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On superstable generic structures

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This manuscript is an expansion of my talk at Kirishima meeting. In this talk, we mainly gave a counter-example of Baldwin's question. Proofs of our results can be found in [18]. So we do not explain all of those details here.

1 Baldwin's question

Many papers [5, 8, 10, 11, 12, 19, 21, 25] have laid out the basics of generic structures in various situations. In particular, this manuscript was influenced by papers of Wagner [25] and Baldwin-Shi [8].

Generic structures Let L be a countable relational language. Let $\mathbf{K}$ be a class of finite L-structures that is closed under substructures. Let $\leq$ be a reflexive and transitive relation on $\mathbf{K}$ satisfying the following:

(C1) $A \leq B \in \mathbf{K}$ implies $A \subset B$;

(C2) $A \leq B \leq C \in \mathbf{K}$ implies $A \leq C$;

(C3) $A, B \leq C \in \mathbf{K}$ implies $A \cap B \leq C$;

(C4) $A \in \mathbf{K}$ implies $\emptyset \leq A$.

Then, for each A, B with $A \subset B$ there is the smallest set $C \leq B$ containing A. We call such a C the *closure* of A in B, and denoted by $\mathrm{cl}_B(A)$. $(\mathbf{K}, \leq)$ has the *amalgamation property* (for short AP), if whenever $A \leq B \in \mathbf{K}$ and $A \leq C \in \mathbf{K}$ then there is a $D \in \mathbf{K}$ such that B and C are closedly embedded in D over A.

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Definition 1.1 A countable L-structure M is said to be $(\mathbf{K}, \leq)$-*generic*, if it satisfies the following:

1. Any finite $A \subset M$ belongs to $\mathbf{K}$;
2. M is *rich*, i.e., For any $A \leq B \in \mathbf{K}$ with $A \leq M$ there is $B' \cong_A B$ with $B' \leq M$;
3. M has *finite closures*, i.e., for any finite $A \subset M$, $|\mathrm{cl}_M(A)|$ is finite.

If $(\mathbf{K}, \leq)$ has AP, then there exists a $(\mathbf{K}, \leq)$-generic M. By the back-and-forth argument, if M, N are $(\mathbf{K}, \leq)$-generic then $M \cong N$. It can be seen also that the generic M is *ultra-homogeneous over closed sets*, i.e., if $B, B' \leq M$ and $B \cong B'$ then $\mathrm{tp}(B) = \mathrm{tp}(B')$.

Ab initio generic structures Let L be a countable relational language, where each $R \in L$ is symmetric and irreflexive, i.e., if $\models R(\bar{a})$ then the elements of $\bar{a}$ are without repetition and $\models R(\sigma(\bar{a}))$ for any permutation σ. Thus, for an L-structure A and $R \in L$ with arity n, R^A can be thought of as a set of n-element subsets of A. For a finite L-structure A, a *predimension* of A is defined by

$$\delta(A) = |A| - \sum_{R \in L} \alpha_R |R^A|$$

where $0 < \alpha_R \leq 1$ for $R \in L$. Write $\delta(B/A) = \delta(BA) - \delta(A)$.

Let $\mathbf{K}^*$ denote the class of all finite L-structures A with $\delta(B) \geq 0$ for every $B \subset A$. For $A \subset B \in \mathbf{K}^*$, define $A \leq B$ to have $\delta(X/A \cap X) \geq 0$ for any finite $X \subset B$. Note that $(\mathbf{K}^*, \leq)$ satisfies (C1)-(C4). Take any $\mathbf{K} \subset \mathbf{K}^*$ closed under substructures. Clearly $(\mathbf{K}, \leq)$ also satisfies (C1)-(C4). So, if $(\mathbf{K}, \leq)$ has AP, then there exists the generic M. M is a generic structure derived from the predimension δ. Such a M is called *ab initio generic.*

Theories having finite closures By definition, an *ab initio* generic structure M has finite closures, however each model of $\mathrm{Th}(M)$ does not always have finite closures. We say that a theory T has *finite closures*, if any model of T has finite closures.

