THE FORMAL DEGREE OF DISCRETE SERIES REPRESENTATIONS OF $GL_N$
($GL_N$の離散系表現の形式的次数について)

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INTRODUCTION

Let $G$ be a connected reductive group defined over a non-Archimedean local field $F$, and $G = G(F)$. It is important to determine the formal degrees of the discrete series representations of $G$ for the explicit Plancherel measure for $G$. In Aubert and Plymen [1], the explicit Plancherel measure for $G = GL_N(D)$ is derived via the work of Silberger and Zink [13], where $D$ is a division algebra over $F$.

For $GL_N(F)$, there are many works, e.g., [16], [8], [4], and [13] of computing formal degrees of discrete series representations. Indeed, the general formal degree formula of the discrete series representations are given in [4], and the explicit values of the formal degrees are computed in [8] in the tame case, and in [13] in the general case.

In this article, we improve the method of [13] by using the results of [4], and compute the values of the formal degrees of the discrete series representations of $GL_N(F)$ (Theorem 1.4), which are expressed in terms of critical exponents (see 1.1 below). These expressions are implicit in the formula of [13, Theorem 1.1]. Thus, our formulas are not essentially new. But our improved method remains valid for some other classical groups. In fact, we obtained analogous results for a symplectic group $Sp_N(F)$ and for a unramified unitary group $U(V, h)$, where $N$ is an even integer $\geq 4$ and $h$ is a non-degenerate Hermitian form of an $N$-dimensional $F$-vector space.

The contents of this article are summarized as follows: In Section 1, we give the improvement of the method of Silberger and Zink [13] for $GL_N(F)$, and in Sectin 2, we present results (Theorem 2.7) obtained using the recent works on Hecke algebras of self-dual simple types of Kariyama and Miyauchi [11] (cf. [10]) for the unramified unitary group $U(V, h)$.

1. AN IMPROVEMENT ON THE METHOD OF SILBERGER-ZINK

1.1. Preliminaries. Let $F$ be a non-Archimedean local field. Let $\mathfrak{o}_F$ be the ring of integers of $F$, $p_F$ its maximal ideal, and $k_F = \mathfrak{o}_F/p_F$ the residue field. We denote by $q = |k_F|$ the cardinality of $k_F$.

Let $N$ be an integer $\geq 2$, and $V$ an $N$-dimensional vector space over $F$. We set $A = \text{End}_F(V)$ and denote by $G = A^\times$ the multiplicative group of $A$. By an appropriate $F$-basis of $V$, we identify $A = M_N(F)$ and $G = GL_N(F)$. 
We use the notations of Bushnell-Kutzko [4]. Let $\mathfrak{A}$ be a hereditary $\mathfrak{o}_F$-order in $A$ with Jacobson radical $\mathfrak{P} = \text{rad}(\mathfrak{A})$. We define a subgroup $\mathcal{K}_\mathfrak{A}$ of $G$ by $\mathcal{K}_\mathfrak{A} = \{g \in G | g\mathfrak{A}g^{-1} = \mathfrak{A}\}$. For an element $\beta$ in $A$, the integer $k_0(\beta, \mathfrak{A})$ is defined in [4, (1.4.5), (1.4.6)].

Following [4, (1.5.5)], a stratum in $A$ is a 4-tuple $[\mathfrak{A}, n, r, \beta]$, where $\mathfrak{A}$ is as above, $n, r$ are integers such that $n > r$, and $\beta \in A$ with $\beta \in \mathfrak{P}^{-n}$.

**Definition 1.1.** ([4, (1.5.5)]) A stratum $[\mathfrak{A}, n, r, \beta]$ is called pure, if the following conditions are satisfied:

1. the algebra $E = F[\beta]$ is a field,
2. $E^\times \subset \mathcal{K}_\mathfrak{A}$,
3. $\beta \in \mathfrak{P}^{-n}\setminus \mathfrak{P}^{1-n}$.

