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Smoluchowski-Poisson equation and harmonic heat flow - quantization observed in nonlinear analysis and diffusion geometry

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1 Introduction

The purpose of the present paper is to describe similarity and difference between harmonic heat flow and Smoluchowski-Poisson equation defined on two-dimensional domain. Starting with stationary states, the former is called the harmonic map, while the latter arises as the point vortex mean field equation.

First, equilibrium statistical mechanics of Gibbs is achieved with Hamiltonian denoted by H. Classical gas molecular dynamics is described by the Hamilton system

$$\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}, \quad 1\leq i\leq N$$

where H casts total energy. Then micro-canonical ensemble indicates the coset in $\mathbf{R}^{6N}/\{H\}$ using the phase variable $x=(q_1, \ldots, q_N, p_1, \ldots, p_N)\in \mathbf{R}^{N}$. Micro-canonical measure

$$d\mu^{H,N}=\frac{1}{\Omega(H)}\cdot\frac{d\Sigma(H)}{|\nabla H|}, \quad \Omega(H)=\int_{\{H(x)=H\}}\frac{d\Sigma(H)}{|\nabla H|}$$

is thus defined with $d\Sigma(H)$ on $\left\{x\in \mathbf{R}^{6N}\mid H(x)=H\right\}$ which satisfies

$$dx=dH\cdot\frac{d\Sigma(H)}{|\nabla H|}$$

by the principle of equal a priori probabilities. Canoncial mechanics, on the other hand, is concerned with iso-thermal system, or closed system in physical chemistry. Canonical ensemble is the coset in $\mathbf{R}^{6N}/\{T\}$ and then micro-canonical measure is formulated by

$$d\mu^{\beta,N} = \frac{e^{-\beta H}dx}{Z(\beta,N)}, \quad Z(\beta,N) = \int_{\mathbf{R}^{6N}} e^{-\beta H}dx$$

where T, $\beta = 1/(kT)$, and k denote temperature, inverse temperature, and the Boltzmann constant, respectively. These two measures are equivalent in thermal equilibrium through the thermodynamical relation

$$\beta = \frac{\partial}{\partial H}\log\Omega(H).$$

Onsager [26] introduced point vortex mean field equation using canonical measure from the point vortex system

$$\alpha_i\frac{dx_i}{dt} = \nabla_{x_i}^{\perp}H_N, \quad \nabla^{\perp} = \begin{pmatrix} \partial/\partial x_2 \\ -\partial/\partial x_1 \end{pmatrix}, \quad x = (x_1, x_2)$$

$$H_N(x_1,\ldots,x_N) = \sum_i \frac{\alpha_i^2}{2}R(x_j) + \sum_{i<j}\alpha_i\alpha_j G(x_i,x_j)$$

defined on simply-connected bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary $\partial\Omega$ where $G = G(x,x')$ is the Green's function of $-\Delta$ provided with $\cdot|_{\partial\Omega} = 0$ and

$$R(x) = \left[G(x,x') + \frac{1}{2\pi}\log|x-x'|\right]_{x'=x}$$

is the Robin function. The above point vortex system is derived from the Euler equation

$$v_t + (v\cdot\nabla)v = -\nabla p, \ \nabla\cdot v = 0 \ \text{ in } \Omega\times(0,T), \quad \nu\cdot v = 0 \ \text{ on } \partial\Omega\times(0,T)$$

that is $\omega = \nabla\times v$, $\omega(dx,t) = \sum_{i=1}^{N}\alpha_i\delta_{x_i(t)}(dx)$. Thus, in the case of equal intensity $\alpha_i = \alpha$ there arises the point vortex mean field equation

$$\rho = \frac{e^{-\beta\psi}}{\int_\Omega e^{-\beta\psi}}, \quad \psi = \int_\Omega G(\cdot,x')\rho(x')dx' \tag{1}$$

as $N \uparrow +\infty$. Mathematical justification is done by [6, 7, 17] under the relative Boltzmann factors $\{Z\}$ bound and unique existence of the solution to (1), re-formulated by

$$-\Delta v = \frac{\lambda e^v}{\int_\Omega e^v} \ \text{ in } \Omega, \quad v = 0 \ \text{ on } \partial\Omega \tag{2}$$

using $\lambda = -\beta$ and $v = e^{-\beta\psi}$. More precisely, weak convergence of canonical measures, equivalence between canonical and micro-canonical ensembles in the mean field limit level, and propagation of caos formulated by

$$\rho_k^n \rightharpoonup \rho^{\otimes k} = \Pi_{i=1}^k \rho(x_i)$$

where ρ_k^n denotes the k-th pdf, see [42]. This property is actually the case for $\lambda < 8\pi$ which is proven by [39]. The case $\lambda \geq 8\pi$ is also exciting as the following theorem [22] indicates.

Theorem 1 *Let $\{(\lambda_k, v_k)\}$ be a solution sequence to (2) satisfying $\lambda_k \to \lambda_0 \in (0, \infty)$, $\|v_k\|_\infty \to \infty$ then it holds that $\lambda_0 = 8\pi\ell$, $\ell \in \mathbf{N}$. Passing to a subsequence we obtain $\mathcal{S} \subset \Omega$, $\sharp\mathcal{S} = \ell$ such that $v_k \to v_0$ locally uniformly in $\overline{\Omega} \setminus \mathcal{S}$ where $v_0(x) = 8\pi \sum_{x_0 \in \mathcal{S}} G(x, x_0)$ and $\mathcal{S} = \{x_1^*, \ldots, x_\ell^*\}$ are the singular limit and the blowup set prescribed by $\nabla_{x_i} H_\ell|_{(x_1,\cdots,x_\ell)=(x_1^*,\cdots,x_\ell^*)} = 0$, $1 \leq i \leq \ell$, respectively, where*

$$H_\ell(x_1, \ldots, x_\ell) = \frac{1}{2}\sum_i R(x_i) + \sum_{i<j} G(x_i, x_j)$$

denotes the original Hamiltonian.

We thus obtain *quantized blowup mechanism* in the total mass λ and also *recursivity* of Hamiltonian.

Geometric background of the above theorem is described by the Liouville integral of (2), see [40]. Thus we obtain a meromorphic function $F = F(z)$, $z \in \Omega \subset \mathbf{R}^2 \cong \mathbf{C}$ satisfying

$$\rho(F) = \left(\frac{\sigma}{8}\right)^{1/2} e^{v/2}, \quad \lambda = \sigma \int_\Omega e^v$$

where $\rho(F) = \dfrac{|F'|}{1 + |F|^2}$ stands for the spherical derivative. Then (2) is transformed to $\rho(F)|_{\partial\Omega} = \left(\frac{\sigma}{8}\right)^{1/2}$, that is to find conformal immesion $\sqrt{8}F$: $\Omega \to S^2$ such that $\left.\dfrac{d\Sigma}{ds}\right|_{\partial\Omega} = \sigma^{1/2}$ where $(S^2, d\Sigma)$ denotes the sphere with $|S^2| = 8\pi$. Since

$$\int_\Omega \left(\frac{d\Sigma}{ds}\right)^2 dx = 8\int_\Omega \rho(F)^2 dx = \int_\Omega \sigma e^v$$

stands for the immersed area of $\sqrt{8}F(\Omega)$, the conclusion

$$\lambda = \int_\Omega \sigma e^v \to 8\pi\ell$$

indicates ℓ-covering of the sphere by this mapping.

At this moment it will be natural to suspect similarity between harmonic map. Given the domain of m-compact manifold (Ω, g) and the target compact manifold $N \hookrightarrow \mathbf{R}^n$ without boundary thus we put

$$H^1(\Omega, N) = \left\{u \in H^1(\Omega, \mathbf{R}^n) \mid u \in N \text{ a.e. on } \Omega\right\}, \quad E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2$$

to define the harmonic map $u \in H^1(\Omega, N)$ by

$$\left.\frac{d}{d\varepsilon}E(\Pi(u+\varepsilon\phi))\right|_{\varepsilon=0} = 0, \quad {}^\forall\phi \in C^\infty(\Omega, \mathbf{R}^n)$$

where $\Pi : U \to N$ is the geodesic projection and U is a tubular neighbourhood of N. There is energy quantization for $m = 2$. The difference, however, is the collision of bubbles classified as bubble on bubble and separated bubble which forms bubble tree.

2 Duality - Symmetry

The Smoluchowski-Poisson equation

$$\begin{aligned} &u_t = \nabla \cdot (\nabla u - u\nabla v), \quad -\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \quad \text{in } \Omega \times (0,T) \\ &\frac{\partial u}{\partial \nu} - u\frac{\partial v}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0,T), \qquad \int_\Omega v = 0 \end{aligned} \tag{3}$$

is a fundamental equation in transport theory which arises in the context of semi-conductor physics, high molecular chemistry, astrophysics, and cell biology [41, 42] where $\Omega \subset \mathbf{R}^n$ denotes bounded domain with smooth boundary. Fundamental properties are the positivity of the solution $u \geq 0$ under that of the initial value $u_0 \geq 0$, total mass conservation

$$\frac{d}{dt}\|u\|_1 = 0 \tag{4}$$

and decrease of the free energy

$$\begin{aligned} &\frac{d}{dt}\mathcal{F}(u) = -\int_\Omega u|\nabla(\log u - v)|^2 \leq 0 \\ &\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\iint_{\Omega\times\Omega} G(x,x')u \otimes u. \end{aligned} \tag{5}$$

Then we obtain formation of collapse with quantized mass. Henceforth $T = T_{\max} \in (0, +\infty]$ denotes the existence time of the solution.

