrak{p}_0}(\alpha) = 0$.

This is also proved similarly as Lemma 15 in [1].

Now we will prove the following lemma on finite fields.

**Lemma 13** Let $p$ be an odd prime and $q = p^f$. Let $\mathbb{F}_q$ be a finite field with $q$ elements other than $\mathbb{F}_3, \mathbb{F}_5$. We let $\eta$ be the quadratic character of $\mathbb{F}_q$, that is, $\eta(0) = 0, \eta(c) = 1$ if $c \in \mathbb{F}_q^{*2}$ and $\eta(c) = -1$ otherwise.

Then for all $a \in \mathbb{F}_q^*$ with $\eta(a) = -1$,

1. there are $b \in \mathbb{F}_q$ and $j \in \mathbb{F}_p$ such that $\eta(b^4 + a)\eta((b + j)^4 + a) = -1$.

Exceptional cases are, $\mathbb{F}_3$ and $a = 2$, and, $\mathbb{F}_5$ and $a = 2$. 

\[\text{Lemma 11} \quad \text{Lemma 12} \quad \text{Lemma 13}\]
Proof. We will first prove the following; for all \( a \in \mathbb{F}_q^* \) with sufficiently large \( q \), we can take \( j = 1 \) in the statement \((\ddagger)\). We use Weil's Theorem [5, p. 225, Theorem 5.41], from which we have that for \( a \in \mathbb{F}_q^* \),

\[
\left| \sum_{c \in \mathbb{F}_q} \eta\{(c^4 + a)((c + 1)^4 + a)\} \right| \leq 7q^{1/2}.
\]

Thus if \( q \) satisfies inequality \( 7q^{1/2} < q - 8 \) then for all \( a \in \mathbb{F}_q^* \) there is \( b \in \mathbb{F}_q \) such that \( \eta(b^4 + a)\eta((b + 1)^4 + a) = -1 \). Hence for all \( \mathbb{F}_q \) with \( q > 64 \) the assertion holds. For the small values of \( q \leq 64 \) we can check the assertion directly.

Note that in the statement \((\ddagger)\) we cannot always take \( j = 1 \) if \( q \leq 64 \); for example in \( \mathbb{F}_7 \) there is no \( b \) such that \( \eta(b^4 + 5)\eta((b + 1)^4 + 5) = -1 \) but in \( \mathbb{F}_7 \) \( \eta(1^4 + 5)\eta((1 + 2)^4 + 5) = -1 \) holds. Note also that we need the assumption \( \eta(a) = -1 \) for \( \mathbb{F}_9 \) since for \( a = 1, 2 \), for which \( \eta(a) = 1 \), the statement \((\ddagger)\) dose not hold.

Lemma 14 Let \( M \in \mathcal{F} \). Let \( a, b \in M^* \). Suppose that \( S_M(a, b) \) contains a non-archimedean place \( p_0 \) such that \( p_0 \nmid 2, \nu_{p_0}(-ab) = 0 \) and \( (\bar{M})_{p_0} \neq \mathbb{F}_3, \mathcal{F}_5 \).

Then \( K \models \neg\forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \).

The proof is similar to that of Lemma 16 in [1].

Proposition 15 Let \( M \in \mathcal{F} \). For \( a, b \in M^* \), if \( S_M(a, b) \) contains no primes dividing 2, then we have \( \mathcal{O}_K \subseteq T_{a, b} \), that is,

\[
K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \rightarrow \varphi(a, b, \alpha) \quad \text{for all} \quad \alpha \in \mathcal{O}_K.
\]

Proof. We first note the following; if we take \( N \in \mathcal{F} \) such that \( a, b \in N^* \) then \( S_N(a, b) \) also contains no primes dividing 2 by Lemma 6. Suppose not. Then there is \( \alpha \in \mathcal{O}_K \) such that

\[
K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \quad \text{but} \quad K_l \models \neg\varphi(a, b, \alpha).
\]

Take \( N \in \mathcal{F} \) such that \( a, b, \alpha \in N \). We have by Lemma 7,

\[
N \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \quad \text{but} \quad N \models \neg\varphi(a, b, \alpha).
\]

Then there is a \( p_0 \in S_N(a, b) \) such that \( \alpha^4 - 1/ab \in N_{p_0}^* \).

We see that \( p_0 \) is not archimedean by Lemma 11 and that \( \nu_{p_0}(-ab) = 0 \) and \( \nu_{p_0}(\alpha) = 0 \) by Lemma 12. If \( (\bar{N})_{p_0} \neq \mathbb{F}_3, \mathcal{F}_5 \), we get a contradiction by Lemma 14.

