Braided differential structure on affine Weyl groups and nil-Hecke algebras

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This article is based on my joint work with A. N. Kirillov [5]. We construct a model of the affine nil-Hecke algebra as a subalgebra of the Nichols-Woronowicz algebra associated to a Yetter-Drinfeld module over the affine Weyl group. We also discuss the Peterson isomorphism between the homology of the affine Grassmannian and the small quantum cohomology ring of the flag variety in terms of the braided differential calculus.

1 Affine nil-Hecke algebra

Let $G$ be a simply-connected semisimple complex Lie group and $W$ its Weyl group. Denote by $\Delta$ the set of the roots. We fix the set $\Delta_+$ of the positive roots by choosing a set of simple roots $\alpha_1, \ldots, \alpha_r$. The Weyl group $W$ acts on the weight lattice $P$ and the coroot lattice $Q^\vee$ of $G$. The affine Weyl group $W_{aff}$ is generated by the affine reflections $s_{\alpha,k}$, $\alpha \in \Delta$, $k \in \mathbb{Z}$, with respect to the affine hyperplanes $H_{\alpha,k} := \{ \lambda \in P \otimes \mathbb{R} \mid \langle \lambda, \alpha \rangle = k \}$. The affine Weyl group is the semidirect product of $W$ and $Q^\vee$, i.e., $W_{aff} = W \ltimes Q^\vee$. The affine Weyl group $W_{aff}$ is generated by the simple reflections $s_1 := s_{\alpha_{1,0}}, \ldots, s_r := s_{\alpha_{r,0}}$ and $s_0 := s_{\theta,1}$ where $\theta = -\alpha_0$ is the highest root. The affine Weyl group $W$ has the presentation as a Coxeter group as follows:

$$W_{aff} = \langle s_0, \ldots, s_r \mid s_0^2 = \cdots = s_r^2 = 1, (s_is_j)^{m_{ij}/2} = 1 \rangle.$$

**Definition 1.1.** The affine nil-Coxeter algebra $A_0$ is the associative algebra generated by $\tau_0, \ldots, \tau_r$ subject to the relations

$$\tau_0^2 = \cdots = \tau_r^2 = 0, \ (\tau_i \tau_j)^{[m_{ij}/2]} \tau_i^{\nu_{ij}} = (\tau_j \tau_i)^{[m_{ij}/2]} \tau_j^{\nu_{ij}},$$
where $\nu_{ij} := m_{ij} - 2[m_{ij}/2]$.

For a reduced expression $x = s_{i_1} \cdots s_{i_l}$ of an element $x \in W_{aff}$, the element $\tau_x := \tau_{i_1} \cdots \tau_{i_l} \in A_0$ is independent of the choice of the reduced expression of $x$. It is known that $\{\tau_x\}_{x \in W_{aff}}$ form a linear basis of $A_0$.

The nil-Coxeter algebra $A_0$ acts on $S := \text{Sym} P_Q$ via
\[
\tau_0(f) := \partial_{\alpha_0}(f) = -(f - s_{\theta,0}f)/\theta,
\]
\[
\tau_i(f) := \partial_{\alpha_i}(f) = (f - s_{\alpha_i,0}f)/\alpha_i, \quad i = 1, \ldots, r,
\]
for $f \in S$.

**Definition 1.2.** ([6]) The nil-Hecke algebra $A$ is defined to be the cross product $A_0 \ltimes S$, where the cross relation is given by
\[
\tau_i f = \partial_{\alpha_i}(f) + s_i(f)\tau_i f \quad f \in S, i = 1, \ldots, r.
\]

The affine Grassmannian $\mathring{\text{Gr}} := G(\mathbb{C}((t)))/G(\mathbb{C}[[t]])$ is homotopic to the loop group $\Omega K$ of the maximal compact subgroup $K \subset G$. Let $T \subset G$ be the maximal torus. An associative algebra structure on the $T$-equivariant homology group $H_*^T(\mathring{\text{Gr}}) \cong H_*^T(\Omega K)$ is induced from the group multiplication
\[
\Omega K \times \Omega K \rightarrow \Omega K.
\]

It is known that the algebra $H_*^T(\mathring{\text{Gr}})$ is commutative. The algebra $H_*^T(\Omega K)$ is called the Pontryagin ring.