Theorem 2 *If $n = 2$ and $T = T_{\max} < +\infty$ in (3) it holds that*

$$u(x,t)dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{6}$$

in $\mathcal{M}(\overline{\Omega})$ as $t \uparrow T$ where

$$\begin{aligned} & 0 \leq f = f(x) \in L^1(\Omega) \cap C(\overline{\Omega} \setminus \mathcal{S}) \\ & \mathcal{S} = \{x_0 \in \overline{\Omega} \mid {}^\exists (x_k, t_k) \to (x_0, T) \textit{such that } u(x_k, t_k) \to +\infty\} \\ & m(x_0) = m_*(x_0) \equiv \begin{cases} 8\pi, & x_0 \in \Omega \\ 4\pi, & x_0 \in \partial\Omega \end{cases} \end{aligned} \tag{7}$$

Since $\|u(t)\|_1 = \|u_0\|_1$ we obtain

$$2\sharp(\mathcal{S} \cap \Omega) + \sharp(\mathcal{S} \cap \partial\Omega) \leq \|u_0\|_1 / (4\pi), \tag{8}$$

and, in particular, the blowup set $\mathcal{S}$ is finite. The strict inequality is actually true in (8).

The above described quantized blowup mechanism was suspected from that of stationary states. Since (4)-(5) this stationary state is defined by

$$\log u - v = \text{constant}, \quad \|u\|_1 = \lambda$$

with the prescribed total mass λ. This property implies

$$u = \frac{\lambda e^v}{\int_\Omega e^v} \tag{9}$$

and hence

$$-\Delta v = \lambda \left(\frac{e^v}{\int_\Omega e^v} - \frac{1}{|\Omega|} \right) \ \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega v = 0 \tag{10}$$

which is reduced to the point vortex mean field equation defined on Riemann surface where the quantized blowup mechanis is observed. This profile that quantization of stationary states implies that of non-stationary states may be called *nonlinear spectral theory.*

Toland duality [48, 49] is observed in (1) between $u = \lambda\rho$ and $v = e^{-\beta\psi}$. In the context of (9)-(10), the variational structure is provided with both u and v, that is the free energy

$$\mathcal{F}(u) = \int_\Omega u(\log u - 1) - \frac{1}{2}\left\langle (-\Delta)^{-1}u, u \right\rangle, \quad u \geq 0, \quad \|u\|_1 = \lambda$$

and the field functional

$$\mathcal{J}_\lambda(v) = \frac{1}{2}\|\nabla v\|_2^2 - \lambda \log \int_\Omega e^v + \lambda(\log \lambda - 1), \quad v \in H^1(\Omega), \ \int_\Omega v = 0$$

where $v = (-\Delta)^{-1}u$ if and only if

$$-\Delta v = u - \frac{1}{|\Omega|}\int_\Omega u \ \text{ in } \Omega, \quad \frac{\partial v}{\partial \nu} = 0 \ \text{ on } \partial\Omega, \quad \int_\Omega v = 0.$$

In fact (3) is the model (B) equation derived from $\mathcal{F}(u)$, that is

$$u_t = \nabla u \cdot \nabla \delta \mathcal{F}(u), \quad u\frac{\partial}{\partial \nu}\delta\mathcal{F}(u)\Big|_{\partial\Omega} = 0.$$

Such duality between field and particles is observed in many models in mathematical physics, see [42]. The other variational structure is based on the symmetry of the Green's function $G(x, x') = G(x', x)$ coming from action-reaction law, that is the weak formulation of (3),

$$\begin{aligned} \frac{d}{dt}\int_\Omega \varphi(x)u(x,t)dx &= \int_\Omega \Delta\varphi(x) \cdot u(x,t)dx \\ &+\frac{1}{2}\iint_{\Omega\times\Omega} \rho_\varphi(x,x')u(x,t)u(x',t)dxdx' \\ \rho_\varphi(x,x') &= \nabla_x G(x,x') \cdot \nabla\varphi(x) + \nabla_{x'}G(x,x') \cdot \nabla\varphi(x') \end{aligned} \tag{11}$$

valid to

$$\varphi \in C^2(\overline{\Omega}), \quad \frac{\partial \varphi}{\partial \nu}\Big|_{\partial\Omega} = 0. \tag{12}$$

3 Blowup Analysis - Partial Regularity

The choice $n = 2$ of Theorem 2 comes from scaling invariance of the equation defined on whole space

$$u_t = \Delta u - \nabla \cdot (u\nabla\Gamma * u), \quad u \geq 0, \quad -\Delta\Gamma = \delta \quad \text{in } \mathbf{R}^n \times (0, T) \tag{13}$$

that is

$$u_\mu(x,t) = \mu^2 u(\mu x, \mu^2 t), \quad \mu > 0.$$

More presicely, this transformation is consistent to the total mass conservation $\|u_\mu\|_1 = \|u\|_1$ if and only if $n = 2$. The critical mass $\lambda_* = 8\pi$ of this dimension is then detected by scaling of free energy, that is

$$\begin{aligned} &\|u_\mu\|_1 = \|u\|_1 \equiv \lambda, \quad \mathcal{F}(u_\mu) = \left(2\lambda - \frac{\lambda^2}{4\pi}\right)\log\mu + \mathcal{F}(u) \\ &u_\mu(x) = \mu^2 u(\mu x), \quad \mu > 0, \end{aligned}$$

where $0 \leq u = u(x) \in L^1(\mathbf{R}^2)$. The above critical dimension and the total mass are associated with the stationary state of (13), $\delta\mathcal{F}(u) = 0$ with $\|u\|_1 = \lambda$, that is

$$\log u - \Gamma * u = \text{constant}, \ u > 0 \quad \text{in } \mathbf{R}^2, \qquad \|u\|_1 = \lambda. \tag{14}$$

In fact, (14) is equivalent to

$$-\Delta v = e^v \quad \text{in } \mathbf{R}^2, \qquad \int_{\mathbf{R}^2} e^v = \lambda < +\infty \tag{15}$$

in terms of $v = \Gamma * u + \text{constant}$, see [41, 42]. Then (15) admits a family of solutions all of which take the value $\lambda = 8\pi$, see [10]. These critical dimension and threshold mass were already noticed heuristically to examine collapse formation and blowup threshold [11]. Mass quantization in non-compact stationary solution sequence on bounded domains was actually clarified later [22, 19]. Then two conjectues made in the context of cell biology, collapse formation [24] and blowup threshold [11], are combined and solved in the affirmative as the quantized blowup mechanism [41, 42, 32].

The above described scaling invariance is observed in the fundamental equations of mathematical physics which induces hierarchical argument called *blowup analysis.* We take a simple example

$$-\Delta v = v^p, \quad v \geq 0, \quad 1 < p < \infty$$

which admits the self-similar transformation

$$v_\mu(x) = \mu^{2/(p-1)} v(\mu x), \quad \mu > 0.$$

Then we can use blowup analysis to show the following theorem [14].

Theorem 3 *Given a bounded domain $\Omega \subset \mathbf{R}^n$ with smooth boundary $\partial\Omega$ and $1 < p < \frac{n+2}{n-2}$, there is $C > 0$ such that any solution $v = v(x)$ to*

$$-\Delta v = v^p, \ v > 0 \ \ in\ \Omega, \quad v = 0 \ \ on\ \partial\Omega$$

admits the estimate $\|v\|_\infty \leq C$.

To prove the above theorem we assume the contrary, existence of $\{v_k\}$ satisfying

$$\begin{aligned} -\Delta v_k &= v_k^p, \ v_k > 0 \ \text{ in } \Omega, \quad v_k = 0 \ \text{ on } \partial\Omega \\ m_k &= v_k(x_k) = \|v_k\|_\infty \to +\infty. \end{aligned}$$

Then $\tilde{v}_k(x) = \mu_k^{\frac{2}{p-1}} v_k(\mu_k x + x_k)$ satisfies the same equation on the rescaled domain where $\mu_k = m_k^{-\frac{2}{p-1}} \downarrow 0$ and hence $\|\tilde{v}_k\|_\infty = \tilde{v}_k(0) = 1$. Elliptic regularity now guarantees the scaled limit $v = v(x)$ satisfying either

$$\begin{aligned} &-\Delta v = v^p, \ \ 0 \leq v \leq v(0) = 1 \quad \text{in } \mathbf{R}^n \\ &-\Delta v = v^p, \ 0 \leq v \leq v(0) = 1 \ \text{ in } \mathbf{R}^n_+, \quad v = 0 \ \text{ on } \partial\mathbf{R}^n_+ \end{aligned} \tag{16}$$

accoding to the approaching speed of x_k toward $\partial\Omega$. The Liouville property, however, guarantees the non-existence of the solution to (16) in case $1 < p < \frac{n+2}{n-2}$, a contradiction. Ingredients of blowup analysis are thus scaling invariance of the model, control of rescaled solution at infinity, hierarchical argument, and classification of scaling limit.

Finiteness of blowup points is the simplest *partial regularity*. It is now well-known that this property is achieved by ε-regularity and monotonicity formula. We take a model harmonic heat flow defined on the torus,

$$\begin{aligned} &u_t - \Delta u = u\,|\nabla u|^2, \quad |u| = 1 \ \text{ in } \Omega \times (0,T) \\ &u = u(x,t) : \Omega \times [0,T) \to S^{n-1} \subset \mathbf{R}^n \end{aligned} \tag{17}$$

where $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$. It holds that

$$\|u_t\|_2^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|_2^2 = \int_\Omega u \cdot u_t |\nabla u|^2 = \frac{1}{2}\int_\Omega \left(\frac{\partial}{\partial t}|u|^2\right)|\nabla u|^2 = 0 \tag{18}$$

which implies the fundamental property, decrease of total energy

$$\frac{dE}{dt} = -\|u_t\|_2^2 \leq 0, \quad E = \frac{1}{2}\|\nabla u\|_2^2. \tag{19}$$

There is a self-similar transformation to $u_\mu(x,t) = u(\mu x, \mu^2 t)$ of (17) consistent to E and the classification of the scaled stationary state which is the origin of energy quantization in the level of stationary states described in the previous section.