It is called simple if, in addition,

4. $r < -k_0(\beta, \mathfrak{A})$.

Thus, for a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, the integer $k_0(\beta, \mathfrak{A})$ satisfies $k_0(\beta, \mathfrak{A}) = -\min\{r \in \mathbb{Z} : [\mathfrak{A}, n, r, \beta] \text{ in not simple}\}$ of [15, (3.6)], and it is called the critical exponent.

1.2. **Simple types.** Hereafter, we assume that the hereditary $\mathfrak{o}_F$-order $\mathfrak{A}$ in a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$ is always principal, that is, there exists an element $z$ in $A$ such that $\mathfrak{P} = z\mathfrak{A} = \mathfrak{A}z$.

Let $e_0 = e(\mathfrak{A}|\mathfrak{o}_F)$ be the $\mathfrak{o}_F$-period of $\mathfrak{A}$, that is, $\mathfrak{P}^{e_0} = \mathfrak{p}_F\mathfrak{A}$, and set

$$f_0 = N/e_0.$$ 

Then each element $x$ of $\mathfrak{A}$ has the block form $x = (x_{ij})_{1 \leq i,j \leq e_0}$ with $x_{ij} \in M_{f_0}(\mathfrak{o}_F)$ if $i \leq j$, and $x_{ij} \in M_{f_0}(\mathfrak{p}_F)$ otherwise.

Let $B$ be the $A$-centralizer of $\beta$, and $\mathfrak{B} = \mathfrak{A} \cap B$. Then it follows from Definition 1.1(2) that $\mathfrak{B}$ is a hereditary $\mathfrak{o}_E$-order in $B$. Let $e_1 = e(\mathfrak{B}|\mathfrak{O}_E)$ be the $\mathfrak{o}_E$-period of $\mathfrak{B}$, defined as is $e_0$ above, where $\mathfrak{o}_E$ is the maximal ideal of $E$.

Associated with a simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, the following three compact open subgroups

$$H^1 = H^1(\beta, \mathfrak{A}) \subset J^1 = J^1(\beta, \mathfrak{A}) \subset J = J(\beta, \mathfrak{A})$$

of $G$ are defined in [4, (3.1)]. Via these groups, a simple type in $G$, denoted by $(J, \lambda)$, is constructed as follows: Take a simple character $\theta$ of $H^1$ (see [4, (3.2)] for the definition). Then it is known that there exists a unique irreducible representation $\eta = \eta(\theta)$ of $J^1$ containing $\theta$. We obtain an extension, $\kappa$, of $\eta$ to $J$, which is called a $\beta$-extension.

Write $G_E = B^\times$. Then $G_E$ is isomorphic to $GL_{N/[E:F]}(E)$. Set

$$f_1 = N/(E : F)e_1.$$ 

For $U(\mathfrak{B}) = \mathfrak{B}^\times \supset U^1(\mathfrak{B}) = 1 + \text{rad}(\mathfrak{B})$, it follows from [4, (3.1.15)] that $J = U(\mathfrak{B})J^1$ and that

$$J/J^1 \simeq U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq GL_{f_1}(k_E)^{e_1},$$
where \( k_F \) denotes the residue field of \( E \). Then \( J/J^1 \) is isomorphic to a Levi subgroup of \( G_{N/F}(k_E) \). Let \( \sigma_0 \) be an irreducible cuspidal \( \lambda = \kappa \otimes \sigma \),
representation of \( GL_{f_1}(k_E) \), and \( \sigma \) the inflation of the representation \( \sigma_0^{\otimes e_1} = \sigma_0 \otimes \cdots \otimes \sigma_0 \) (\( e_1 \)-times) of \( J/J^1 \) to \( J \). Now the simple type \( (J, \lambda) \) in \( G \) is defined by

\[
\lambda = \kappa \otimes \sigma,
\]

in \([4, (5.5.10)(a)]\).