Let M be an *ab initio* generic structure such that $\mathrm{Th}(M)$ has finite closures, and $\mathcal{M}$ a big model of $\mathrm{Th}(M)$. For a finite $A \subset \mathcal{M}$, a *dimension* of A is defined by $d(A) = \delta(\mathrm{cl}_{\mathcal{M}}(A))$. For finite $A, B \subset \mathcal{M}$, put

$d(A/B) = d(A \cup B) - d(B)$. For an infinite B, let $d(A/B) = \inf\{d(A/B_0) : B_0$ is a finite subset of $B\}$. For $A, B, C \subset \mathcal{M}$ with $B \cap C \subset A$, we say that B and C are *free* over A (write $B \bot_A C$), if $R^{ABC} = R^{AB} \cup R^{AC}$ for each $R \in L$. *The free amalgamation* of B and C over A, denoted by $B \oplus_A C$, is the structure $B \cup C$ with $B \bot_A C$.

Examples and Question The following are examples of *ab initio* generic structures:

- L is finite, and the generic is saturated: An $\aleph_0$-categorical stable pseudoplane (Hrushovski [13]), A strongly minimal structure with a new geometry (Hrushovski [14]), An $\aleph_1$-categorical non-Desarguesian projective plane (Baldwin [4]), An almost strongly minimal generalized n-gon (Debonis-Nesin [9], Tent [23]), A minimal but not strongly minimal structure with arbitrary finite dimension (Ikeda [15]).
- L is finite, and the generic is not saturated: A sparse random graph (Shelah-Spencer [22], Baldwin-Shelah [7], Laskowski [20]).
- L is infinite, and the generic is saturated: A stable small structure with infinite weight (Herwig [12]).

All known examples are either strictly stable or ω-stable. Therefore the following question arises naturally.

Question 1.2 (Baldwin [3, 6]) Is there an *ab initio* generic structure which is superstable but not ω-stable?

2 Results

Here we deal with an *ab initio* generic graph M with coefficient 1: Let $L = \{R(*, *)\}$ and $\delta(A) = |A| - |R^A|$.

Proposition 2.1 Let M be an *ab initio* generic graph with coefficient 1. Then $\mathrm{Th}(M)$ is λ-stable for each $\lambda \geq |S(\mathrm{Th}(M))|$.

Sketch of Proof. Let $\mathcal{M}$ be a big model. Take any $N \prec \mathcal{M}$ with $|N| = \lambda$, and take any $p \in S(N)$. For $b \models p$, there is a finite $A \subset N$ with $d(b/N) = d(b/A)$. Let $B = \mathrm{cl}(bA)$. We can assume that $B \oplus_A N \leq \mathcal{M}$. Note that $\mathrm{Th}(M)$ is not always ultra-homogeneous over closed sets. As $\alpha = 1$, $\mathrm{tp}(B/N)$ is determined by $\mathrm{tp}(B/A)$. Hence $|S(N)| \leq |N|^{<\omega} \cdot |S(\mathrm{Th}(M))| = \lambda$.

Remark 2.2 The case of $\alpha = 1$ is particular. When α is rational with $\alpha < 1$, the above statement does not necessarily hold. However, if M is saturated, it can be shown that $\mathrm{Th}(M)$ is ω-stable.

First Example Here we construct an *ab initio* generic graph which has coefficient 1 and is not saturated.

A graph $A = \{a_0, a_1, ..., a_k\}$ is called a *line*, if the relations of A are $R(a_0, a_1), ..., R(a_{k-1}, a_k)$. A graph $A = \{a_0, a_1, ..., a_k\}$ is called a *cycle*, if the relations of A are $R(a_0, a_1), ..., R(a_{k-1}, a_k), R(a_k, a_0)$. A connected acyclic graph is called a *tree*.

Let $\mathbf{T}$ be the class of all finite trees. Let $\mathbf{C}$ be the class of all cycles. Let $\mathbf{K}_1 = \{A_0 \oplus \cdots \oplus A_n : A_0, ..., A_n \in \mathbf{T} \cup \mathbf{C}, n \in \omega\}$. Clearly $\mathbf{K}_1$ is closed under substructures. Moreover, the following lemma can be seen easily.