We have *ε-regularity* so that there is $\varepsilon_0 > 0$ such that $u = u(x,t)$ is smooth in $B_{R/2} \times [0,T]$ provided that

$$\sup_{t\in[0,T]} E(u(\cdot,t), B_R) < \varepsilon_0 \tag{20}$$

where $E(u,R) = \frac{1}{2}\int_{B_R} |\nabla u|^2$, $B_R = B(0,R)$. Inequality (20) means smallness of local energy in global-in-time which is reduced to the initial state by the *monotonicity formula*

$$E(u(\cdot,T), B_R) \leq E(u_0, B_{2R}) + CE_0 T/R^2, \quad E_0 = \frac{1}{2}\|\nabla u_0\|_2^2. \tag{21}$$

Theory of partial regularity is thus composed of ε-regularity derived from standard local parabolic theory and monotonicity formula which trades-off space and time variables.

4 $2D$ Smoluchowski-Poisson Equation

Global-in-time existence of the solution to (3) is related to the fundamental properties (4)-(5), that is the Trudinger-Moser inequality. This inequality is associated with the scaling property described above, but the critical mass is reduced to a half because the boundary blowup is involved, that is

$$\inf \{\mathcal{F}(u) \mid u \geq 0, \ \|u\|_1 = 4\pi\} > -\infty \tag{22}$$

valid to $n = 2$. Then we obtain the following theorem.

Theorem 4 ([2, 13, 21]) *If $\lambda = \|u_0\|_1 < 4\pi$, then $T = +\infty$ in (13).*

Formation of collapse with mass estimate from below is done by localizing the above theorem.

1. Using nice cut-off functions, we show the formation of collapse at each *isolated* blowup point $x_0 \in \mathcal{S}$ in the form of

$$\lim_{R\downarrow 0} \liminf_{t\uparrow T} \|u(t)\|_{L^1(\Omega\cap B(x_0,R))} \geq m_*(x_0) \tag{23}$$

provided that $T < +\infty$.

2. The Gagliardo-Nirenberg inequality guarantees ε-regularity. There is thus an absolute constant $\varepsilon_0 > 0$ such that

$$\lim_{R\downarrow 0} \limsup_{t\uparrow T} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} < \varepsilon_0 \quad \Rightarrow \quad x_0 \notin \mathcal{S}$$

or equivalently

$$x_0 \in \mathcal{S} \Rightarrow \limsup_{t\uparrow T} \|u(\cdot,t)\|_{L^1(\Omega\cap B(x_0,R))} \geq \varepsilon_0, \quad {}^\forall R > 0. \tag{24}$$

3. If we can replace $\limsup_{t\uparrow T}$ by $\liminf_{t\uparrow T}$ in (24), we obtain $\sharp\mathcal{S} < +\infty$ by the total mass conservation $\|u(t)\|_1 = \|u_0\|_1 = \lambda$. Then (23) arises for any $x_0 \in \mathcal{S}$ because any blowup point is now isolated.

4. Above replacement is justified by the weak formulation (11)-(12). In more details, we have

$$\left| \frac{d}{dt} \int_\Omega u(\cdot,t)\varphi \right| \leq C_\varphi(\lambda + \lambda^2) \tag{25}$$

by

$$\rho_\varphi \in L^\infty(\Omega \times \Omega), \tag{26}$$

recall $n = 2$, which takes place of the monotonicity formula. Thus $u(x,t)dx = \mu(dx,t)$ is extended $\mu(dx,t) \in C_*([0,T], \mathcal{M}(\overline{\Omega}))$ up to $t = T$. Then (23) implies

$$\mu(dx,T) = \sum_{x_0\in\mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{27}$$

with $m(x_0) \geq m_*(x_0)$ and $0 \leq f = f(x) \in L^1(\Omega)$.

The reverse inequality

$$m(x_0) \leq m_*(x_0) \tag{28}$$

regarded as a localization of blowup threshold. Global blowup criterion follows from the weak formulation (26) and the second moment of which plot-type argument is founded in [3]. In the case of (13), thus, the equality

$$\frac{d}{dt} \int_{\mathbf{R}^2} |x|^2 u(\cdot,t) = 4\lambda - \frac{\lambda^2}{2\pi}$$

holds under $u_0 \in L^1(\mathbf{R}^2, (1+|x|^2)dx)$ which implies $T < +\infty$ in case $\lambda > 8\pi$. Employing the method of localization to the above described second moment argument, we obtain the following theorem [30].

Theorem 5 *There is a constant* $\eta = \eta(m) > 0$ *such that if we have* $x_0 \in \mathbf{R}^2$ *and* $0 < R \ll 1$ *satisfying*

$$\frac{1}{R^2} \int_{\Omega \cap B(x_0, 2R)} |x - x_0|^2 u_0 < \eta, \quad m = \int_{\Omega \cap B(x_0, R)} u_0 > m_*(x_0) \tag{29}$$

then $T = T_{\max} < +\infty$.

The following are the proof of (28).

1. We take the backward self-similar variables defined by

$$y = (x - x_0)/(T - t)^{1/2}, \quad s = -\log(T - t), \qquad t < T. \tag{30}$$

Then the transformation consistent to the ODE blowup rate

$$z(y, s) = (T - t)u(x, t), \quad w(y, s) = v(x, t) \tag{31}$$

induces

$$\begin{aligned} & z_s = \nabla \cdot (\nabla z - z \nabla(w + |y|^2/4)) \;\; \text{in } Q \\ & -\Delta w = z - \frac{\lambda}{|\Omega_s|} \text{ in } Q, \quad \frac{\partial w}{\partial \nu} = 0 \;\; \text{on } \Gamma, \quad \int_{\Omega_s} w = 0 \end{aligned} \tag{32}$$

where $Q = \bigcup_{s > -\log T} \Omega_s \times \{s\}$, $\Gamma = \bigcup_{s > -\log T} \partial\Omega_s \times \{s\}$, and $\Omega_s = (T - t)^{-1/2}(\Omega - \{x_0\})$. Similarly to the pre-scaled case there is *generation of weak solution* so that any $s_k \uparrow +\infty$ admits $\{s'_k\} \subset \{s_k\}$ and $\zeta(dy, s)$ such that

$$z(y, s + s'_k)dy \rightharpoonup \zeta(dy, s) \tag{33}$$

in $C_*(-\infty, +\infty; \mathcal{M}_0(\mathbf{R}^2))$. Here, 0-extension of $z(y, s)$ is taken where it is not defined, $\mathcal{M}_0(\mathbf{R}^2) = C_0(\mathbf{R}^2)'$, and $C_0(\mathbf{R}^2)$ denotes the set of continuous functions on $\mathbf{R}^2 \cup \{\infty\}$ taking 0 at ∞. This $\zeta(dy, s)$ is a (finite) Radon measure on $\mathbf{R}^2$ for each s. Contrary to the pre-scaled case, $y = \infty$ is excluded in convergence (33) to derive collapse mass estimate above.

2. Taking even extension in case $x_0 \in \partial\Omega$, the above $\zeta(dy, s)$ becomes a weak solution to

$$z_s = \nabla \cdot (\nabla z - z \nabla(\Gamma * z + |y|^2/4)) \quad \text{in } \mathbf{R}^2 \times (-\infty, +\infty). \tag{34}$$

There is thus $0 \le \mathcal{K} = \mathcal{K}(\cdot, s) \in \mathcal{E}'$, $\|\mathcal{K}(\cdot, s)\|_{\mathcal{E}'} \le \lambda^2$, a.e. s such that

$$\begin{aligned} &\mathcal{K}(\cdot, s)|_X = \zeta(dy, s) \otimes \zeta(dy', s) \\ &\frac{d}{ds}\langle \varphi, \zeta(dy, s)\rangle = \langle \Delta\varphi + \frac{1}{2} y \cdot \nabla\varphi, \zeta(dy, s)\rangle + \frac{1}{2}\langle \rho_\varphi^0, \mathcal{K}(\cdot, s)\rangle \quad \text{a.e. } s \end{aligned}$$

for $\varphi \in C_0^2(\mathbf{R}^2)$ where $s \in (-\infty, +\infty) \mapsto \langle \varphi, \zeta(dy, s)\rangle$ is locally absolutely continuous. Here

$$\rho_\varphi^0 = \rho_\varphi^0(y, y') = -\frac{y - y'}{2\pi|y - y'|^2} \cdot (\nabla\varphi(y) - \nabla\varphi(y'))$$

and $\mathcal{E}$ is the closure of the linear hull of $\mathcal{E}_0$ defined by

$$\begin{aligned} \mathcal{E}_0 &= \{\rho_\varphi + \psi \mid \varphi \in C_0^2(\mathbf{R}^2), \ \psi \in X\} \\ X &= C_0(\mathbf{R}^2 \times \mathbf{R}^2) \oplus [(C_0(\mathbf{R}^2) \oplus \mathbf{R}) \otimes \mathbf{R}] \oplus [\mathbf{R} \otimes (C_0(\mathbf{R}^2) \oplus \mathbf{R})] \end{aligned}$$

with $C_0(\mathbf{R}^2 \times \mathbf{R}^2)$ standing for the set of continuous functions on $(\mathbf{R}^2 \cup \{\infty\}) \times (\mathbf{R}^2 \cup \{\infty\})$ vanishing at $[(\mathbf{R}^2 \cup \{\infty\}) \times \{\infty\}] \cup [\{\infty\} \times (\mathbf{R}^2 \cup \{\infty\})]$.

3. We use *parabolic envelope*, infinitely large parabolic region. Henceforth $\varphi = \varphi_{x_0,R}$ is a function satisfying (12) and

$$\varphi_{x_0,R}(x) = \begin{cases} 1, & \overline{\Omega} \cap \overline{B(x_0, R/2)} \\ 0, & \overline{\Omega} \setminus B(x_0, R). \end{cases}$$

First, we refine (25) as

$$\left| \frac{d}{dt} \int_\Omega u(\cdot, t)\varphi_{x_0,R} \right| \le C(\lambda + \lambda^2)R^{-2} \tag{35}$$

with a constant $C > 0$ independent of $0 < R \le 1$ which implies

$$\begin{aligned} &\left| \int_\Omega \varphi_{x_0,R} u(\cdot, t) - \langle \varphi_{x_0,R}, \mu(dx, T)\rangle \right| \le C_\lambda R^{-2}(T - t) \\ &\lim_{b\uparrow+\infty} \limsup_{t\uparrow T} \left| \langle \varphi_{x_0,b(T-t)^{1/2}}, \mu(dx, t)\rangle - m(x_0) \right| = 0. \end{aligned} \tag{36}$$

It says that infinitely wide parabolic region in xt space associated with backward self-similar variables contains the whole blowup mechanism.