In particular, a simple type \( (J, \lambda) \) in \( G \) of level zero is defined in \([4, (5.5.10)(b)]\). This is a special case of \([4, (5.5.10)(a)]\), by setting \( E = F \), \( \mathfrak{B} = \mathfrak{A} \), \( J^t = U^t(\mathfrak{A})(t = 0, 1) \), and \( \theta, \eta, \kappa \) all trivial. Thus, \( J/J^1 \) is isomorphic to \( GL_{N/e_1}(k_F)^{e_1} \) for \( e_1 = e(\mathfrak{A}|\mathfrak{o}_F) \), and \( \lambda \) is the inflation of a representation \( \sigma_0^{\otimes e_1} \), where we set \( f_1 = N/e_1 \) and \( \sigma_0 \) is an irreducible cuspidal representation of \( GL_{f_1}(k_F) \), as above.

A simple type \( (J, \lambda) \) in \( G \) is called maximal, if \( e_1 = e(\mathfrak{A}|\mathfrak{o}_E) = 1 \).

1.3. Discrete series representations of \( G = GL_N(F) \). Let \( e_1 \) be a positive integer dividing \( N \), and \( \rho \) an irreducible supercuspidal representation of \( G' = GL_{N/e_1}(F) \). Then there exists a maximal simple type \( (J_0, \lambda_0) \) in \( G' \) containing \( \lambda_0 \) by \([4, (8.4.1)]\).

Let \( M \) be a Levi subgroup of \( G = GL_N(F) \) that is isomorphic to \( G'^{e_1} = G' \times \cdots \times G' \) (\( e_1 \)-times), and \( P = MN \) a parabolic subgroup of \( G \) with Levi factor \( M \) and with unipotent radical \( N \). Then \( \rho^{\otimes e_1} \) is an irreducible supercuspidal representation of \( M \cong G'^{e_1} \). Set \( J_M = J_0^{e_1} \), and \( \lambda_M = \lambda_0^{\otimes e_1} \). Then \( (J_M, \lambda_M) \) is a \( [M, \rho^{\otimes e_1}]_M \)-type in the sense of \([5, (8.1)]\), and by \([6, Proposition 1.4]\), there exists an irreducible representation \( \lambda_P \) of a compact open subgroup \( J_P \) of \( G \) associated with the parabolic subgroup \( P \) such that \( (J_P, \lambda_P) \) is a \( G \)-cover of \( (J_M, \lambda_M) \).

Thus, \( (J_P, \lambda_P) \) is a \( [M, \rho^{\otimes e_1}]_{G} \)-type. The pair \( (J_P, \lambda_P) \) is derived, as in \([4, (7.2.17)]\), from a simple type \( (J, \lambda) \) in \( G \) associated with a simple stratum \( \mathfrak{A}, n, 0, \beta \) in \( A \) as in Section 1.2, and satisfies \( \text{Ind}_{J_P}^{G} \lambda_P \simeq \lambda \).

By \([17]\), the induced representation

\[
\text{Ind}_{P}^{G}(|\det |(1-e_1)/2 \rho \otimes \cdots \otimes |\det |(e_1-1)/2 \rho)
\]

contains a unique irreducible discrete series representation, say \((\pi, \mathcal{V})\), of \( G \). Hence, by \([4, (7.3.14)]\), \((\pi, \mathcal{V})\) contains \((J, \lambda)\) and so \((J_P, \lambda_P)\).

1.4. A formal degree formula. Let \((\pi, \mathcal{V})\) be the irreducible discrete series representation of \( G \) in the previous section that contains a simple type \((J, \lambda)\) in \( G \) associated with a simple stratum \( \mathfrak{A}, n, 0, \beta \) in \( A \) with \( E = F[\beta] \) as in Section 1.2. Let \( B \) be the \( A \)-centralizer of \( \beta \), and \( \mathfrak{B} = B \cap \mathfrak{A} \).

