On superstable generic structures

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This manuscript is an expansion of my talk at Kirishima meeting. In this talk, we mainly gave a counter-example of Baldwin’s question. Proofs of our results can be found in [18]. So we do not explain all of those details here.

1 Baldwin’s question

Many papers [5, 8, 10, 11, 12, 19, 21, 25] have laid out the basics of generic structures in various situations. In particular, this manuscript was influenced by papers of Wagner [25] and Baldwin-Shi [8].

Generic structures Let $L$ be a countable relational language. Let $K$ be a class of finite $L$-structures that is closed under substructures. Let $\leq$ be a reflexive and transitive relation on $K$ satisfying the following:

(C1) $A \leq B \in K$ implies $A \subset B$;

(C2) $A \leq B \leq C \in K$ implies $A \leq C$;

(C3) $A, B \leq C \in K$ implies $A \cap B \leq C$;

(C4) $A \in K$ implies $\emptyset \leq A$.

Then, for each $A, B$ with $A \subset B$ there is the smallest set $C \leq B$ containing $A$. We call such a $C$ the closure of $A$ in $B$, and denoted by $\text{cl}_B(A)$. $(K, \leq)$ has the amalgamation property (for short AP), if whenever $A \leq B \in K$ and $A \leq C \in K$ then there is a $D \in K$ such that $B$ and $C$ are closely embedded in $D$ over $A$.

*The author is supported by JSPS Grants-in-Aid for Scientific Research No. 19540150.
Definition 1.1 A countable $L$-structure $M$ is said to be $(K, \leq)$-generic, if it satisfies the following:

1. Any finite $A \subset M$ belongs to $K$;
2. $M$ is rich, i.e., For any $A \leq B \in K$ with $A \leq M$ there is $B' \cong_A B$ with $B' \leq M$;
3. $M$ has finite closures, i.e., for any finite $A \subset M$, $|\text{cl}_M(A)|$ is finite.

If $(K, \leq)$ has AP, then there exists a $(K, \leq)$-generic $M$. By the back-and-forth argument, if $M, N$ are $(K, \leq)$-generic then $M \cong N$. It can be seen also that the generic $M$ is ultra-homogeneous over closed sets, i.e., if $B, B' \leq M$ and $B \cong B'$ then $\text{tp}(B) = \text{tp}(B')$.

Ab initio generic structures Let $L$ be a countable relational language, where each $R \in L$ is symmetric and irreflexive, i.e., if $\models R(\bar{a})$ then the elements of $\bar{a}$ are without repetition and $\models R(\sigma(\bar{a}))$ for any permutation $\sigma$. Thus, for an $L$-structure $A$ and $R \in L$ with arity $n$, $R^A$ can be thought of as a set of $n$-element subsets of $A$. For a finite $L$-structure $A$, a predimension of $A$ is defined by

$$\delta(A) = |A| - \sum_{R \in L} \alpha_R |R^A|$$

where $0 < \alpha_R \leq 1$ for $R \in L$. Write $\delta(B/A) = \delta(BA) - \delta(A)$.

Let $K^*$ denote the class of all finite $L$-structures $A$ with $\delta(B) \geq 0$ for every $B \subset A$. For $A \subset B \in K^*$, define $A \leq B$ to have $\delta(X/A \cap X) \geq 0$ for any finite $X \subset B$. Note that $(K^*, \leq)$ satisfies (C1)-(C4). Take any $K \subset K^*$ closed under substructures. Clearly $(K, \leq)$ also satisfies (C1)-(C4). So, if $(K, \leq)$ has AP, then there exists the generic $M$. $M$ is a generic structure derived from the predimension $\delta$. Such a $M$ is called ab initio generic.

Theories having finite closures By definition, an ab initio generic structure $M$ has finite closures, however each model of Th$(M)$ does not always have finite closures. We say that a theory $T$ has finite closures, if any model of $T$ has finite closures.

Let $M$ be an ab initio generic structure such that Th$(M)$ has finite closures, and $\mathcal{M}$ a big model of Th$(M)$. For a finite $A \subset \mathcal{M}$, a dimension of $A$ is defined by $d(A) = \delta(\text{cl}_\mathcal{M}(A))$. For finite $A, B \subset \mathcal{M}$, put
\[ d(A/B) = d(A \cup B) - d(B). \] For an infinite \( B \), let \( d(A/B) = \inf\{d(A/B_0) : B_0 \text{ is a finite subset of } B\}. \) For \( A, B, C \subset \mathcal{M} \) with \( B \cap C \subset A \), we say that \( B \) and \( C \) are free over \( A \) (write \( B \perp_A C \)), if \( R^{ABC} = R^{AB} \cup R^{AC} \) for each \( R \in L \). The free amalgamation of \( B \) and \( C \) over \( A \), denoted by \( B \oplus_A C \), is the structure \( B \cup C \) with \( B \perp_A C \).