Suppose that \( (\bar{N})_{p_0} = \mathcal{F}_5 \). Since \( (a, b)_{p_0} = -1 \) and \( N \models \psi(1) \), we have \( -1/ab \in N_{p_0}^* \) and \( 1 - 1/ab \in N_{p_0}^* \), hence \( -1/ab \equiv 2 \mod p_0 \). Since \( \nu_{p_0}(\alpha) = 0 \), we have
\[ \alpha^4 \equiv 1 \pmod{p_0}. \] Then we have \( \alpha^4 - 1/ab \equiv 3 \pmod{p_0} \), hence \( \alpha^4 - 1/ab \not\in N_{p_0}^{*2} \), a contradiction.

Suppose that \( (\tilde{N})_{p_0} = \mathcal{F}_3 \). We first deal with the case where \( p_0 \) is not ramified in \( N/\mathbb{Q} \). Then \( 3 \) is a prime element of \( N_{p_0} \) and we can write \( -1/ab = 2 + s_13 + s_23^2 + \cdots \), where \( s_i \in \{0, 1, 2\} \). We note that \( N \models \varphi(a, b, n) \) for all \( n \in \mathbb{N} \). If \( s_1 = 0 \), then \[ 2^4 - 1/ab = (s_2 + 2)3^2 + \cdots, 7^4 - 1/ab = s_23^2 + \cdots \] and \( 11^4 - 1/ab = (s_2 + 1)3^2 + \cdots \). Thus we have one of these three must be contained in \( N_{p_0}^{*2} \), a contradiction. Likewise if \( s_1 = 1 \), then \( 4^4 - 1/ab = (s_2 + 2)3^2 + \cdots, 13^4 - 1/ab = s_23^2 + \cdots \) and \( 5^4 - 1/ab = (s_2 + 1)3^2 + \cdots \) and \( 10^4 - 1/ab = (s_2 + 1)3^2 + \cdots \). Thus in the case where \( p_0 \) is not ramified in \( N/\mathbb{Q} \), we get contradictions.

Secondly We deal with the case where \( p_0 \) is ramified in \( N/\mathbb{Q} \). Let \( \nu_{p_0}(3) = e \) and let \( \pi \) be a prime element of \( N_{p_0} \). We can write \( -1/ab = 2 + s_1\pi + s_2\pi^2 + \cdots \), where \( s_i \in \{0, 1, 2\} \). We may write \( \alpha = 1 + c_1\pi + c_2\pi^2 + \cdots \) where \( c_i \in \{0, 1, 2\} \), since if \( \alpha \equiv 2 \pmod{p_0} \) then \( -\alpha \equiv 1 \pmod{p_0} \). Since \( N \models \neg \varphi(a, b, \alpha) \), we have \( N \models \neg \varphi(a, b, \alpha - n) \) for all \( n \in \mathbb{N} \). But \( (\alpha - 1)^4 - 1/ab \equiv 2 \pmod{p_0} \), hence there must be another prime \( p_1 \in S_N(a, b) \) with \( (\alpha - 1)^4 - 1/ab \in N_{p_1}^{*2} \). \( p_1 \) must be a prime lying above \( 3 \) and \( \alpha \equiv 2 \pmod{p_1} \). And we have \( (\alpha - (3k + 1))4 - 1/ab \equiv 2 \pmod{p_0} \) and \( (\alpha - (3k + 2))4 - 1/ab \equiv 0 \pmod{p_0} \). Likewise \( (\alpha - (3k + 1))4 - 1/ab \equiv 0 \pmod{p_1} \) and \( (\alpha - (3k + 2))4 - 1/ab \equiv 2 \pmod{p_1} \). Since there are finitely many primes in \( S_N(a, b) \), we must have for some \( k \) \( (\alpha - (3k + 2))4 - 1/ab \equiv 0 \pmod{p_0} \) and \( (\alpha - (3k + 2))4 - 1/ab \in N_{p_0}^{*2} \).

We have \( s_1 + c_1 \equiv 0 \pmod{p_0} \) since \( \alpha^4 - 1/ab = (s_1 - c_1)\pi + \cdots \). And we have \( s_1 - c_1 \equiv 0 \pmod{p_0} \) since \( (\alpha - (3k + 2))4 - 1/ab = (s_1 - c_1)\pi + \cdots \). Thus we have \( s_1 \equiv 0 \pmod{p_0} \) and \( c_1 \equiv 0 \pmod{p_0} \). Likewise we have \( s_2 \equiv 0 \pmod{p_0} \) and \( c_2 \equiv 0 \pmod{p_0} \). We can proceed to \( \pi^{e-1} \). It follows that \( -1/ab = 2 + s_e\pi^e + s_{e+1}\pi^{e+1} + \cdots \). Then we have \( 2^4 - 1/ab = (s_e + 2)3^2 + \cdots, 7^4 - 1/ab = s_e3^2 + \cdots \) and \( 11^4 - 1/ab = (s_e + 1)3^2 + \cdots \), a contradiction. \( \square \)

We will deal with primes dividing \( 2 \).