We thus obtain

$$\hat{m}(x_0) = \zeta(\mathbf{R}^2, s), \quad -\infty < s < +\infty \tag{37}$$

for

$$\hat{m}(x_0) = \begin{cases} m(x_0), & x_0 \in \Omega \\ 2m(x_0) & x_0 \in \partial\Omega \end{cases}$$

which means that collapse mass is equal to total mass of rescaled weak limit with $y = \infty$ excluded in the limit process.

4. We take the scale-back $\zeta(dy, s) = e^{-s}A(dy', s')$, $y' = e^{-s/2}y$, $s' = -e^{-s}$ to (34) and obtain weak solution to

$$\begin{aligned} &A_s = \nabla \cdot (\nabla A - A\nabla\Gamma * A), \quad A \geq 0 && \text{in } \mathbf{R}^2 \times (-\infty, 0) \\ &A(\mathbf{R}^2, s) = \hat{m}(x_0) && -\infty < s < 0. \end{aligned}$$

Here we use

$$\left| \frac{d}{ds} \langle \varphi, A(dy, s) \rangle \right| \leq C_\varphi, \quad A(dy, s) \geq 0, \quad A(\mathbf{R}^2, s) = m(x_0)$$

to take the translation weak limit where $\varphi \in C_0^2(\mathbf{R}^2) \oplus \mathbf{R}$. Any $s_k \uparrow +\infty$ thus admits $\{s_k'\} \subset \{s_k\}$ such that $A(dy, s - s_k') \rightharpoonup a(dy, s)$ in $C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$ where $a = a(dy, s)$ is a weak solution to

$$\begin{aligned} &a_s = \nabla \cdot (\nabla a - a\nabla\Gamma * a), \quad a \geq 0 && \text{in } \mathbf{R}^2 \times (-\infty, +\infty) \\ &a(\mathbf{R}^2, s) = \hat{m}(x_0) && -\infty < s < +\infty \end{aligned} \tag{38}$$

and $\mathcal{M}(\mathbf{R}^2) = [C_0(\mathbf{R}^2) \oplus \mathbf{R}]'$. Here we use $\mathcal{M}(\mathbf{R}^2)$ to envelope the rescaled total mass which gurantees $a(\mathbf{R}^2, s) = \hat{m}(x_0)$. Then (28) is derived similarly to the classical solution done by [18], see below.

5. First we use the local second moment. We obtain

$$\frac{d}{ds}\langle c(|y|^2) + 1, a(dy, s)\rangle \leq C\langle c(|y|^2) + 1, a(dy, s)\rangle + \delta\hat{m}(x_0)(4 - \frac{\hat{m}(x_0)}{2\pi})$$

a.e. s with $C > 0$, $\delta > 0$ where

$$\begin{gathered} 0 \leq c'(s) \leq 1, \ s \geq 0, \quad -1 \leq c(s) \leq 0, \ s \geq 0 \\ c(s) = \begin{cases} s - 1, & 0 \leq s \leq 1/4 \\ 0, & s \geq 4. \end{cases} \end{gathered}$$

In case $\hat{m}(x_0) > 8\pi$, therefore, the condition

$$\langle c(|y|^2) + 1, a(dy, 0)\rangle < \eta = \frac{\delta\hat{m}(x_0)}{2\pi C}(\hat{m}(x_0) - 8\pi)$$

implies

$$\langle c(|y|^2) + 1, a(dy, s)\rangle < 0, \qquad s \gg 1$$

a contradiction. Hence it must hold that

$$\langle c(|y|^2) + 1, a(dy, 0)\rangle \geq \eta. \tag{39}$$

6. Problem (38) is invariant under the transformation

$$a^\mu(dy', s') = \mu^2 a(dy, s), \quad y' = \mu y, \; s' = \mu s, \quad \mu > 0$$

and so is true for η in (39). In case $\hat{m}(x_0) > 8\pi$, therefore, we obtain

$$\begin{aligned} &\langle c(|y|^2) + 1, a^\mu(dy, 0)\rangle \geq \eta \\ &\langle c(\mu^{-2}|y|^2) + 1, a(dy, 0)\rangle \geq \eta, \qquad {}^\forall \mu > 0. \end{aligned}$$

Then we apply the dominated convergence theorem using

$$0 \leq c(\mu^{-2}|y|^2) + 1 \leq 1, \quad c(\mu^{-2}|y|^2) + 1 \to 0, \quad {}^\forall y \in \mathbf{R}^2$$

as $\mu \uparrow +\infty$ which implies $\eta \leq 0$, a contradiction.

Having proven (6)-(7), now we can show the following theorem.

Theorem 6 ([28, 23]) *Every $x_0 \in \mathcal{S}$ is of type II. More strongly, we have*

$$\lim_{t \uparrow T} (T - t) \|u(\cdot, t)\|_{L^\infty(B(x_0, b(T-t)^{1/2}))} = +\infty \tag{40}$$

for any $b > 0$ and

$$z(y, s + s')dy \rightharpoonup 8\pi\delta_0(dy) \tag{41}$$

in $C_*(-\infty, +\infty; \mathcal{M}(\mathbf{R}^2))$ as $s' \uparrow +\infty$.

Relation (41) is called formation of *sub-collapse* which says that total blowup mechanism is enclosed in *hyper-parabola*, infinitesimally small parabolic region. In fact it is easy to see that (41) implies (40), while the latter is proven as follows.

1. Similarly to (35) we obtain

$$\left| \frac{d}{dt} \int_\Omega |x - x_0|^2 u(\cdot, t)\varphi_{x_0,R} \right| \leq C(\lambda + \lambda^2)$$

which implies

$$\int_\Omega |x-x_0|^2 u(\cdot,t)\varphi_{x_0,R} \le C_\lambda(T-t) + \int_\Omega |x-x_0|^2 \varphi_{x_0,R} f(x)dx$$

and hence

$$\int_\Omega \left|\frac{x-x_0}{R(t)}\right|^2 \varphi_{x_0,bR(t)} \cdot u(\cdot,t) \le C + 4b^2 \left\langle \varphi_{x_0,bR(t)}, f\right\rangle$$
$$R(t) = (T-t)^{1/2}, \quad b > 0. \tag{42}$$

2. Given $t_k \uparrow T$, we take $\{s_k'\} \subset \{s_k\}$ satisfying (33) for $s_k = -\log(T - t_k)$. Inequality (42) implies $\langle |y|^2 \varphi_{x_0,b}, \zeta(dy,s)\rangle \le C$ and hence

$$\left\langle |y|^2, \zeta(dy,s)\right\rangle \le C, \quad -\infty < s < \infty \tag{43}$$

with $b \uparrow +\infty$.

3. Putting $I(s) = \left\langle |y|^2, \zeta(dy,s)\right\rangle$, now we obtain

$$\frac{dI}{ds} = 4\hat{m}(x_0) - \frac{\hat{m}(x_0)^2}{2\pi} + I = I \qquad \text{a.e. } s \in \mathbf{R} \tag{44}$$

by (32) and (37) which implies $\left\langle |y|^2, \zeta(dy,s)\right\rangle = 0$, $-\infty < s < \infty$ by (43). It follows that

$$\zeta(dy,s) = 8\pi\delta_0(dy) \tag{45}$$

and hence (40).

Here we mention the problem of quantization of blowup in infinite time of the solution to (13) which says that $T = +\infty$ with $\lim_{t\uparrow+\infty} \|u(t)\|_\infty = +\infty$ will imply $\lambda = \|u_0\|_1 \in 8\pi\mathbf{N}$. This expected property indicates the non-stationary redisual vanishing which is assured for non-compact sequence of stationary states [19]. We also expect that the movement of collapse formed in infinite time is subject to a Hamiltonian. These profiles are actually the cases of radially symmetric solutions and that of the first critical mass, respectively, see [25]. The fundamental property of sup + inf inequality valid to the stationary problem, however, will not hold to the non-stationary problem. In [1], the repulsive self-interaction is studied where (32) is replaced by

$$v_t = \nabla \cdot (\nabla v + v\nabla(\Gamma * z + |x|^2/4)) \qquad \text{in } \mathbf{R}^n \times (0,+\infty).$$

In this case there is exponential decay of a relative entropy.

5 Harmonic Heat Flow

The following theorem is obtained by the above described ε-regularity and monotonicity formula for the model harmonic heat flow [34].

Theorem 7 *There is a global-in-time H^1-solution to (17) with finite singular points in $\Omega \times [0, +\infty)$ where $\Omega = \mathbf{R}^2/a\mathbf{Z} \times b\mathbf{Z}$.*

We show the key ingredient, monoticity formula (21). This property is regarded as bounded variation in time of local energy. For the harmonic heat flow, this property is derived from the definite sign of the density of E, that is $|\nabla u|^2$, more precisely,

$$\frac{1}{2}\sup_{t\in[0,T)} \|\nabla u(\cdot,t)\|_2^2 + \int_0^T \|u_t(\cdot,s)\|_2^2\,ds \le \frac{1}{2}\|\nabla u_0\|_2^2 = E_0 \tag{46}$$

by (18). We have also

$$\int_\Omega |u_t|^2\varphi^2 + \int_\Omega \nabla u \cdot \nabla(u_t\varphi^2) = \frac{1}{2}\int_\Omega \left[\frac{\partial}{\partial t}|u|^2\right] |\nabla u|^2\varphi^2 = 0$$

with

$$\begin{aligned}\int_\Omega \nabla u \cdot \nabla(u_t\varphi^2) &= \int_\Omega (\nabla u \cdot \nabla u_t)\varphi^2 + \int_\Omega [(u_t\cdot\nabla)u]\cdot\nabla\varphi^2 \\ &= \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2 + \int_\Omega [(u_t\cdot\nabla)u]\cdot\nabla\varphi^2,\end{aligned}$$

and, therefore,

$$\int_\Omega |u_t|^2\varphi^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2 + \int_\Omega [(u_t\cdot\nabla)u]\cdot\nabla\varphi^2 = 0$$

for each $\varphi \in C^1(\Omega)$. It holds that

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2\right| &\le \|u_t\|_2^2\|\varphi\|_\infty^2 + \|u_t\|_2\cdot\left\{\int_\Omega |\nabla u|^2|\nabla\varphi^2|^2\right\}^{1/2} \\ &\le \|u_t\|_2^2\|\varphi\|_\infty^2 + \sqrt{2}\|u_t\|_2\cdot E_0^{1/2}\|\nabla\varphi^2\|_\infty \end{aligned}\tag{47}$$

and hence

$$\int_0^T \left|\frac{d}{dt}\int_\Omega |\nabla u|^2\varphi^2\right| dt < +\infty \tag{48}$$

by (46). Then (21) follows from (46)-(47).