Lemma 2.3 $\mathbf{K}_1$ has *the free amalgamation property*, i.e., if $A \leq B \in \mathbf{K}_1, A \leq C \in \mathbf{K}_1$ and $B \perp_A C$, then $B \oplus_A C \in \mathbf{K}_1$.

By Lemma 2.3, we can take the $(\mathbf{K}_1, \leq)$-generic M_1. Let $\mathcal{M}_1$ be a big model. By compactness, $\mathcal{M}_1$ has infinite lines without endpoints as connected components. So we have the following lemma.

Lemma 2.4 M_1 is not saturated.

It is seen that any connected component of $\mathcal{M}_1$ is isomorphic to either a cycle, an infinite line without endpoints, or a tree with $\deg = \infty$. Then we have the following lemma.

Lemma 2.5 $\mathrm{Th}(M_1)$ is small.

By Proposition 2.1 and Lemma 2.4, 2.5, we have the following theorem.

Theorem 2.6 ([18]) There is an *ab initio* generic graph which has coefficient 1 and is not saturated. Moreover, the theory is ω-stable.

Second Example As an answer to Question 1.2, we construct an *ab initio* generic graph with coefficient 1 such that the theory is superstable but not ω-stable.

The construction is as follows. Let $F_0 = \{a_0\}$ and $F_1 = \{a_1, b_1\}$ be graphs with no relations. For $n \in \omega$ and $\eta \in {}^n2$, a graph $E_\eta = (E_\eta, R^{E_\eta})$ is defined as follows:

- $F^k_{\eta(k)} \cong F_{\eta(k)}$ for each k with $0 \leq k \leq n$;
- $E_\eta = \{e_k : -n \leq k \leq n\} \cup \bigcup_{0 \leq k \leq n} F^k_{\eta(k)}$;
- $R^{E_\eta} = \{(e_k, e_{k+1}) : -n \leq k \leq n-1\} \cup \{(e_k, a) : a \in F^k_{\eta(k)}, 0 \leq k \leq n\}$.

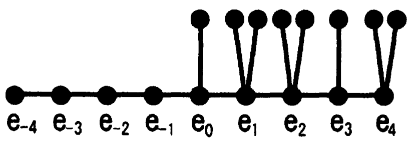

Figure 1: The graph E_η where $n = 4$ and $\eta = (01101)$

Take a 1-1 onto map $f : {}^{\omega>}2 \to \omega - \{0, 1, 2\}$. Using f and E_η, a graph $D_\eta = (D_\eta, R^{D_\eta})$ is defined as follows:

- $e^i_{-n}E^i_\eta \cong e_{-n}E_\eta$ for each i with $0 \leq i < f(\eta)$;
- $D_\eta = \bigcup_{0 \leq i < f(\eta)} E^i_\eta$;
- $R^{D_\eta} = \bigcup_{0 \leq i < f(\eta)} R^{E^i_\eta} \cup \{(e^0_{-n}, e^1_{-n}), ..., (e^{f(\eta)-2}_{-n}, e^{f(\eta)-1}_{-n}), (e^{f(\eta)-1}_{-n}, e^0_{-n})\}$.

Let $\mathbf{T}$ be the class of all finite trees. Let $\mathbf{D}$ be the class of all finite substructures of D_η for every $n \in \omega$ and $\eta \in {}^n2$. Let $\mathbf{K}_2 = \{A_0 \oplus \cdots \oplus A_n : A_0, ..., A_n \in \mathbf{T} \cup \mathbf{D}, n \in \omega\}$.

Lemma 2.7 $\mathbf{K}_2$ has the amalgamation property.