**Lemma 16** Let \( M \in \mathcal{F} \). Let \( a, b \in M^{*}, \alpha \in \mathfrak{o}_M \) and \( p_0 \in S_M(a, b) \) with \( p_0 \mid 2 \) such that

1. \( K \models \forall c(\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \) and
2. \( \alpha^4 - 1/ab \in M_{p_0}^{*2} \) hold.

Then \( \nu_{p_0}(-ab) = \pm 2 \).

The proof is similar to that of Lemma 18 in [1].

We shall prove a similar result to Lemma 14.
Lemma 17 Let $M \in \mathcal{S}$ and $a, b \in M^*$. Suppose that $S_n(a, b)$ contains a $p_0$ such that $p_0 | 2$ and $\nu_{p_0}(-ab) = -2$.

Then $K_l \models \neg \forall c (\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1))$.

The proof is similar to that of Lemma 19 in [1].

Thus we get the following proposition. The proof is similar to that of Proposition 15.

Proposition 18 Let $l$ be an odd prime such that $l \equiv -1 \pmod{4}$. For $a, b \in F_n^*$, if $S_n(a, b)$ contains no primes $p$ such that $p | 2$ and $\nu_p(-ab) = 2$, then we have $\mathfrak{O}_{K_l} \subseteq T_{a, b}$, that is,

$$K_l \models \forall c (\varphi(a, b, c) \rightarrow \varphi(a, b, c + 1)) \rightarrow \varphi(a, b, \alpha) \text{ for all } \alpha \in \mathfrak{O}_{K_l}.$$ 

Since $\psi(K) = \bigcap_{(a, b) \in \Delta} T_{a, b}$, Proposition 18 implies $\psi(K) = \bigcap_{(a, b) \in \Delta} T_{a, b}$, where $\Delta$ is the set of $(a, b) \in K^* \times K^*$ such that for some $M$ with $a, b \in M$, $S_M(a, b)$ contains a prime $p$ with $p | 2$ and $\nu_p(-ab) = 2$. Such $a$ and $b$ exist, for example, let $a = 2$ and $b = 10$.

Let $M \in \mathcal{S}$ and $(2) = p_1 \cdots p_k$ in $M$. Put $P_M = \bigcap_{i}((1 + p_i) \cup p_i)$. Then $P_M$ is a subring of $\mathcal{O}_M$ containing 1. Let $P_K = \bigcup\{P_M : M \in \mathcal{S}\}$. $P_K$ is a subring of $\mathcal{O}_K$ containing 1.

Theorem 19 $\psi(K) = P_K$.

The proof is similar to that of Proposition 20 in [1].

Example 20

1. $K_l = \bigcup_n \mathbb{Q}(\cos(2\pi/l^n))$ with $l$ a prime and with $l \equiv -1 \pmod{4}$.

2. $K_W = \prod_{l \in W} K_l$. ($W = \{l \text{ a prime : } l \equiv -1 \pmod{4}\}$)

3. $K_0 = \mathbb{Q}(\{\cos(2\pi/l) : l \text{ a prime, } l \equiv -1 \pmod{4}\})$.

5 Undecidability results

Let $K_l = \bigcup_n \mathbb{Q}(\cos(2\pi/l^n))$. In [1] we proved that if $l$ is a prime such that $l \equiv -1 \pmod{4}$ and 2 is a prime of $\mathcal{O}_{K_l}$, then $K_l$ is undecidable. But in 2000 C.R. Videla [12] proved that $K_l$ is undecidable for every prime $l$. He considered $K/F$ a pro-$p$ Galois extension over a number field $F$ and using Rumely’s formula in [6] he proved that $\mathcal{O}_{K_l}$ is definable with parameters. Then he also used the results of Kronecker and J. Robinson.

Kronecker [3] determined all sets of conjugate algebraic integers in the interval $c - 2 \leq x \leq c + 2$, provided that $c$ is a rational integer; they have the form

$$x = c + 2\cos(2k\pi/m) \text{ with } 0 \leq k \leq m/2 \text{ and } (k, m) = 1.$$
Note that if \( m = 1, 2, 3, 4 \), then \( x = c + 2, c - 2, c \pm 1, c \) respectively. Furthermore it is known that an interval of length less than 4 can contain only finitely many complete sets of conjugate algebraic integers. (See [11].)

Therefore we see that the interval \((0, 4)\) contains infinitely many complete conjugate sets of totally real algebraic integers and that no sub-interval does.