Let \( e_1 = e(\mathfrak{B}|\mathfrak{o}_E) \) and \( f_1 = N/([E : F]e_1) \) be as in Section 1.2. Let \( K/E \) be an unramified extension of degree \( f_1 \), and set \( C^\times = GL_{e_1}(K) \).

Let \( \mathcal{I} \) be an Iwahori subgroup of \( C^\times \), and \( 1_{\mathcal{I}} \) the trivial representation of \( \mathcal{I} \). Then the Hecke algebras \( \mathcal{H}(G, \lambda) \) and \( \mathcal{H}(C^\times, 1_{\mathcal{I}}) \) for \((J, \lambda)\) and
$(\mathcal{I}, 1_{\mathcal{I}})$, respectively, are obtained (cf. [4, Section 4]). It follows from [4, (5.5.16), (5.6.6)] that there exists a support-preserving isomorphism

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^\times, 1_{\mathcal{I}})$$

(cf. [4, (7.6.18)]). Via this isomorphism $\Psi$, the equivalence class of the irreducible discrete series representation $(\pi, \mathcal{V})$ of $G$ corresponds to that of an unramified twist $\phi \cdot St_{C^\times}$ of the Steinberg representation $St_{C^\times}$ of $C^\times$ (cf. [13, p.13]). In the notation of [4, (7.7.1)], we have

$$\pi \simeq \mathfrak{U}_\Psi(\phi \cdot St_{C^\times})$$

where $\mathfrak{U}_\Psi$ is defined as in [4, (7.6.21)]. Denote by $St_G$ the Steinberg representation of $G$, and by $\deg(St_G, dx)$ the formal degree of $St_G$ relative to a Haar measure $dx$ on $G$. Since the formal degrees of $St_{C^\times}$ and $\phi \cdot St_{C^\times}$ are the same, by [4, (7.7.11)], we obtain the following result.

**Theorem 1.2.** (cf. [13, Proposition 3.6]) Let notations and assumptions be as above. Then for arbitrary measures $dx$ on $G$ and $dy$ on $C^\times$,

$$\frac{\dim(\lambda)}{e(E|F)} \text{vol}(K^\times \mathcal{I}/K^\times, dy) \deg(St_{C^\times}, dy) = \text{vol}(F^\times J/F^\times, dx) \deg(\pi, dx),$$

where $e(E|F)$ denotes the index of ramification of $E/F$.

**1.5. A deformation of the formal degree formula.** There exists a hereditary $\sigma_F$-order $\mathfrak{A}_m$ in $A$ such that $U(\mathfrak{A}_m) = \mathfrak{A}_m^\times$ is an Iwahori subgroup of $G$ that is contained in the parahoric subgroup $U(\mathfrak{A})$. Now, we first normalize $dx$ on $G$ so that $\text{vol}(F^\times U(\mathfrak{A}_m)/F^\times, dx) = 1$. Then

$$\text{vol}(F^\times U(\mathfrak{A})/F^\times, dx) = (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \text{vol}(F^\times U(\mathfrak{A}_m)/F^\times, dx) = (U(\mathfrak{A}) : U(\mathfrak{A}_m)),$$

and by [3, Proposition 5.3], the formal degree of the Steinberg representation $St_G$ of $G$ is given by

$$\deg(St_G, dx) = \frac{1}{N} \tilde{W}_{A_{N-1}}(q^{-1})^{-1},$$

where $\tilde{W}_{A_{N-1}}(t)$ is the Poincaré series of type $A_{N-1}$ (see [12]). Thus, we obtain Macdonald’s formula:

$$\text{vol}(F^\times U(\mathfrak{A})/F^\times, dx) \deg(St_G, dx) = \frac{1}{N} (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \tilde{W}_{A_{N-1}}(q^{-1})^{-1}$$

which is the same as that of [13, 3.7]. This formula holds for any Haar measure $dx$ on $G$. We again normalize $dx$ on $G$ so that