**Examples and Question** The following are examples of *ab initio* generic structures:

- \( L \) is finite, and the generic is saturated: An \( \aleph_0 \)-categorical stable pseudoplane (Hrushovski [13]), A strongly minimal structure with a new geometry (Hrushovski [14]), An \( \aleph_1 \)-categorical non-Desarguesian projective plane (Baldwin [14]), An almost strongly minimal generalized n-gon (Debonis-Nesin [9], Tent [23]), A minimal but not strongly minimal structure with arbitrary finite dimension (Ikeda [15]).

- \( L \) is finite, and the generic is not saturated: A sparse random graph (Shelah-Spencer [22], Baldwin-Shelah [7], Laskowski [20]).

- \( L \) is infinite, and the generic is saturated: A stable small structure with infinite weight (Herwig [12]).

All known examples are either strictly stable or \( \omega \)-stable. Therefore the following question arises naturally.

**Question 1.2 (Baldwin [3, 6])** Is there an *ab initio* generic structure which is superstable but not \( \omega \)-stable?

**2 Results**

Here we deal with an *ab initio* generic graph \( M \) with coefficient 1: Let \( L = \{R(\ast, \ast)\} \) and \( \delta(A) = |A| - |R^A| \).

**Proposition 2.1** Let \( M \) be an *ab initio* generic graph with coefficient 1. Then \( \text{Th}(M) \) is \( \lambda \)-stable for each \( \lambda \geq |S(\text{Th}(M))| \).
Sketch of Proof. Let $\mathcal{M}$ be a big model. Take any $N \prec \mathcal{M}$ with $|N| = \lambda$, and take any $p \in S(N)$. For $b \models p$, there is a finite $A \subset N$ with $d(b/N) = d(b/A)$. Let $B = \text{cl}(bA)$. We can assume that $B \oplus_A N \leq \mathcal{M}$. Note that $\text{Th}(M)$ is not always ultra-homogeneous over closed sets. As $\alpha = 1$, $\text{tp}(B/N)$ is determined by $\text{tp}(B/A)$. Hence $|S(N)| \leq |N|^{<\omega} \cdot |S(\text{Th}(M))| = \lambda$.

Remark 2.2 The case of $\alpha = 1$ is particular. When $\alpha$ is rational with $\alpha < 1$, the above statement does not necessarily hold. However, if $M$ is saturated, it can be shown that $\text{Th}(M)$ is $\omega$-stable.

First Example Here we construct an ab initio generic graph which has coefficient 1 and is not saturated.

A graph $A = \{a_0, a_1, \ldots, a_k\}$ is called a line, if the relations of $A$ are $R(a_0, a_1), \ldots, R(a_{k-1}, a_k)$. A graph $A = \{a_0, a_1, \ldots, a_k\}$ is called a cycle, if the relations of $A$ are $R(a_0, a_1), \ldots, R(a_{k-1}, a_k), R(a_k, a_0)$. A connected acyclic graph is called a tree.

Let $T$ be the class of all finite trees. Let $C$ be the class of all cycles. Let $K_1 = \{A_0 \oplus \cdots \oplus A_n : A_0, \ldots, A_n \in T \cup C, n \in \omega\}$. Clearly $K_1$ is closed under substructures. Moreover, the following lemma can be seen easily.

Lemma 2.3 $K_1$ has the free amalgamation property, i.e., if $A \leq B \in K_1, A \leq C \in K_1$ and $B \perp_A C$, then $B \oplus_A C \in K_1$.

By Lemma 2.3, we can take the $(K_1, \leq)$-generic $M_1$. Let $\mathcal{M}_1$ be a big model. By compactness, $\mathcal{M}_1$ has infinite lines without endpoints as connected components. So we have the following lemma.

Lemma 2.4 $M_1$ is not saturated.

It is seen that any connected component of $\mathcal{M}_1$ is isomorphic to either a cycle, an infinite line without endpoints, or a tree with $\deg = \infty$. Then we have the following lemma.

Lemma 2.5 $\text{Th}(M_1)$ is small.

By Proposition 2.1 and Lemma 2.4, 2.5, we have the following theorem.

Theorem 2.6 ([18]) There is an ab initio generic graph which has coefficient 1 and is not saturated. Moreover, the theory is $\omega$-stable.
**Second Example**  As an answer to Question 1.2, we construct an *ab initio* generic graph with coefficient 1 such that the theory is superstable but not $\omega$-stable.