These facts are used by J. Robinson in [9]. Her results concerns the integral closure of \( \mathbb{Z} \) inside totally real fields, not necessarily finite over \( \mathbb{Q} \). She calls such a ring a totally real algebraic integer ring. In 1962 she proved the following: The natural numbers can be defined arithmetically in any totally real algebraic integer ring \( A \) such that there is a smallest interval \((0, s)\) with \( s \) real or \( \infty \), which contains infinitely many complete conjugate sets of numbers of \( A \). But we can say more. We recall that \( \mathbb{Z}^{ir} \) denotes the ring of all totally real algebraic integers.

**Theorem 21** Let \( R \) be a subring of \( \mathbb{Z}^{ir} \) containing \( \mathbb{Z} \) such that there is a smallest interval \((0, s)\) with \( s \) real or \( \infty \), which contains infinitely many complete conjugate sets of numbers of \( R \). Here \( s \) need not be in \( R \). Then \( \mathbb{N} \) is definable in \( R \).

In particular such a ring is undecidable.

The proof of J. Robinson just works. See [9, pp. 300–301].

Thus it follows that for every positive integer \( l > 1 \), \( \mathcal{O}_{K_{l}} \) is undecidable, from which Videla proved that \( K_{l} \) is undecidable. Note that even if the defining formula contains parameters it is possible to define \( \mathbb{N} \). See [12].

We give alternative proof of this fact in the case where \( l \) is a prime with \( l \equiv -1 \) (mod 4). We know that \( \psi(K_{l}) \) is a subring of \( \mathbb{Z}^{ir} \) containing \( \mathbb{Z} \) if \( l \) is a prime such that \( l \equiv -1 \) (mod 4). Furthermore we know by [11, p. 312], that \( 2 + 2 \cos(2\pi/l^n) \) are units in \( \mathcal{O}_{K_{l}} \), and that \( 1 + 2 \cos(2\pi/l^n) \) are units in \( \mathcal{D}_{K_{l}} \) if \( l \neq 3 \), and \( |N_{F_n/\mathbb{Q}}(1 + 2 \cos(2\pi/3^n))| = 3 \) for \( n \geq 2 \). Hence we see that \( 2 + 2 \cos(2\pi/l^n) \) are not in \( \psi(K_{l}) \) if \( l^n \neq 3 \). On the other hand \( 4 + 4 \cos(2\pi/l^n) \) are in \( \bigcap_{i} \mathcal{P}_{i}^{(2)} \), hence in \( \psi(K_{l}) \). Thus we see that the interval \((0, 8)\) contains infinitely many complete conjugate sets of numbers of \( \psi(K_{l}) \) and the interval \((0, 4)\) does not. We show that \((0, 8)\) is actually such a smallest interval for \( \psi(K_{l}) \).

**Lemma 22** Let \( l \) be an odd prime such that \( l \equiv -1 \) (mod 4). Then \((0, 8)\) is a smallest interval of the form \((0, c)\) which contains infinitely many complete conjugate sets of numbers of \( \psi(K_{l}) \).

**Proof.** We know that \( K_{l} \) has only finitely many primes lying above 2. (See Lemma 13 in [1].) Thus \( \psi(K_{l}) = P_{K_{l}} = \bigcap_{i}((1 + \mathcal{P}_{i}) \cup \overline{\mathcal{P}_{i}}) \), where \( \mathcal{P}_{1}, \ldots, \mathcal{P}_{k} \) are primes of \( K_{l} \) lying above 2. We easily see that \( \psi(K_{l}) \) is a union of \( 2^k \) cosets of \( \mathcal{D}_{K_{l}}/2\mathcal{D}_{K_{l}} \).

Suppose that \((0, 8)\) is not such a smallest interval. Then some interval \((0, \delta)\) with \( \delta < 8 \) contains infinitely many complete conjugate sets of numbers of \( \psi(K_{l}) \). Then we have that some coset, say \( \alpha + 2\mathcal{D}_{K_{l}} \), contains infinitely many complete conjugate
sets of numbers. It follows that an interval of length less than 4 contains infinitely many complete conjugate sets of algebraic integers, a contradiction.

Let \( K_\Delta = \prod_{\ell \in \Delta} K_\ell \) where \( \Delta \) is a finite set of primes. From the result of Videla we deduce that \( K_\Delta \) is undecidable. If \( \Delta \) is a finite set of primes with \( l \equiv -1 \pmod{4} \), then we can give another proof similarly.

Nevertheless we can give a new undecidable infinite algebraic extension of \( \mathbb{Q} \) by our method. Let \( V \) be a set of Sophie Germain primes, that is, a prime \( p \) such that \( 2p + 1 \) is again a prime. It is considered that there are infinitely many Sophie Germain primes but it is not proved. Let \( K_V = \mathbb{Q}(\{\cos(2\pi/l) : l \in V\}) \). Then we have \( \psi(K_V) = (1 + 2\mathfrak{O}_{K_V}) \cup \mathfrak{O}_{K_V} \), hence \( K_V \) is undecidable.

References