Inequality (48) assures collapse formation with finite blowup points,

$$|\nabla u(x,t)|^2 dx \rightharpoonup \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx \tag{49}$$

as $t \uparrow T$ in $\mathcal{M}(\Omega)$ with $m(x_0) \geq \varepsilon_0$, $0 \leq f = f(x) \in L^1(\Omega)$ where $\mathcal{S} = \Omega \setminus \mathcal{B}$,

$$\mathcal{B} = \{x_0 \in \Omega \mid \limsup_{t \uparrow T} \|\nabla v(\cdot,t)\|_{L^2(B(x_0,R))} < +\infty, \ ^{\exists}R > 0\}.$$

Inequality (47) implies also

$$\left| \int_\Omega |\nabla u(\cdot,t)|^2 \varphi_{x_0,R}^2 - \int_\Omega f(x)\varphi_{x_0,R}^2 - m(x_0) \right| \leq 2C \int_t^T \|u_t(\cdot,s)\|_2^2 ds$$
$$+2\sqrt{2}E_0^{1/2} \left\{ \int_t^T \|u_t(\cdot,s)\|_2^2 ds \right\}^{1/2} C(T-t)^{1/2}/R, \quad x_0 \in \mathcal{S}$$

with $C > 0$ independent of $0 < R \ll 1$. Then it follows that

$$\lim_{t \uparrow T} \int_\Omega |\nabla u(\cdot,t)|^2 \varphi_{x_0,bR(t)}^2 = m(x_0)$$

from (46) again, where $b > 0$ is arbitrary and $R(t) = (T-t)^{1/2}$. Hyper-parabola thus arises and we obtain type (II) blowup rate at each $x_0 \in \mathcal{S}$, see Theorem 12 below concerning (51) for the proof. Inequality (46), furthermore, implies

$$\liminf_{t \uparrow T} (T-t) \|u_t(t)\|_2^2 \to 0 \tag{50}$$

which assures a stationary profile of $u(\cdot,t)$ as $t \uparrow T$, see [50].

The convergence (49) is compatible to $u(\cdot,t) \rightharpoonup u(\cdot,T)$, $|\nabla u(\cdot,T)|^2 = f$ as $t \uparrow T$ in $H^1(\Omega)$ because this convergence is valid also in $C^1_{loc}(\Omega \setminus \mathcal{S})$. Then the solution is extended beyond $t = T$ with the updated initial value $u(\cdot,T) \in H^1(\Omega)$. This process ends after finitely many times. The solution in Theorem 7 is thus constructed with weak continuity in $H^1(\Omega)$.

The concentration lemma [31] guarantees to generalize the initial value $u_0 = u_0(x) \geq 0$ to L^1 function in (3). This solution is defined as the limit of regular solutions for regularized initial values and becomes regular for $0 < t \ll 1$. If we take $0 \leq f = f(x) \in L^1(\Omega)$ in (6), we can extend the blowup solution beyond $t = T$ and this process ends with finitely many times. This solution may be comparable to Struwe's solution for harmonic heat flow in Theorem 7, although its continuity is not certain in standard function spaces.

6 Mass Quantization in Higher Dimension

Degenerate parabolic equation is introduced by [8] to describe the motion of the mean field of many self-interacting particles, that is

$$
\begin{aligned}
u_t &= \frac{m-1}{m}\Delta u^m - \nabla \cdot (u \nabla \Gamma * u), \quad u \geq 0 \qquad \text{in } \mathbf{R}^n \times (0,T) \\
\Gamma(x) &= \frac{1}{\omega_{n-1}(n-2)\,|x|^{n-2}}
\end{aligned}
\tag{51}
$$

with $n \geq 3$, and ω_{n-1} denoting the area of the boundary of the unit ball in $\mathbf{R}^n$ so that $-\Delta\Gamma = \delta$. First, the particle density at $(x,t) \in \mathbf{R}^n \times (0,T)$ with the velocity $v \in \mathbf{R}^n$ is denoted by $0 \leq f = f(x,v,t)$ which satisfies the kinetic equation

$$
f_t + v \cdot \nabla_x f - \nabla\varphi \cdot \nabla_v f = -\nabla_v \cdot j
$$

provided with the general dissipation flux term $-\nabla_v \cdot j$, where φ is the gravitational potential generated by f. We have the density-pressure relation and the Poisson equation

$$
p = p(\mu, \theta), \quad \Delta\varphi = \mu \tag{52}
$$

where p and θ stand for the pressure and the temperature, respectively. The dissipation flux term $-\nabla_v \cdot j$, next, is determined by the maximum entropy production principle, so that f maximize the local entropy

$$
S = \int_{\mathbf{R}^n} s(f(x,v,t))dv
$$

under the constraint

$$
\mu(x,t) = \int_{\mathbf{R}^n} f(x,v,t)dv, \quad p(x,t) = \frac{1}{n}\int_{\mathbf{R}^n} |v|^2 f(x,v,t)dv.
$$

Averaging f over the velocities $v \in \mathbf{R}^n$ and the passage to the limit of large friction or large times lead to

$$
\mu_t = \nabla[D_* \cdot (\nabla p + \mu\nabla\varphi)], \tag{53}
$$

that is a hydrodynamical limit of self-gravitating particles whereby the total mass $\lambda = \int_{\mathbf{R}^n} \mu(x,t)dx$ is conserved during the evolution. We have, thus, several mean field equations according to the entropy function $s(f)$ subject to the law of partition of macroscopic states of particles into mezoscopic states, that is the entropies of Boltzmann, Fermi-Dirac, Bose-Einstein, and so forth. System (52)-(53) is still under-determined, and there are several theories to prescribe the temperature θ. In the canonical statistics one takes the iso-thermal setting, and hence the temperature θ is a constant. In the micro-canonical statistics, on the other hand, θ is a function of t while the time-independent total energy $E = \frac{n}{2}\int_\Omega pdx + \frac{1}{2}\int_\Omega \mu\varphi dx$ is prescribed in advance. Rényi-Tsallis' entropy $S = \frac{-1}{q-1}\int_{\mathbf{R}^n}(f^q - f)dv$ is q-analogue of the Boltzmann entropy. Adopting this entropy, (52) takes $p = \kappa\theta^{1-\frac{\gamma n}{2}}\mu^{1+\gamma}$ where $\kappa > 0$ is a constant and $\frac{1}{\gamma} = \frac{1}{q-1} + \frac{n}{2}$, see [9, 4]. Normalizing physical constants and taking the iso-thermal setting, we can reduce (52)-(53) to the degenerate parabolic equation (51) where the new unknown u is a positive constant times μ and $\frac{1}{m-1} = \frac{1}{q-1} + \frac{n}{2}$. When $n = 3$ and $q = \frac{5}{3}$, the case $m = 2 - \frac{2}{n} = \frac{4}{3}$ arises to (51). Equation (51) with $m = 2 - \frac{2}{n}$ shares, actually, similar variational and scaling properties to the Smoluchowski-Poisson equation (13) with $n = 2$.

The solution to (51) which we handle with is the weak solution formulated by [43]. Given the initial value

$$0 \leq u_0 \in L^1(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n), \qquad u_0^m \in H^1(\mathbf{R}^n), \tag{54}$$

we thus take the approximate solution $u_\varepsilon = u_\varepsilon(x,t)$ satisfying

$$u_{\varepsilon t} = \frac{m-1}{m}\Delta(u_\varepsilon + \varepsilon)^m - \nabla\cdot(u_\varepsilon \nabla\Gamma * u_\varepsilon) \qquad \text{in } \mathbf{R}^n \times (0,T)$$

for $0 < \varepsilon \ll 1$ with sufficiently regular initial value $u_{0\varepsilon} = u_{0\varepsilon}(x)$ and take the limit process $\varepsilon \downarrow 0$. If the existence time of the weak solution u denoted $T = T_{\max} \in (0, +\infty]$ is finite, then

$$\lim_{t\uparrow T} \|u(t)\|_\infty = +\infty. \tag{55}$$

For the proof, first, $u_\varepsilon = u_\varepsilon(x,t)$ is extended as far as $\|u_\varepsilon(t)\|_\infty$ is bounded, while

$$B \equiv \sup_{t\in(0,T)} \|u_\varepsilon(t)\|_\infty \leq \frac{\|u_0\|_\infty}{1 - T\|u_0\|_\infty}, \qquad 0 < T < \|u_0\|_\infty^{-1}$$

holds by the L^∞-energy method [27]. Then we derive several estimates of $u_\varepsilon(\cdot,t)$ uniform in $0 < \varepsilon \ll 1$ and $0 \leq t \leq T$ to take the limit. Then the existence time of the weak solution $u = u(\cdot, t)$ is bounded from below using $\|u_0\|_\infty$, and, consequently, it extends in t as far as $\|u(t)\|_\infty$ is bounded. In particular, (55) arises in the case $T = T_{\max} < +\infty$. The scheme to construct weak solution is due to [38, 36, 35] where the Besse potential is used instead of Γ. We compensate the lack of decay at the infinity of Γ by the decomposition $\nabla\Gamma * u = g_1 + g_2$,

$$g_1 = [\nabla\Gamma \cdot \chi_{\mathbf{R}^n \setminus B(0,1)}] * u, \quad g_2 = [\nabla\Gamma \cdot \chi_{B(0,1)}] * u \tag{56}$$

and the Calderón-Zygmund estimate

$$\|D^2 u\|_p \leq C(n,p)\|\Delta u\|_p, \qquad u \in W^{2,p}(\mathbf{R}^n), \ 1 < p < \infty.$$

We emphasize that the blowup criterion (55) is concerned with the weak solution. In [20], for instance, the unique existence of the classical solution local in time is proven for $n = 1$, $m = 3$, and $(-\Delta + 1)\Gamma = \delta$, provided with the regular non-negative initial value $u_0 = u_0(x)$. If $u_0 \not\equiv 0$ has compact support, furthermore, this classical solution breaks down in finite time T_c exposing the profile $\lim_{t\uparrow T_c} \|\partial_x u(t)\|_\infty = +\infty$, $\limsup_{t\uparrow T_c} \|u(t)\|_p < +\infty$, $1 \leq p \leq \infty$. Thus we have the continuation after $t = T_c$ of this classical solution as a weak solution. Several arguments described below are formal because of the lack of sufficient regularity of the solution althoug are justified by the approximate solution.