We regard the $T$-equivariant homology $H_*^T(\mathring{\text{Gr}})$ as an $S$-algebra by identifying $S = H_*^T(pt)$. The diagonal embedding
\[
\Omega K \rightarrow \Omega K \times \Omega K
\]
induces a coproduct on $H_*^T(\mathring{\text{Gr}})$.

**Proposition 1.1.** ([10]) The $T$-equivariant homology $H_*^T(\mathring{\text{Gr}})$ is isomorphic to the centralizer $Z_A(S)$ of $S$ in $A$ as Hopf algebras.
2 Nichols-Woronowicz algebra for affine Weyl groups

Let $M$ be a vector space over a field of characteristic zero and $\psi : M^\otimes 2 \rightarrow M^\otimes 2$ be a fixed linear endomorphism satisfying the braid relations $\psi_i \psi_{i+1} \psi_i = \psi_{i+1} \psi_i \psi_{i+1}$ where $\psi_i : M^\otimes n \rightarrow M^\otimes n$ is a linear endomorphism obtained by applying $\psi$ to the $i$-th and $(i+1)$-st components. Denote by $s_i$ the simple transposition $(i, i+1) \in S_n$. For any reduced expression $w = s_{i_1} \cdots s_{i_l} \in S_n$, the endomorphism $\Psi_w = \psi_{i_1} \cdots \psi_{i_l} : M^\otimes n \rightarrow M^\otimes n$ is well-defined. The Woronowicz symmetrizer [11] is given by $\sigma_n := \sum_{w \in S_n} \Psi_w$.

**Definition 2.1.** ([11]) The Nichols-Woronowicz algebra associated to a braided vector space $M$ is defined by

$$\mathcal{B}(M) := \bigoplus_{n \geq 0} M^\otimes n / \text{Ker}(\sigma_n),$$

where $\sigma_n : M^\otimes n \rightarrow M^\otimes n$ is the Woronowicz symmetrizer.

**Definition 2.2.** A vector space $M$ is called a Yetter-Drinfeld module over a group $\Gamma$, if the following conditions are satisfied:

1. $M$ is a $\Gamma$-module,
2. $M$ is $\Gamma$-graded, i.e. $M = \bigoplus_{g \in \Gamma} M_g$, where $M_g$ is a linear subspace of $M$,
3. for $h \in \Gamma$ and $v \in M_g$, $h(v) \in M_{hgh^{-1}}$.

The Yetter-Drinfeld module $M$ over a group $\Gamma$ is naturally braided with the braiding $\psi : M^\otimes 2 \rightarrow M^\otimes 2$ defined by $\psi(a \otimes b) = g(b) \otimes a$ for $a \in M_g$ and $b \in M$.

In the following we are interested in the Yetter-Drinfeld module over the affine Weyl groups $W_{aff}$. Denote by $t_\lambda \in W_{aff}$ the translation by $\lambda \in Q^\vee$. We define a Yetter-Drinfeld module $V_{aff}$ over $W_{aff}$ by

$$V_{aff} := \bigoplus_{\alpha \in \Delta, k \in \mathbb{Z}} \mathbb{Q} \cdot [\alpha, k]/([\alpha, k] + [-\alpha, -k]),$$

where the $W_{aff}$ acts on $V_{aff}$ by

$$w[\alpha, k] := [w(\alpha), k], \quad w \in W, \quad t_\lambda[\alpha, k] := [\alpha, k + (\alpha, \lambda)], \quad \lambda \in Q^\vee.$$

The $W_{aff}$-grading is given by $\deg_{W_{aff}}([\alpha, k]) := s_{\alpha, k}$. Then it is easy to check the conditions in Definition 2.1. Now we have the Nichols-Woronowicz algebra $\mathcal{B}_{aff} := \mathcal{B}(V_{aff})$ associated to the Yetter-Drinfeld module $V_{aff}$. 
Let us define the extension $\mathcal{B}_{aff}(S) = \mathcal{B}_{aff} \ltimes S$ by the cross relation

$$[\alpha, k]f = \partial_\alpha f + s_{\alpha,0}(f)[\alpha, k], \quad [\alpha, k] \in V_{aff}, f \in S.$$ 

Proposition 2.1. There exists a homomorphism $\varphi : \mathfrak{A} \to \mathcal{B}_{aff}(S)$ given by $\tau_0 \mapsto [\alpha_0, -1], \tau_i \mapsto [\alpha_i, 0], i = 1, \ldots, r$, and $f \mapsto f$, $f \in S$.