$$\deg(St_G, dx) = 1.$$

Then

$$\text{vol}(F^\times U(\mathfrak{A})/F^\times, dx) = \frac{1}{N} (U(\mathfrak{A}) : U(\mathfrak{A}_m)) \tilde{W}_{A_{N-1}}(q^{-1})^{-1}$$
On the other hand, for $C^\times = GL_{e_1}(K)$, set
\[ f = f(K|F) = [k_K : k_F], \]
where $k_K$ is the residue field of $K$. Then, from Macdonald’s formula for $St_{C^\times}$, we also obtain
\[ \text{vol}(K^\times I/K^\times, dy) \deg(St_{C^\times}, dy) = \frac{1}{e_1} \widetilde{W}_{A_{e_1}-1}(q^{-f})^{-1}. \]
Hence, since $N/e_1 = [K : F] = fe(E|F)$ and
\[ \text{vol}(F^\times J/F^\times, dx) = \text{vol}(F^\times U(\mathfrak{A})/F^\times, dx)(U(\mathfrak{A}) : J)^{-1}, \]
we obtain
\[ \frac{\deg(St_{C^\times}, dy) \text{vol}(K^\times I/K^\times, dy)}{e(E|F)} = \frac{\widetilde{W}_{A_{N-1}}(q^{-1})}{\widetilde{W}_{A_{e_1}-1}(q^{-f})}(U(\mathfrak{A}) : J) \frac{(U(\mathfrak{A}_m) : U^1(\mathfrak{A}))}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))}(U^1(\mathfrak{A}) : J^1). \]

**Lemma 1.3.** Let notations and assumptions be as above. Then the formal degree $\deg(\pi, dx)$ is equal to
\[ \left( \frac{f}{\widetilde{W}_{A_{e_1}-1}(q^{-f})} \right)(U(\mathfrak{A}_m) : U^1(\mathfrak{A})) \left( \frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} \right) \]
\[ \times \left( (U^1(\mathfrak{A}) : J^1) \dim(\eta) \right). \]

**Proof.** By definition, $\dim(\lambda) = \dim(\eta) \dim(\sigma)$. Thus, the lemma follows from Eqs. (1.1) to (1.3).

**1.6. Calculation of the factors of $\deg(\pi, dx)$.** By the definition in [12],
\[ \frac{f}{\widetilde{W}_{A_{e_1}-1}(q^{-f})} = \frac{q^N - 1}{q^{N/e} - 1} \frac{(q^f - 1)^{e_1}}{(q^{f(E|F)i} - 1)^{e_1}}, \]
where we set $e = e(E|F)$.

Since $U(\mathfrak{A})/U^1(\mathfrak{A}) \simeq GL_{f_0}(k_F)^{e_0}$ in Section 1.2, the quotient group $U(\mathfrak{A}_m)/U^1(\mathfrak{A})$ is isomorphic to the product of $e_0$-copies of a Borel subgroup $B_0$ of $GL_{f_0}(k_F)$. Hence,
\[ (U(\mathfrak{A}_m) : U^1(\mathfrak{A})) = |B_0|^{e_0} = \{(q - 1)^{f_0}q^{\frac{1}{2}f_0(f_0 - 1)}\}^{e_0}. \]

By the definition of $\sigma_0$,
\[ \dim(\sigma) = (\dim(\sigma_0))^{e_1} = \prod_{i=1}^{f_1-1} (q^{f(E|F)i} - 1)^{e_1}, \]
and \((U(\mathfrak{B}) : U^1(\mathfrak{B})) = |GL_{f_1}(k_E)|^{e_1}\). Hence,
\[
\frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} = \frac{q^{-\frac{1}{2}f(E|F)e_1f_1(f_1-1)}}{(q^f-1)^{e_1}}.
\]

1.7. **Calculation of the factor** \((U^1(\mathfrak{U}) : J^1)\dim(\eta)\). Since by definition \(U^1(\mathfrak{U}) = 1 + \mathfrak{P} \supseteq J^1 = 1 + \mathfrak{J}^1 \supseteq H^1 = 1 + \mathfrak{H}^1\), it follows from \([4, (5.1.1)]\) that the last factor \((U^1(\mathfrak{U}) : J^1)\dim(\eta)\) is equal to \((\mathfrak{P} : 0^1)(\mathfrak{J}^1 : H^1)^{1/2}\) = \((\mathfrak{P} : 3^1)^{1/2}(\mathfrak{P} : \text{REJECT}^1)^{1/2}\).