First, (51) is a *model B equation*, see [42], associated with the *free energy*

$$\mathcal{F}(u) = \int_{\mathbf{R}^n} \frac{u^m}{m} dx - \frac{1}{2} \langle \Gamma * u, u \rangle, \qquad u \geq 0.$$

In fact we have

$$\langle v, \delta\mathcal{F}(u) \rangle = \frac{d}{ds}\mathcal{F}(u + sv)\bigg|_{s=0} = \left(v, u^{m-1} - \Gamma * u\right)$$

using the L^2-inner product (,). Under this identification $\delta\mathcal{F}(u) = u^{m-1} - \Gamma * u$ we can write (51) as $u_t = \nabla \cdot u\nabla\delta\mathcal{F}(u)$ in $\mathbf{R}^n \times (0, T)$ which implies the the total mass conservation and the decrease of the free energy

$$\|u(t)\|_1 = \|u_0\|_1 = \lambda, \quad \frac{d}{dt}\mathcal{F}(u) = -\int_{\mathbf{R}^n} u\,|\nabla\delta\mathcal{F}(u)|^2 \leq 0. \tag{57}$$

Next we examine scaling invariance compatible to total mass conservation, that is

$$u_\mu(x,t) = \mu^n u(\mu x, \mu^n t), \quad \mu > 0 \tag{58}$$

valid to $m = 2 - \frac{2}{n}$. The Trudinger-Moser inequality

$$\inf\{\mathcal{F}(u) \mid u \geq 0, \text{ supp } u \subset B(0,R), \int_{\mathbf{R}^n} u = \lambda_*\} > -\infty \tag{59}$$

actually arises for each $R > 0$ to this exponent $m = 2 - \frac{2}{n}$, see [52, 51, 33], and the threshold value λ_* is detected by the stationary state of (51),

$$u^{m-1} - \Gamma * u = \text{constant} \ \text{ in } \{u > 0\}, \qquad \int_{\mathbf{R}^n} u = \lambda. \tag{60}$$

Then $v = \Gamma * u + \text{constant}$ satisfies

$$-\Delta v = v_+^q \quad \text{in } \mathbf{R}^n, \qquad \int_{\mathbf{R}^n} v_+^q = \lambda \tag{61}$$

where $m = 1 + \frac{1}{q}$ and hence $q = \frac{n}{n-2}$. Problem (61) for this exponent is invariant under the scaling transformation $v_\mu(x) = \mu^{n-2} v(\mu x)$, $\mu > 0$.

This stationary problem admits a family of solutions each of which is necessarily radially symmetric, and determines the threshold value $\lambda = \lambda_*$ uniquely. There is also mass quantization to the non-compact stationary solution sequence on bounded domains [51, 47]. Threshold value λ_* is also prescribed by the best constant $C(n)$ of the Hardy-Littlewood-Sobolev inequality

$$|\langle f, \Gamma * f\rangle| \leq C(n) \|f\|_m^m \|f\|_1^{2/n}, \qquad m = 2 - \frac{2}{n},$$

see [5]. There is a difference between (61) and the two-dimensional problem (15), however, that v_+^q has a compact support. The other difference is scaling propery of free energy $\mathcal{F}(u_\mu) = \mu^{n-2}\mathcal{F}(u)$ which refines (59) as

$$j_* = \inf\{\mathcal{F}(u) \mid 0 \leq u \in L^m(\mathbf{R}^n), \int_{\mathbf{R}^n} u = \lambda_*\} = 0. \tag{62}$$

If $\|u_0\|_1 = \lambda < \lambda_*$ is the case we obtain $\limsup_{t\uparrow T} \|u(t)\|_m < +\infty$ by (57) and (62). Moser's iteration scheme guarantees $\limsup_{t\uparrow T} \|u(t)\|_\infty < +\infty$ and hence $T = +\infty$. Assuming

$$|x|^2 u_0 \in L^1(\mathbf{R}^n), \tag{63}$$

on the other hand, we have

$$\begin{aligned} \frac{d}{dt}\int_{\mathbf{R}^n} |x|^2 u(\cdot,t) &= \frac{m-1}{m} \cdot 2n \int_{\mathbf{R}^n} u^m(\cdot,t) - (n-2)\langle \Gamma * u, u\rangle \\ &= 2(n-2)\mathcal{F}(u) \end{aligned} \tag{64}$$

which implies $T < +\infty$ in case $\mathcal{F}(u_0) < 0$. Since (62) is sharp and

$$\inf\{\mathcal{F}(u) \mid 0 \leq u \in L^m(\mathbf{R}^n), \int_{\mathbf{R}^n} u = \lambda\} = -\infty$$

each $\lambda > \lambda_*$ admits $u_0 = u_0(x) \geq 0$ with compact support such that $\mathcal{F}(u_0) < 0$ and $\|u_0\|_1 = \lambda$.

Theorem 8 **([5, 44])** *For $\lambda_* > 0$ determined by the dimension $n \geq 3$ it holds that if $u_0 = u_0(x)$ is the initial value satisfying (54), (63), and $\|u_0\|_1 < \lambda_*$, then $T = +\infty$ in (51) for $m = 2 - \frac{2}{n}$. Each $\lambda > \lambda_*$, on the other hand, takes $u_0 = u_0(x)$ such that (54), (63), $\|u_0\|_1 = \lambda$, and $T < +\infty$.*

Localization of the above criteria, particularly, deriving monotonicity formula, however, has not been achieved. This difficulty is due to the above described different relation from $2D$ Smolchowski-Poisson equation, between total mass, free energy, and second moment of u. In fact we have $x \cdot \nabla\Gamma = -\frac{1}{2\pi}$ and $x \cdot \nabla\Gamma = -(n-2)\Gamma$ for $\Gamma(x)$ defined by (13) and (51), respectively.
The blowup set is defined by $\mathcal{S} = \mathbf{R}^n \setminus \mathcal{B}$,

$$\mathcal{B} = \{x_0 \in \mathbf{R}^n \mid {}^{\exists}r > \text{ such that } \limsup_{t\uparrow T} \|u(t)\|_{L^\infty(B(x_0,r))} < +\infty\}$$

which is non-empty because weak solution $u = u(x,t)$ satisfies the blowup criterion (55) for $T < +\infty$. To confirm the blowup rate, next, we write (51) as

$$u_t = \frac{m-1}{m}\Delta u^m - \nabla u \cdot \nabla\Gamma * u + u^2$$

and take the ODE part $\dot{\zeta} = \zeta^2$. It follows that $\zeta(t) = (T-t)^{-1}$ which defines type I blowup rate of $u = u(x,t)$, that is $\|u(t)\|_\infty = O((T-t)^{-1})$. Then we say that $x_0 \in \mathcal{S}$ is type I if $\liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(x_0,r_0))} < +\infty$ for some $r_0 > 0$ and type II in the other case. Then we obtain finiteness of type II blowup points, recall, in $2D$ Smoluchowski-Poisson equation (13) any $x_0 \in \mathcal{S}$ is type II.

Theorem 9 **([46])** *Let $u_0 = u_0(x)$ be the initial value satisfying (54) and (63), and assume $T < +\infty$ for the above described weak solution $u = u(x,t)$ to (51) with $m = 2 - \frac{2}{n}$. Then $\mathcal{S}$ is bounded and $\mathcal{S}_{II}$ is finite where*

$$\mathcal{S}_{II} = \left\{x_0 \in \mathcal{S} \mid \lim_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(x_0,r_0))} = +\infty, \ {}^{\forall}r_0 > 0\right\}.$$

The first step to prove Theorem 9 is the ε-regularity stated below.

Theorem 10 ([45]) *We have $\varepsilon_0 > 0$, $C > 0$ independent of $x_0 \in \mathbf{R}^n$, $0 < R \ll 1$ such that*

$$\sup_{t\in(0,T)} \|u(t)\|_{L^1(B(x_0,R))} < \varepsilon_0 \quad \Rightarrow \quad \sup_{t\in(0,T)} \|u(t)\|_{L^\infty(B(x_0,R/2))} \leq C \qquad (65)$$

where $u = u(x,t)$ is a weak solution to (51).