Proof. It is enough to check the Coxeter relations among $\varphi(\tau_0), \ldots, \varphi(\tau_r)$ in $\mathcal{B}_{aff}(S)$ based on the classification of the affine root systems. This is done by the direct computation of the symmetrizer for the subsystems of rank 2 in the similar manner to [1, Section 6].

Example 2.1. Here we list the Coxeter relations in $\mathcal{B}_{aff}$ involving $[\theta, 1] = -[\alpha_0, -1]$ for the root systems of rank 2. Let $(\epsilon_1, \ldots, \epsilon_r)$ be an orthonormal basis of the $r$-dimensional Euclidean space. Put $[ij, k] := [\epsilon_i - \epsilon_j, k]$, $[i,j,k] := [\epsilon_i + \epsilon_j, k]$, $[i,k] := [\epsilon_i, k]$ and $[\alpha] := [\alpha, 0]$.

(i) (Type $A_2$ case)

$$[13, 1][23][13, 1] + [23][13, 1][23] = 0, \quad [13, 1][12][13, 1] + [12][13, 1][12] = 0$$

(ii) (Type $B_2$ case)

$$[12, 1][2][\overline{12}, 1][2] = [2][\overline{12}, 1][2][\overline{12}, 1]$$

(iii) (Type $G_2$ case) Let $\alpha_1, \alpha_2$ be the simple roots for $G_2$-system. We assume that $\alpha_1$ is a short root and $\alpha_2$ is a long one. Then we have $\theta = 3\alpha_1 + 2\alpha_2$.

$$[\theta, 1][\alpha_2][\theta, 1] + [\alpha_2][\theta, 1][\alpha_2] = 0.$$ 

3 Model of nil-Hecke algebra

The connected components of $P \otimes \mathbb{R} \setminus \cup_{\alpha \in \Delta_+, k \in \mathbb{Z}} H_{\alpha,k}$ are called alcoves. The affine Weyl group $W_{aff}$ acts on the set of the alcoves simply and transitively.

Definition 3.1. ([8]) (1) A sequence $(A_0, \ldots, A_l)$ of alcoves $A_i$ is called an alcove path if $A_i$ and $A_{i+1}$ have a common wall and $A_i \neq A_{i+1}$.

(2) An alcove path $(A_0, \ldots, A_l)$ is called reduced if the length $l$ of the path is minimal among all alcove paths connecting $A_0$ and $A_l$.

(3) We use the symbol $A_i \xrightarrow{\beta,k} A_{i+1}$ when $A_i$ and $A_{i+1}$ have a common wall of the form $H_{\beta,k}$ and the direction of the root $\beta$ is from $A_i$ to $A_{i+1}$.
The alcove $A^o$ defined by the inequalities $\langle \lambda, \alpha_0 \rangle \geq -1$ and $\langle \lambda, \alpha_i \rangle \geq 0$, $i = 1, \ldots, r$, is called the fundamental alcove. For a reduced alcove path $\gamma : A_0 = A^o \xrightarrow{\beta_1,k_1} \cdots \xrightarrow{\beta_l,k_l} A_l$, we define an element $[\gamma] \in B_{aff}$ by

$$[\gamma] := [-\beta_1, -k_1] \cdots [-\beta_l, -k_l].$$

When $A_l = x^{-1}(A^o)$ for $x \in W_{aff}$, we will also use the symbol $[x]$ instead of $[\gamma]$, since $[\gamma]$ depends only on $x$ thanks to the Yang-Baxter relation.