The construction is as follows. Let $F_0 = \{a_0\}$ and $F_1 = \{a_1, b_1\}$ be graphs with no relations. For $n \in \omega$ and $\eta \in {}^n 2$, a graph $E_\eta = (E_\eta, R^{E_\eta})$ is defined as follows:

- $F_{\eta(k)}^k \cong F_{\eta(k)}$ for each $k$ with $0 \leq k \leq n$;
- $E_\eta = \{e_k : -n \leq k \leq n\} \cup \bigcup_{0 \leq k \leq n} F_{\eta(k)}^k$;
- $R^{E_\eta} = \{(e_k, e_{k+1}) : -n \leq k \leq n - 1\} \cup \{(e_k, a) : a \in F_{\eta(k)}^k, 0 \leq k \leq n\}$.

![Graph](image)

**Figure 1:** The graph $E_\eta$ where $n = 4$ and $\eta = (01101)$

Take a 1-1 onto map $f : \omega \rightarrow \omega - \{0, 1, 2\}$. Using $f$ and $E_\eta$, a graph $D_\eta = (D_\eta, R^{D_\eta})$ is defined as follows:

- $e_{-n}^i E_\eta^i \cong e_{-n} E_\eta$ for each $i$ with $0 \leq i < f(\eta)$;
- $D_\eta = \bigcup_{0 \leq i < f(\eta)} E_\eta^i$;
- $R^{D_\eta} = \bigcup_{0 \leq i < f(\eta)} R^{E_\eta^i} \cup \{(e_{-n}^0, e_{-n}^1), \ldots, (e_{-n}^{f(\eta)-2}, e_{-n}^{f(\eta)-1}), (e_{-n}^{f(\eta)-1}, e_{-n}^0)\}$.

Let $T$ be the class of all finite trees. Let $D$ be the class of all finite substructures of $D_\eta$ for every $n \in \omega$ and $\eta \in {}^n 2$. Let $K_2 = \{A_0 \oplus \cdots \oplus A_n : A_0, \ldots, A_n \in T \cup D, n \in \omega\}$.

**Lemma 2.7** $K_2$ has the amalgamation property.
Figure 2: The graph $D_\eta$ where $f(\eta) = 6$

**Sketch of Proof.** Suppose that $A \leq B \in \mathbf{K}_2$ and $A \leq C \in \mathbf{K}_2$. We can assume that $B$ and $C$ are connected, $B \perp_A C$ and $A \neq \emptyset$. If both $B$ and $C$ have no cycles, then we have $D \in T \subset \mathbf{K}_2$. So we can assume that either $B$ or $C$ has a cycle. Then any cycle in $B$ or $C$ must be contained in $A$. Moreover it has the unique $n$-cycle for some $n \in \omega$. Let $\eta = f^{-1}(n)$. We can assume that $A \leq D_\eta$. Then both of $B$ and $C$ can be closely embedded over $A$ in $D_\eta \in \mathbf{K}_2$. Hence $B$ and $C$ are amalgamated over $A$.

Figure 3: $B$ and $C$ can be closely embedded over $A$ in $D_{f^{-1}(5)}$. 
By Lemma 2.7, we can take the \((K_2, \leq)\)-generic \(M_2\). Let \(M_2\) be a big model. For \(\beta \in \omega 2\), a graph \(E_\beta\) is defined as the following figure:

\[
E_\beta
\]

\[
\begin{array}{ccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 1 & 1 & 0 & 1 & & \\
\end{array}
\]

Figure 4: The graph \(E_\beta\) where \(\beta = (01101 \ldots)\)

By compactness, in a big model \(M_2\), there are continuously many \(E_\beta\)'s as connected components. Hence we have the following lemma.

**Lemma 2.8** \(|S(Th(M_2))| = 2^{\aleph_0}\)

By Proposition 2.1 and Lemma 2.11, we have the following theorem.

**Theorem 2.9** ([18]) There is an \textit{ab initio} generic structure which is super-stable but not \(\omega\)-stable.

In Kirishima meeting, Baldwin suggested to me that the following question should arise naturally.

**Question 2.10** Is there an \textit{ab initio} generic structure which is \textit{small} and superstable but not \(\omega\)-stable?

This question is still open.

**Saturated Generic Structures** We have a negative answer to Question 1.2 under the assumption that \(L\) is finite and the generic is saturated. To get this result, we need the following lemma. The proof of the lemma is similar to that of Lemma 2.4 in [1].

**Lemma 2.11** Let \(M\) be an \textit{ab initio} generic structure and \(\mathcal{M}\) a big model of \(\text{Th}(M)\). Suppose that \(M\) is saturated. If \(A \leq B \leq M\) and \(B \cap \text{acl}(A) = A\), then \(B \cup \text{acl}(A) \leq \mathcal{M}\).

The following theorem is a generalization of that of [17], and the proof is a modification of [1].
Theorem 2.12 ([18]) Let $M$ be an ab initio generic $L$-structure. If $L$ is finite and $M$ is saturated, then $\text{Th}(M)$ is strictly stable or $\omega$-stable.

Question 2.13 Let $M$ be an ab initio generic structure in a countable relational language. If $M$ is saturated, then is the theory strictly stable or $\omega$-stable?

References