This amounts to \(\dim \pi_{\beta}^1\) in [13, 7.1]. We compute this similarly to as done in [13].

For the simple stratum \([\mathfrak{A}, n, 0, \beta]\) in \(A\), \([\mathfrak{A}, n, r, \beta]\) with \(r = -k_0(\beta, \mathfrak{A})\) is a pure stratum, and it determines a family \([\mathfrak{A}, n, r_i, \gamma_i], 0 \leq i \leq s\) of simple strata that satisfy the conditions \([4, (2.4.2)]\). This family is called a *defining sequence* for the pure stratum \([\mathfrak{A}, n, r, \beta]\). Hence we get a family of pairs \((r_i, \gamma_i), 0 \leq i \leq s\), such that

1. \([\mathfrak{A}, n, r_i, \gamma_i]\) is simple, \(0 \leq i \leq s\);
2. \([\mathfrak{A}, n, r_0, \gamma_0]\) \sim \([\mathfrak{A}, n, r, \beta]\);
3. \(0 < r = r_0 < r_1 < \cdots < r_s < r_{s+1} = n\);
4. \(r_{i+1} = -k_0(\gamma_i, \mathfrak{A})\), and for \(0 \leq i \leq s-1\), \([\mathfrak{A}, n, r_{i+1}, \gamma_{i+1}]\) \sim \([\mathfrak{A}, n, r_{i+1}, \gamma_i]\);
5. \(-k_0(\gamma_s, \mathfrak{A}) = n\) or \(\infty\).

These conditions include all the conditions in \([4, (2.4.2)]\) except condition (vi).

Denote by \(A_{\gamma_i}\) the \(A\)-centralizer of \(\gamma_i\). Then by using the family \((r_i, \gamma_i), 0 \leq i \leq s\), we define \(J = J(\beta) = J(\beta, \mathfrak{A})\) and \(H = H(\beta) = H(\beta, \mathfrak{A})\) inductively as follows:

\[
J(\gamma_i) = \mathfrak{A} \cap A_{\gamma_i} + J(\gamma_{i+1}) \cap \mathfrak{P}^{[(r_{i+1}+1)/2]},
\]

\[
H(\gamma_i) = \mathfrak{A} \cap A_{\gamma_i} + H(\gamma_{i+1}) \cap \mathfrak{P}^{[r_{i+1}/2]+1},
\]

for \(-1 \leq i \leq s\), where we set \(\gamma_{-1} = \beta\) and \(J(\gamma_{s+1}) = H(\gamma_{s+1}) = \mathfrak{A}\), (see \([4, (3.1.7), (3.1.8)]\)).

For a real number \(r\), set \(\bar{r} = [r/2] + 1, \quad r = [(r + 1)/2]\),

where \([x]\) denotes the greatest integer \(\leq x\), for a real number \(x\). From a filtration