For a braided vector space $M$, it is known that an element $a \in M$ acts on $\mathcal{B}(M^*)$ as a braided differential operator (see [1], [9]). Let us identify $M^*$ with $M$ via the $W_{aff}$-invariant inner product $(\cdot, \cdot)$ given by

$$(\alpha, k, \beta, l) = \begin{cases} 1, & \text{if } \alpha = \beta \text{ and } k = l, \\ 0, & \text{otherwise}, \end{cases}$$

for $\alpha, \beta \in \Delta_+$, $k, l \in \mathbb{Z}$. In our case, the differential operator $\overrightarrow{D}_{[\alpha,k]}, [\alpha,k] \in V_{aff}$, acting from the right is determined by the following characterization:

1. $\overrightarrow{D}_{[\alpha,k]} = 0$, $c \in \mathbb{Q}$,
2. $([\alpha,k]) \overrightarrow{D}_{[\beta,l]} = ([\alpha,k], [\beta,l]),$
3. $(FG) \overrightarrow{D}_{[\alpha,k]} = F(G \overrightarrow{D}_{[\alpha,k]}) + (F \overrightarrow{D}_{[\alpha,k]}) s_{\alpha,k}(G),$

for $\alpha, \beta \in \Delta$, $k, l \in \mathbb{Z}$, $F, G \in B_{aff}$. The operator $\overrightarrow{D}_{[\alpha,k]}$ extends to the one acting on $B_{aff}(S)$ by the commutation relation $f \cdot \overrightarrow{D}_{[\alpha,k]} = \overrightarrow{D}_{[\alpha,k]} \cdot s_{\alpha,k}(f)$, $f \in S$.

We use the abbreviation $\overrightarrow{D}_0 := \overrightarrow{D}_{[\alpha_0,-1]}$, $\overrightarrow{D}_i := \overrightarrow{D}_{[\alpha_i,0]}$, $i = 1, \ldots, r$. For $x \in W_{aff}$, fix a reduced decomposition $x = s_{i_1} \cdots s_{i_r}$. We define the corresponding braided differential operator $\overrightarrow{D}_x$ acting on $B_{aff}$ by the formula

$$\overrightarrow{D}_x := \overrightarrow{D}_{i_1} \cdots \overrightarrow{D}_{i_r},$$

which is also independent of the choice of the reduced decomposition of $x$ because of the braid relations.

**Lemma 3.1.** For $x \in W_{aff}$, take a reduced alcove path $\gamma$ from the fundamental alcove $A^o$ to $x^{-1}(A^o)$. Then, we have $(\gamma) \overrightarrow{D}_x = 1$.

**Proof.** Let us take a reduced path

$$\gamma : A_0 = A^o \xrightarrow{\beta_1,k_1} A_1 \xrightarrow{\beta_2,k_2} \cdots \xrightarrow{\beta_l,k_l} A_l = x^{-1}(A^o).$$
Define a sequence $\sigma_1, \ldots, \sigma_l \in W_{aff}$ inductively by

$$\sigma_1 := s_{\beta_1, k_1}, \quad \sigma_{j+1} := \sigma_j s_{\beta_{j+1}, k_{j+1}} \sigma_j.$$ 

Then it is easy to see that $\sigma_\nu(A_j) \neq A^\circ, 1 \leq \nu \leq j - 1, \sigma_j(A_j) = A^\circ$ and the walls $\sigma_j(H_{\beta_{j+1}, k_{j+1}})$ are corresponding to simple roots. Hence, $\sigma_1, \ldots, \sigma_l$ are simple reflections. This sequence gives a reduced expression $x = \sigma_l \cdots \sigma_1$. Put $\sigma_i = s_{\alpha_{i_j}}$. Since the direction of $\beta_{j+1}$ is chosen to be from $A_j$ to $A_{j+1}$, we have

$$[\gamma] \overrightarrow{D}_x = ([\beta_1, k_1]) \overrightarrow{D}_{i_1} \cdot (\sigma_1([\beta_2, k_2])) \overrightarrow{D}_{i_2} \cdots (\sigma_{l-1}([\beta_l, k_l])) \overrightarrow{D}_{i_l} = 1.$$

Example 3.1. (1) ($A_2$-case) The standard realization is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2 - \varepsilon_3, \alpha_0 = \varepsilon_3 - \varepsilon_1$. Consider the translation $t_{\alpha_1}$ by the simple root $\alpha_1$. If we take a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{\alpha_1, 0} A_1 \xrightarrow{\alpha_2, 1} A_2 \xrightarrow{\alpha_0, 1} A_3 \xrightarrow{\alpha_1, 2} A_4 = t_{\alpha_1}(A^\circ),$$