For the proof of Theorem 10 we use the decomposition (56),

$$v = \Gamma * u = v_1 + v_2, \quad v_1 = \left[\Gamma \cdot \chi_{\mathbf{R}^n \setminus B(0,1)}\right] * u, \quad v_2 = \left[\Gamma \cdot \chi_{B(0,1)}\right] * u$$

combined with the relation $v = G * v_1 + G * v_2 + G * u$ where $G = G(x)$ is the Bessel potential satisfying $(-\Delta + 1)G = \delta$. This potential, studied by [37], decays exponentially at infinity. Direct consequence of Theorem 10 is the boundedness of the blowup set $\mathcal{S}$ derived from (64). In fact we have

$$\int_{\mathbf{R}^n} |x|^2 u(\cdot,t) \leq C(T,u_0) = 2(n-2)T\mathcal{F}(u_0) + \int_{\mathbf{R}^n} |x|^2 u_0$$

and hence $\limsup_{t\uparrow T} \int_{|x|>R} u(\cdot,t) \leq R^{-2}C(T,u_0)$. We obtain $\mathcal{S} \subset \mathbf{R}^n \setminus B(0,R)$, taking $R \gg 1$ as $C(T,u_0)R^{-2} < \varepsilon_0$,

The constant C in (65) is involved by the initial value. This propery, however, is compensated by the parabolic regularity concerning local norms of the solution noticed by [31] to (13).

Theorem 11 ([46]) *Given $r \in [2,\infty)$ and $R > 0$, we have $0 < \varepsilon_r \ll 1$ such that*

$$\sup_{t\in(0,1)} \|u(t)\|_{L^1(B(x_0,2R))} < \varepsilon_r \quad \Rightarrow \quad \|u(t)\|_{L^r(B(x_0,R))} \leq t^{-1}, \quad 0 < t \leq 1.$$

Given $x_0 \in \mathcal{S}$ and $0 < R \ll 1$, we take $0 \leq \varphi = \varphi_{x_0,R}(x) \in C_0^\infty(\mathbf{R}^n)$ satisfying $\operatorname{supp}\varphi \subset \overline{B(x_0,R)}$, $\varphi = 1$ on $B(x_0,R/2)$. Put $A(t) = \int_{\mathbf{R}^n} \varphi u(\cdot,t)$. First, it holds that

$$\begin{aligned} \left|\frac{d}{dt}\int_{\mathbf{R}^n} \varphi u\right|^2 &= \left|\int_{\mathbf{R}^n} u\nabla(u^{m-1} - \Gamma * u)\cdot\nabla\varphi\right|^2 \leq \int_{\mathbf{R}^n} u\,|\nabla\varphi|^2 \\ &\cdot \int_{\mathbf{R}^n} u\left|\nabla(u^{m-1} - \Gamma * u)\right|^2 \cdot \leq -\lambda\,\|\nabla\varphi\|_\infty^2 \frac{d}{dt}\mathcal{F}(u) \end{aligned} \qquad (66)$$

which means

$$(A')^2 \leq -\frac{\|\nabla\varphi\|_\infty^2 \lambda}{2(n-2)} H'', \quad H(t) = \int_{\mathbf{R}^n} |x|^2 u(\cdot,t). \tag{67}$$

If

$$\lim_{t\uparrow T} \mathcal{F}(u(t)) > -\infty \tag{68}$$

is the case, therefore, it follows that

$$\int_0^T \left| \frac{d}{dt}\int_{\mathbf{R}^n} \varphi u \right| dt \leq T^{1/2} \left\{ \int_0^T \left| \frac{d}{dt}\int_{\mathbf{R}^n} \varphi u \right|^2 dt \right\}^{1/2} < +\infty$$

and hence $\lim_{t\uparrow T} A(t) = \lim_{t\uparrow T} \int_{\mathbf{R}^n} \varphi u(\cdot,t)$ exists. Since Theorem 10 guarantees

$$\liminf_{t\uparrow T} A(t) = \limsup_{t\uparrow T} A(t) \geq \limsup_{t\uparrow T} \|u(t)\|_{L^1(B(x_0,R))} \geq \varepsilon_0,$$

we obtain $\lim_{R\downarrow 0} \liminf_{t\uparrow T} \|u(t)\|_{L^1(B(x_0,2R))} \geq \varepsilon_0$ for any $x_0 \in \mathcal{S}$, and hence the finiteness of $\mathcal{S}$ by the total mass conservation.

In the other case of $\lim_{t\uparrow T} \mathcal{F}(u(t)) = -\infty$, we have $\mathcal{F}(u(t_0)) < 0$ for some $t_0 \in [0,T)$. We may assume $t_0 = 0$ without loss of generality. Inequality (64) then implies $\frac{dH}{dt} < 0$ and the existence of $H(T) = \lim_{t\uparrow T} H(t) \geq 0$.

Lemma 1 *It holds that*

$$\sup_{t'\in[t,\frac{t+T}{2}]} A(t') \leq A(t) + C(H(t) - H(T))^{1/2}. \tag{69}$$

Proof: Inequality (67) implies

$$\int_t^{t'} (t'-s)A'(s)^2 ds \leq \frac{\|\nabla\varphi\|_\infty^2 \lambda}{2(n-2)} (H(t) - H(t'))$$

for $0 \leq t \leq t' < T$ by $H'(t) \leq 0$, and, therefore, it holds that

$$\begin{aligned} \left| A(\frac{t+t'}{2}) - A(t) \right|^2 &= \left| \int_t^{\frac{t+t'}{2}} A'(s)ds \right|^2 \leq \int_t^{\frac{t+t'}{2}} (t'-s)^{-1} ds \\ &\cdot \int_t^{t'} (t'-s)A'(s)^2 ds \leq \frac{\log 2}{2} \cdot \frac{\|\nabla\varphi\|_\infty^2}{n-2} \cdot \lambda \cdot (H(t) - H(t')) \\ &\leq \frac{\log 2}{2} \cdot \frac{\|\nabla\varphi\|_\infty^2}{n-2} \cdot \lambda \cdot (H(t) - H(T)), \quad t' \in [t,T) \end{aligned}$$

which implies

$$A(\frac{t+t'}{2}) \leq A(t) + C(H(t) - H(T))^{1/2}, \quad t' \in [t, T)$$

and hence (69). ∎

Now we use the scaling property (58).

Lemma 2 *Each $r_0 > 0$ admits $t_0 \in [0, T)$ and $C > 0$ such that*

$$\|u(\cdot, t_1)\|_{L^1(B(x_0, r_0))} < \varepsilon_0/2 \quad \Rightarrow \quad \sup_{t \in (t_1 + \frac{1}{8}(T-t_1), t_1 + \frac{3}{8}(T-t_1))} (T-t)\|u(t)\|_{L^\infty(B(x_0, (T-t_1)^{1/n}))} \leq C \tag{70}$$

where $x_0 \in \mathbf{R}^n$ and $t_1 \in [t_0, T)$.

Proof: We have $A(t_1) < \varepsilon_0/2$, $A(t) = \int_{\mathbf{R}^n} \varphi_{x_0, r_0} u(\cdot, t)$ by the assumption (70) and hence

$$0 < T - t_0 \ll 1, \; t_1 \in [t_0, T) \quad \Rightarrow \quad \sup_{t' \in [t_1, \frac{T+t_1}{2}]} A(t') < \varepsilon_0 \tag{71}$$

by Lemma 1. Here we use the scaling property (58) and take $\mu > 0$ and $\tilde{u}(x, t)$ by

$$\tilde{u}(x, t) = \mu^n u(\mu x + x_0, \mu^n t + t_1), \quad \mu^n + t_1 = \frac{T + t_1}{2}.$$

It holds that

$$\tilde{u}_t = \frac{m-1}{m}\Delta \tilde{u}^m - \nabla \cdot (\tilde{u}\nabla\Gamma * \tilde{u}), \quad \tilde{u} \geq 0 \quad \text{in } \mathbf{R}^n \times (0, 1)$$

$$\mu^n = \frac{T - t_1}{2}, \qquad \sup_{t \in (0,1)} \|\tilde{u}(t)\|_{L^1(B(0, r_0\mu^{-1}))} < \varepsilon_0$$

by (71). Now we use Theorem 11 and then Moser's iteration scheme applied to the proof of Theorem 10. We obtain

$$\sup_{t \in [1/4, 3/4]} \|\tilde{u}(t)\|_{L^\infty(B(0,1))} \leq C_1$$

similarly, because $r_0\mu^{-1} \geq 2$ holds for $0 < T - t_1 \ll 1$ which means

$$\sup_{t \in (t_1 + \frac{1}{8}(T-t_1), t_1 + \frac{3}{8}(T-t_1))} (T - t_1)\|u(t)\|_{L^\infty(B(x_0, (T-t_1)^{1/n}))} \leq C_1.$$

Then (70) follows for $C = \frac{3}{4}C_1$. ∎

The proof of Theorem 2 is complete by showing

$$\inf_{x_0 \in \mathcal{S}_{II}} \lim_{r \downarrow 0} \liminf_{t \uparrow T} \|u(t)\|_{L^1(B(x_0,r))} \geq \varepsilon_0/2.$$

In fact, then $\sharp \mathcal{S}_{II} < +\infty$ follows from the total mass conservation. Assuming the contrary, we have $x_0 \in \mathcal{S}_{II}$, $r_0 > 0$, and $t_j \uparrow T$ such that

$$\|u(t_j)\|_{L^1(B(x_0,2r_0))} < \varepsilon_0/2$$

for $j = 1, 2, \cdots$. Then we obtain $\sup_{y \in B(x_0,r_0)} \|u(t_j)\|_{L^1(B(y,r_0))} < \varepsilon_0/2$, and, therefore,

$$\sup_{t \in (t_j + \frac{1}{8}(T-t_j), t_j + \frac{3}{8}(T-t_j))} (T-t)\|u(t)\|_{L^\infty(B(y,(T-t_j)^{1/n}))} \leq C$$

by Lemma 2, where $y \in B(x_0, r_0)$ is arbitrary. This inequality implies

$$\sup_{t \in (t_j + \frac{1}{8}(T-t_j), t_j + \frac{3}{8}(T-t_j))} (T-t)\|u(t)\|_{L^\infty(B(x_0,r_0))} \leq C$$
$$\liminf_{t \uparrow T} (T-t)\|u(t)\|_{L^\infty(B(x_0,r_0))} < +\infty$$

that is $x_0 \in \mathcal{S}_I$, a contradiction. ∎

The proof of the next theorem is valid to Theorem 6.