\[
\mathfrak{P} = \mathfrak{P}^1 \supset \mathfrak{P}^{1_1} \supset \mathfrak{P}^{r_2} \supset \mathfrak{P}^{1_2} \supset \cdots \supset \mathfrak{P}^{r_s} + \mathfrak{J}^1 \supset \mathfrak{P}^n + \mathfrak{J}^1 = \mathfrak{J}^1
\]
in \( \mathfrak{P} \supset \mathfrak{J}^1 \), we get
\[
(\mathfrak{P} : \mathfrak{J}^1) = \prod_{i=-1}^{s} (\mathfrak{P}^r_i + \mathfrak{J}^1 : \mathfrak{P}^r_{i+1} + \mathfrak{J}^1)
\]
\[
= \prod_{i=-1}^{s} \frac{(\mathfrak{P}^r_i : \mathfrak{P}^r_{i+1})}{(\mathfrak{P}^r_i \cap \mathfrak{J} : \mathfrak{P}^r_{i+1} \cap \mathfrak{J})}
\]
\[
= \prod_{i=-1}^{s} \frac{(\mathfrak{P}^r_i : \mathfrak{P}^r_{i+1})}{(\mathfrak{P}^r_i \cap A_{\gamma_i} : \mathfrak{P}^r_{i+1} \cap A_{\gamma_i})},
\]
where we set \( r_{-1} = 1 \). The last line follows from [4, (3.1.8), (3.1.10)]. For \(-1 \leq i \leq s\), set
\[
d_i = r_{i+1} - r_i.
\]
Since \( \mathfrak{P} \) is principal,
\[
(\mathfrak{P}^r_i : \mathfrak{P}^r_{i+1}) = (\mathfrak{A} : \mathfrak{P})^{d_i} = |M_{f_0}(k_F)|^{e_0 d_i} = q^{N f_0 d_i}.
\]
Set \( f'_i = f(F[\gamma_i]|F), \quad e'_i = e(F[\gamma_i]|F) \), and
\[
m_i = (N/|F[\gamma_i] : F|)/(e_0/e'_i) = f_0/f'_i.
\]
Then
\[
(\mathfrak{P}^r_i \cap A_{\gamma_i} : \mathfrak{P}^r_{i+1} \cap A_{\gamma_i}) = |M_{m_i}(k_{F[\gamma_i]})|^{(e_0/e'_i)d_i} = q^{N f_0 d_i/|F[\gamma_i]:F|}.
\]
Hence,
\[
(\mathfrak{P} : \mathfrak{J}^1) = q^{\mu}; \quad \mu = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i]:F]^{-1})d_i.
\]
Similarly, setting \( d'_i = \overline{r_{i+1}} - \overline{r_i} \), we get
\[
(\mathfrak{P} : \mathfrak{H}^1) = q^{\mu'}; \quad \mu' = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i]:F]^{-1})d'_i.
\]
Finally, we obtain \( (\mathfrak{P} : \mathfrak{J}^1)^{1/2}(\mathfrak{P} : \mathfrak{H}^1)^{1/2} = q^{\nu/2} \) with
\[
(1.7) \quad \nu = \sum_{i=-1}^{s} N f_0 (1 - [F[\gamma_i]:F]^{-1})(r_{i+1} - r_i).
\]

1.8. Main Theorem for \( GL_N(F) \). We are now ready to determine the explicit formal degree formula for an irreducible discrete series representation \((\pi, \mathcal{V})\) of \( G \) as in Section 1.3.

Theorem 1.4. ([13, Theorem 1.1]) Let \((\pi, \mathcal{V})\) be an irreducible discrete series representation of \( G \) that contains a simple type \((J, \lambda)\) in \( G \) associated with a simple stratum \([A, n, 0, \beta]\) in \( A \) as in Section 1.3. Let
Let \( dx \) be a Haar measure on \( G \) such that \( \deg(\text{St}_G, dx) = 1 \). For a family \( (r_i, \gamma_i), \ 0 \leq i \leq s \), as in Section 1.7, set

\[
\Delta = \frac{1}{e(\mathfrak{A}|0_F)} \sum_{i=-1}^{s} (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i)
\]

where we reset \( (r_{-1}, \gamma_{-1}) = (0, \beta) \) and \( r_{s+1} = n \). Then this positive rational number \( \Delta \) does not depend on the choice of defining sequence, and

\[
deg(\pi, dx) = f \cdot \frac{q^N - 1}{q^{N/e} - 1} \cdot q^\frac{1}{2}[N^2\Delta - N(1 - 1/e)],
\]

where \( f = f(K|F) \) and \( e = e(K|F) = e(E|F) \).