then we have $[\gamma] = [23][21, -1][31, -1][21, -2]$. On the other hand, the differential operator corresponding to $t_{-\alpha_1}$ is given by $\overrightarrow{D}_2 \overrightarrow{D}_0 \overrightarrow{D}_2 \overrightarrow{D}_1$, where $\overrightarrow{D}_0 = \overrightarrow{D}_{[31, -1]}, \overrightarrow{D}_1 = \overrightarrow{D}_{[12]}, \overrightarrow{D}_2 = \overrightarrow{D}_{[23]}$. It is easy to check by direct computation

$$([23][21, -1][31, -1][12, 2]) \overrightarrow{D}_2 \overrightarrow{D}_0 \overrightarrow{D}_0 \overrightarrow{D}_1 \overrightarrow{D}_2 \overrightarrow{D}_1 = 1.$$ 

(2) ($B_2$-case) The standard realization is given by $\alpha_1 = \varepsilon_1 - \varepsilon_2, \alpha_2 = \varepsilon_2, \alpha_0 = -\varepsilon_1 - \varepsilon_2$. Let us consider the translation $t_{2\varepsilon_1}$ and a reduced path

$$\gamma : A_0 = A^\circ \xrightarrow{[12, 1]} A_1 \xrightarrow{[2, 1]} A_2 \xrightarrow{[12, 1]} A_3 \xrightarrow{[12, 2]} A_4 \xrightarrow{[1, 2]} A_5 \xrightarrow{[12, 2]} A_6 = t_{2\varepsilon_1}(A^\circ).$$

Then we have

$$[\gamma] = (-[12, 1])(-[2, 1])(-[12, 1])(-[12, 2])(-[1, 2])(-[12, 2])$$

$$= [12, 1][2, 1][12, 1][12, 2][1, 2][12, 2].$$

The differential operator corresponding to $t_{-2\varepsilon}$ is given by

$$\overrightarrow{D}_{t_{-2\varepsilon}} = \overrightarrow{D}_0 \overrightarrow{D}_2 \overrightarrow{D}_0 \overrightarrow{D}_1 \overrightarrow{D}_2 \overrightarrow{D}_1.$$ 

So we have

$$[\gamma] \overrightarrow{D}_{t_{-2\varepsilon}} = ([12, 1][2, 1][12, 1][12, 2][1, 2][12, 2]) \overrightarrow{D}_0 \overrightarrow{D}_2 \overrightarrow{D}_0 \overrightarrow{D}_1 \overrightarrow{D}_2 \overrightarrow{D}_1 = 1.$$
Theorem 3.1. The algebra homomorphism $\varphi : A \rightarrow B_{aff}(S)$ is injective.

Proof. The nil-Hecke algebra $A$ is also $W_{aff}$-graded. Since the homomorphism $\varphi : A \rightarrow B_{aff}(S)$ preserves the $W_{aff}$-grading, it is enough to check $\varphi(\tau_x) \neq 0$, for $x \in W_{aff}$ in order to show the injectivity of $\varphi$. On the other hand, $B_{aff}^{op}$ acts on $B_{aff}$ itself via the braded differential operators. Let $\gamma$ be a reduced alcove path from $A^o$ to $x^{-1}(A^o)$. Then we have $([\gamma])^x D_x = 1$ from Lemma 3.1. This shows $D_x \neq 0$, so $\varphi(\tau_x) \neq 0$.

This theorem implies the following (see Proposition 1.1):

Corollary 3.1. The $T$-equivariant Pontryagin ring $H^T_{*}(\hat{Gr})$ is a subalgebra of $B_{aff}(S)$.

By taking the non-equivariant limit, we also have:

Corollary 3.2. The Pontryagin ring $H_{*}(\hat{Gr})$ is a subalgebra of $B_{aff}$.

4 Affine Bruhat operators

We denote by $x \rightarrow y$ the cover relation in the Bruhat ordering of $W_{aff}$, i.e. $y = xs_{\alpha,k}$ for some $\alpha \in \Delta$ and $k \in \mathbb{Z}$, and $l(y) = l(x) + 1$.