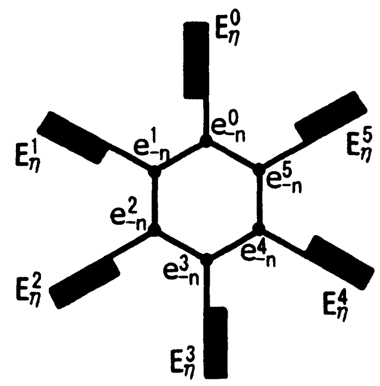

Figure 2: The graph D_η where $f(\eta) = 6$

Sketch of Proof. Suppose that $A \leq B \in \mathbf{K}_2$ and $A \leq C \in \mathbf{K}_2$. We can assume that B and C are connected, $B \perp_A C$ and $A \neq \emptyset$. If both B and C have no cycles, then we have $D \in \mathbf{T} \subset \mathbf{K}_2$. So we can assume that either B or C has a cycle. Then any cycle in B or C must be contained in A. Moreover it has the unique n-cycle for some $n \in \omega$. Let $\eta = f^{-1}(n)$. We can assume that $A \leq D_\eta$. Then both of B and C can be closedly embedded over A in $D_\eta \in \mathbf{K}_2$. Hence B and C are amalgamated over A.

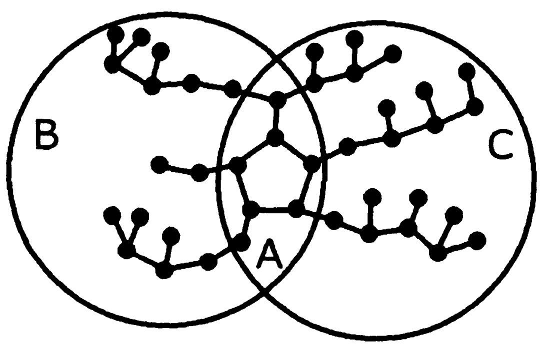

Figure 3: B and C can be closedly embedded over A in $D_{f^{-1}(5)}$.

By Lemma 2.7, we can take the $(\mathbf{K}_2, \leq)$-generic M_2. Let $\mathcal{M}_2$ be a big model. For $\beta \in {}^{\omega}2$, a graph E_β is defined as the following figure:

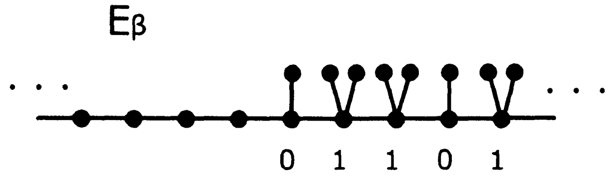

Figure 4: The graph E_β where $\beta = (01101\cdots)$

By compactness, in a big model $\mathcal{M}_2$, there are continuously many E_β's as connected components. Hence we have the following lemma.

Lemma 2.8 $|S(\mathrm{Th}(M_2))| = 2^{\aleph_0}$

By Proposition 2.1 and Lemma 2.11, we have the following theorem.

Theorem 2.9 ([18]) There is an *ab initio* generic structure which is superstable but not ω-stable.

In Kirishima meeting, Baldwin suggested to me that the following question should arise naturally.

Question 2.10 Is there an *ab initio* generic structure which is *small* and superstable but not ω-stable?

This question is still open.

Saturated Generic Structures We have a negative answer to Question 1.2 under the assumption that L is finite and the generic is saturated. To get this result, we need the following lemma. The proof of the lemma is similar to that of Lemma 2.4 in [1].

Lemma 2.11 Let M be an *ab initio* generic structure and $\mathcal{M}$ a big model of $\mathrm{Th}(M)$. Suppose that M is saturated. If $A \leq B \leq \mathcal{M}$ and $B \cap \mathrm{acl}(A) = A$, then $B \cup \mathrm{acl}(A) \leq \mathcal{M}$.

The following theorem is a generalization of that of [17], and the proof is a modification of [1].

Theorem 2.12 (**[18]**) Let M be an *ab initio* generic L-structure. If L is finite and M is saturated, then $\mathrm{Th}(M)$ is strictly stable or ω-stable.

Question 2.13 Let M be an *ab initio* generic structure in a countable relational language. If M is saturated, then is the theory strictly stable or ω-stable?

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