Theorem 12 *If (68) holds, then each $x_0 \in \mathcal{S}$ is type II. We have, more strongly,*

$$\lim_{t \uparrow T} (T-t) \|u(t)\|_{L^\infty(B(x_0, b(T-t)^{1/n})} = +\infty \tag{72}$$

for any $b > 0$.

Proof: We have shown that the formation of collapse arises in this case. By (66), furthermore, it holds that

$$\int_0^T \left| \frac{d}{dt} \int_\Omega \varphi u \right| dt \leq C(T\lambda)^{1/2} \|\nabla \varphi\|_\infty.$$

Putting $\varphi = \varphi_{x_0,R}$, we obtain

$$|\langle \varphi_{x_0,R}, u(t)\rangle - \langle \varphi_{x_0,R}, \mu(dx,T)\rangle| \leq C(T\lambda)^{1/2} R^{-1}(T-t)$$
$$\mu(dx,T) = \sum_{x_0 \in \mathcal{S}} m(x_0)\delta_{x_0}(dx) + f(x)dx. \tag{73}$$

Given $b > 0$, we can take $R = b(T-t)$ for $0 < T-t \ll 1$ in (73), and again

$$\lim_{b\uparrow+\infty} \limsup_{t\uparrow T} \left| \int_{B(x_0,b(T-t))} u(\cdot,t) - m(x_0) \right| = 0 \tag{74}$$

follows for any $b > 0$. Using $v(y,s) = (T-t)u(x,t)$, $y = (x-x_0)/(T-t)^{1/n}$, $s = -\log(T-t)$, inequality (74) reads;

$$\lim_{b\uparrow+\infty} \limsup_{s\uparrow+\infty} \left| \int_{B(0,be^{-\frac{n-1}{n}s})} v(\cdot,s) - m(x_0) \right| = 0. \tag{75}$$

Since $\int_{\mathbf{R}^n} v(\cdot,s) = \lambda$, $s > -\log T$, any $t_k \uparrow T$ admits $\{s_k'\} \subset \{s_k\}$ for $s_k = -\log(T-t_k)$ such that

$$v(y,s_k')dy \rightharpoonup \zeta(dy), \quad \zeta(dy) \geq m(x_0)\delta_0(dy) \tag{76}$$

in $\mathcal{M}(\mathbf{R}^n)$ by (75). Relations (76) imply $\lim_{k\to\infty} \|v(s_k')\|_{L^\infty(B(0,b))} = +\infty$ for any $b > 0$ and hence (72). ∎

Scaling property combined with the compactness of solution sequence derives other aspects of type II blowup points. The following theorem may be comparable to non-degeneracy of the blowup point concerning the semilinear parabolic equation with sub-critical nonlinearity [15].

Theorem 13 **([46])** *We have* $\sharp\mathcal{S}_{*,\xi} < +\infty$, $\xi > 0$ *where*

$$\mathcal{S}_{*,\xi} = \{x_0 \in \mathbf{R}^n \mid {}^\exists y(t) \to x_0,\ {}^\exists b > 0,\ |y(t)-x_0| = O((T-t)^{1/n}) \\ \liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(y(t),b(T-t)^{1/n}))} \geq \xi\}.$$

Proof: If $\inf_{x_0\in\mathcal{S}_{*,\xi}} \lim_{r\downarrow 0} \liminf_{t\uparrow T} \|u(t)\|_{L^1(B(x_0,r))} > 0$ is not the case we have $x_k \in \mathcal{S}_*$, $r_k > 0$, $0 < T - t_{jk} < \frac{1}{jk}$, $j,k = 1,2,\cdots$ such that

$$\int_{B(x_k,4r_k)} u(x,t_{jk})dx < \min\left\{\frac{\varepsilon_0}{2}, \frac{1}{2k}\right\}.$$

Let k be fixed. Since $x_k \in \mathcal{S}_*$, there is $y_k(t) \to x_k$, $b_k > 0$, $L_k > 0$ such that

$$|y_k(t) - y_k(s)| \leq L_k|t-s|^{1/n}, \qquad 0 < T-t,\ T-s \ll 1 \\ \liminf_{t\uparrow T}(T-t)\|u(t)\|_{L^\infty(B(y_k(t),b_k(T-t)^{1/n}))} \geq \xi.$$

We have $j_k \gg 1$ such that $B(y_k(t_{jk}), 3r_k) \subset B(x_k, 4r_k)$, $j \geq j_k$ and hence

$$\sup_{y \in B(y_k(t_{jk}), r_k)} \|u(t_{jk})\|_{L^1(B(y,2r_k))} < \min\left\{\frac{\varepsilon_0}{2}, \frac{1}{2k}\right\}. \tag{77}$$

In particular, it holds that

$$\sup_{y \in B(y_k(t_{jk}), r_k)} \|u(t_{jk})\|_{L^1(B(y,4(T-t_{jk})^{1/n}))} < \frac{\varepsilon_0}{2}$$

with j_k replaced larger if necessary which implies

$$\sup_{t \in (t_{jk}+\frac{1}{8}(T-t_{jk}), t_{jk}+\frac{3}{8}(T-t_{jk}))} (T-t)\|u(t)\|_{L^\infty(B(y_k(t_{jk}), r_k))} \leq C \tag{78}$$

by Lemma 2. We obtain, also,

$$\sup_{t \in (t_{jk}, \frac{1}{2}(T+t_{jk}))} \|u(t)\|_{L^1(B(y_k(t_{jk}), 2r_k))} < \frac{1}{k} \tag{79}$$

by (77) and Lemma 1 under the same agreement. Inequalities (78)-(79) imply

$$\sup_{t \in (\frac{1}{4}, \frac{3}{4})} \|u_{jk}(t)\|_{L^\infty(B(0, \mu_{jk}^{-1} r_k))} \leq C, \quad \sup_{t \in [0,1]} \|u_{jk}(t)\|_{L^1(B(0, 2\mu_{jk}^{-1} r_k))} < \frac{1}{k}$$

$$\mu_{jk} = \frac{1}{2}(T - t_{jk}), \quad u_{jk}(x,t) = \mu_{jk}^n u(\mu_{jk} x + y_k(t_{jk}), \mu_{jk}^n t + t_{jk}).$$

Since the parabolic regularity does not apply to the family $u_{jk} = u_{jk}(x,t)$ we come back to the approximate solution. Then passing to a subsequence of $\{j\}$ denoted by the same symbol, we have $u_{jk} \to u_k$ locally uniformly in $\mathbf{R}^n \times [\frac{3}{8}, \frac{5}{8}]$ as $j \to \infty$ for $k = 1, 2, \cdots$ by diagonal argument where $u_k = u_k(x,t)$ is a solution to (51) satisfying

$$\sup_{t \in [\frac{3}{8}, \frac{5}{8}]} \|u_k(t)\|_{L^\infty(\mathbf{R}^n)} \leq C, \quad \sup_{t \in [\frac{3}{8}, \frac{5}{8}]} \|u_k(t)\|_{L^1(\mathbf{R}^n)} \leq \frac{1}{k}.$$

Then it holds that $u_k \to 0$ locally uniformly in $\mathbf{R}^n \times [\frac{3}{8}, \frac{5}{8}]$ as $k \to \infty$. Given $0 < \eta < \xi$, therefore, we have $\|u_k\|_{L^\infty(B(0,2b) \times [\frac{3}{8}, \frac{5}{8}])} < \frac{\eta}{2}$ for a k sufficiently large, and, then, there arises $j_{b,\eta,k}$ such that $\|u_{jk}\|_{L^\infty(B(0,2b) \times [\frac{3}{8}, \frac{5}{8}])} < \eta$ for any $j \geq j_{b,\eta,k}$. This inequality implies

$$\sup_{t \in (t_{jk}+\frac{3}{16}(T-t_{jk}), t_{jk}+\frac{5}{16}(T-t_{jk}))} (T-t)\|u(t)\|_{L^\infty(B(y_k(t_{jk}), \mu_{jk} b))} < \eta$$

$$\liminf_{t \uparrow T} (T-t)\|u(t)\|_{L^\infty(B(y_k(t), b(T-t)^{1/n}))} \leq \eta$$

sending $j \to \infty$, a contradiction. ∎

The following theorem suggests that $\mathcal{S}$ may be a positive dimensional set if $\sharp\mathcal{S} = +\infty$. Since the lower dimensionally reduced solution does not blowup because of the total mass conservation, one may suspect the validity of $\sharp\mathcal{S} < +\infty$. Here we say that $x(t) \in \mathbf{R}^n$ attains a positive local maximum if $u(\cdot, t)$ is positive in a neighborhood of $x(t)$ in x-space and $x = x(t)$ takes a local maximum of $u(\cdot, t)$.

Theorem 14 *If $\sharp\mathcal{S} = +\infty$, there are infinite number of $x_0 \in \mathcal{S}$ such that*

$$\limsup_{t\uparrow T} \frac{dist(x(t), x_0)}{(T-t)^{1/n}} = +\infty, \tag{80}$$

provided that $x(t)$ attains a positive local maxmimum of $u(\cdot, t)$ such that

$$\limsup_{t\uparrow T} u(x(t), t) = +\infty. \tag{81}$$

Proof: If $\sharp\mathcal{S} = +\infty$ is the case, there are infinite number of $x_0 \in \mathcal{S} \setminus \mathcal{S}_{*,1}$ by Theorem 13. Since $x(t)$ attains a positive local maximum of $u(\cdot, t)$ it follows that $\dot{m} \leq m^2$ for $m(t) = u(x(t), t)$, see [12], and hence

$$m(t) = u(x(t), t) \geq (T-t)^{-1}, \qquad 0 \leq t < T$$

holds by (81). In particular we have

$$\liminf_{t\uparrow T} (T-t) \|u(t)\|_{L^\infty(B(x(t), b(T-t)^{1/n})} \geq 1, \qquad {}^\forall b > 0.$$

Hence $x_0 \in \mathcal{S}_{*,1}$ if (80) is not the case, a contradiction. ∎

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