We will use some terminology from [7]. Denote by $\tilde{Q}$ the set of antidominant elements in $Q^\vee$. An element $x \in W_{aff}$ can be expressed uniquely as a product of form $x = wt_{v\lambda} \in W_{aff}$ with $v, w \in W, \lambda \in \tilde{Q}$. We say that $x = wt_{v\lambda}$ belongs to the ”$v$-chamber”. An element $\lambda \in \tilde{Q}$ is called superregular when $|\langle \lambda, \alpha \rangle| > 2(#W) + 2$ for all $\alpha \in \Delta_+$. If $\lambda \in \tilde{Q}$ is superregular, then $x = wt_{v\lambda}$ is called superregular. The subset of superregular elements in $W_{aff}$ is denoted by $W_{aff}^{sreg}$. We say that a property holds for sufficiently superregular elements $W_{aff}^{sreg} \subset W_{aff}$ if there is a positive constant $k \in \mathbb{Z}$ such that the property holds for all $x \in W_{aff}^{sreg}$ satisfying the following condition:

$$y \in W_{aff}, y < x, \text{ and } l(x) - l(y) < k \Rightarrow y \in W_{aff}^{sreg}.$$ 

The meaning of $W_{aff}^{sreg}$ depends on the context, see [7, Section 4] for the details. For $v \in W$, consider the $S$-submodule $M^{sreg}_v$ in $B_{aff}$ generated by the sufficiently superregular elements $[x]$ where $x$ belongs to the $v$-chamber.

Lemma 4.1. Let $x \in W_{aff}$. For $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$, we have

$$[x] D_{[\alpha,k]} = \begin{cases} [xs_{\alpha,k}], & \text{if } l(x) = l(xs_{\alpha,k}) + 1, \\ 0, & \text{otherwise}. \end{cases}$$
The fundamental alcove $A^\circ$ is contained in the region $\{ \lambda \in P \otimes \mathbb{R} | \langle \lambda, \alpha \rangle < k \}$ for $\alpha \in \Delta$ and $k \in \mathbb{Z}_{>0}$. Let us choose any reduced path $\gamma : A_0 \stackrel{\beta_1,k_1}{\to} \cdots \stackrel{\beta_i,k_i}{\to} A_l = x^{-1}(A^\circ)$ with $k_i \geq 0$. If $l(x) > l(xs_{\alpha,k})$, then $(\beta_i,k_i) = (\alpha,k)$ for some $i$. Take the largest $i$ and consider the path

$$\gamma' : A_0 \stackrel{\beta_1,k_1}{\to} \cdots \stackrel{\beta_i-1,k_i-1}{\to} A_{i-1} \stackrel{\beta_i+1,k_i+1}{\to} s_{\alpha,k}(A_{i+1}) \stackrel{\beta_{i+2},k_{i+2}}{\to} \cdots$$

where $(\beta'_j,k'_j)$ is determined by the condition $s_{\alpha,k}(H_{\beta_j,k_j}) = H_{\beta'_j,k'_j}$. If $l(x) = l(xs_{\alpha,k}) + 1$, then the path $\gamma'$ is a reduced path. In this case, we have $[x]^{-1}D_{[\alpha,k]} = [xs_{\alpha,k}]$.

If $l(x) > l(xs_{\alpha,k}) + 1$, the above path $\gamma'$ is not reduced and $[x]^{-1}D_{[\alpha,k]} = 0$. When $l(x) < l(xs_{\alpha,k})$, the element $[\alpha,k]$ does not appear in the monomial $[\gamma]$, so we have $[x]^{-1}D_{[\alpha,k]} = 0$.

**Proposition 4.1.** ([7, Proposition 4.1]) Let $\lambda \in \tilde{Q}$ be superregular. For $x = wt_{v\lambda}$ and $y = xs_{v\alpha,-n}$ with $v,w \in W$, we have the cover relation $y \to x$ if and only if one of the following conditions holds:

1. $l(wv) = l(ws_{\alpha}) - 1$ and $n = \langle \lambda, \alpha \rangle$, giving $y = ws_{v(\alpha)}t_{v(\lambda)}$,
2. $l(wv) = l(ws_{\alpha}) + \langle \alpha^\vee, 2\rho \rangle - 1$ and $n = \langle \lambda, \alpha \rangle + 1$, giving $y = ws_{v(\alpha)}t_{v(\lambda + \alpha^\vee)}$,
3. $l(v) = l(vs_{\alpha}) + 1$ and $n = 0$, giving $y = ws_{v(\alpha)}t_{vs_{\alpha}(\lambda)}$,
4. $l(v) = l(vs_{\alpha}) - \langle \alpha^\vee, 2\rho \rangle + 1$ and $n = -1$, giving $y = ws_{v(\alpha)}t_{vs_{\alpha}(\lambda + \alpha^\vee)}$.

In [7], the first kind of the conditions (1) and (2) are called the near relation because $x$ and $y$ belong to the same chamber. In this paper we denote the near relation by $y \to_{\text{near}} x$.

The affine Bruhat operator $B^\mu : S(W_{aff}^{ssreg}) \to S(W_{aff}^{ssreg}), \mu \in P$, due to Lam and Shimozono [7, Section 5] is an $S$-linear map defined by the formula

$$B^\mu(x) = (\mu - wv\mu)x + \sum_{\alpha \in \Delta_+} \sum_{xs_{v(\alpha),k} \to_{\text{near}} x} \langle \alpha^\vee, \mu \rangle xs_{v(\alpha),k}$$

for $x = wt_{v\lambda} \in W_{aff}^{ssreg}$. We also introduce the operator $\beta^\mu_v, \mu \in P$, acting on each $M^{ssreg}_v$ by

$$\beta^\mu_v([x]) := (\mu - wv\mu)[x] + [x] \sum_{\alpha \in \Delta_+, k > 1} \langle \alpha^\vee, \mu \rangle [D_{v(\alpha),k}]$$
where $x = wt_v \lambda \in W_{aff}^{ssreg}$. Denote by $W_{aff}^{ssreg}(v)$ the subset of $W_{aff}$ consisting of the superregular elements belonging to the $v$-chamber. Fix a left $S$-module isomorphism

$$\iota : S(W_{aff}^{ssreg}(v)) \to M_v^{ssreg}$$

$$x \mapsto [x].$$

**Proposition 4.2.** For each $v \in W$ and a sufficiently superregular element $x \in W_{aff}^{ssreg}(v)$,

$$\beta_v^\mu([x]) = \iota(B^\mu(x)).$$

**Proof.** This can be shown by using Lemma 4.1 and Proposition 4.1.

$$\beta_v^\mu([x]) = (\mu - wv \mu)[x] + [x] \sum_{\alpha \in \Delta_+, k > 1} \langle \alpha^\vee, \mu \rangle [D_{[\alpha], k}]$$

$$= (\mu - wv \mu)[x] + \sum_{\alpha \in \Delta_+} \sum_{k > 1, l(x_{[\alpha], k]) = l(x) - 1} \langle \alpha^\vee, \mu \rangle [x_{S_{\alpha}, k}]$$

$$= (\mu - wv \mu)[x] + \sum_{\alpha \in \Delta_+} \sum_{x_{\alpha} \in \Delta_{S_{\alpha}, k} \text{ near } x} \langle \alpha^\vee, \mu \rangle [x_{S_{\alpha}, k}] = \iota(B^\mu(x)).$$

**Remark 4.1.** In [4] the authors introduced the quantization operators $\eta_\alpha$ acting on the model of $H^\ast(G/B) \otimes \mathbb{C}[q_1, \ldots, q_r]$ realized as a subalgebra of $B_W \otimes \mathbb{C}[q_1, \ldots, q_{n-1}]$. For a superregular element $\lambda \in \tilde{Q}$ and $w \in W$, consider a homomorphism $\theta_w^\lambda$ from the $\lambda$-small elements (see [7, Section 5]) of $H^\ast(G/B) \otimes \mathbb{C}[q]$ to $B_{aff}$ defined by

$$\theta_w^\lambda(q^\mu \sigma^v) := [vw^{-1}t_{w(\lambda+\mu)}],$$

where $\sigma^v$ is the Schubert class of $G/B$ corresponding to $v \in W$ and $q^\mu = q_1^{\mu_1} \cdots q_r^{\mu_r}$ for $\mu = \sum_{i=1}^r \mu_i \alpha_i^\vee$. The following is an interpretation of the formula of [7, Proposition 5.1] in our setting:

$$\theta_w^\lambda(\eta_\alpha(\sigma)) = \beta_w^{\sigma_\alpha}(\theta_w^\lambda(\sigma)).$$

In [5, Section 5], the comparison between the operators $\beta_v^\mu$ and the quantum Bruhat representation of the quantized Fomin-Kirillov quadratic algebra $\mathcal{E}_n^q$ is discussed.
References