Proof. Denote by \( \deg(\pi, dx)_{p'} \) the \( p \)-prime part of \( \deg(\pi, dx)/f \). From Lemma 1.3 and from Eqs. (1.4) to (1.6), it follows immediately that

\[
f \cdot \deg(\pi, dx)_{p'} = f \cdot \frac{q^N - 1}{q^{N/e} - 1}.
\]

Since the \( q \)-power of (1.5) times (1.6) is equal to

\[
\frac{1}{2}e_0f_0(f_0 - 1) - \frac{1}{2}f(E|F)e_1f_1(f_1 - 1)
\]

\[
= \frac{1}{2}N\left(1 - \frac{1}{e}ight) + \frac{1}{2}Nf_0\left(1 - \frac{1}{[E : F]}ight),
\]

the sum of this value and the value \( \nu \) of (1.7) is reduced to the \( q \)-power of the right-hand side of the formula in the assertion.

It follows directly from [4, (2.1.4)] that \( \Delta \) does not depend on the choice of defining sequence. The proof is complete.

In Theorem 1.4, if \( e_1 = e(\mathfrak{B}|0_E) = 1 \), the irreducible discrete series representation \( (\pi, \mathcal{V}) \) of \( G \) is supercuspidal. Thus, Theorem 1.4 contains the formal degree formula for an irreducible supercuspidal representation of \( G \) containing a maximal simple type \( (J, \lambda) \) as follows:

Corollary 1.5. Let \( (\pi, \mathcal{V}) \) be an irreducible supercuspidal representation of \( G \) containing a maximal simple type \( (J, \lambda) \) in \( G \), and \( \{(r_i, \gamma_i) \} : 0 \leq i \leq s \) a family as in Theorem 1.3. Then

\[
deg(\pi, dx) = f \cdot \frac{q^N - 1}{q^{N/e} - 1} \cdot q^\frac{1}{2}[N^2\Delta - N(1 - 1/e)],
\]

for a Haar measure \( dx \) on \( G \) with \( \deg(\text{St}_G, dx) = 1 \), where

\[
\Delta = \frac{1}{e} \sum_{i=-1}^{s} (1 - [F[\gamma_i] : F]^{-1})(r_{i+1} - r_i).
\]

Remarks 1.6. (i) It is shown by [4, (6.2.2)] that an irreducible supercuspidal representation \( (\pi, \mathcal{V}) \) of \( G \) containing a maximal simple type
(J, \lambda) in G of positive level is equivalent to c-Ind_{E^xJ}^{G}\Lambda for an extension \Lambda of \lambda. From this fact, we obtain
\[
\text{deg}(\pi, dx) = \frac{\dim(\lambda)}{\text{vol}(E^xJ/F^x, dx)}
\]
for any Haar measure dx on G, (cf. [7, 5.9]), independently from Theorem 1.2, and can similarly show Corollary 1.5.

(ii) In Theorem 1.4, the level zero case is implicit. In this case, $E = F$, $\mathfrak{B} = \mathfrak{A}$, $J^t = U^t(\mathfrak{A})(t = 0, 1)$ and $\eta$ is trivial, as in Section 1.2. Thus, in the formula for $\text{deg}(\pi,dx)$ of Lemma 1.3, we have $(U^1(\mathfrak{A}) : J^1) \dim(\eta) = 1$, and can get
\[
\text{deg}(\pi, dx) = \frac{N}{e(\mathfrak{A}|F)}
\]
similarly to as done in Section 1.6, for such a Haar measure dx on G as above.

(iii) This formula follows also from Theorem 1.4, by setting $e = 1$, $\Delta = 0$ and $\lambda = \lambda_1$, since $E = F$, $J = U(\mathfrak{A})$ and so the defining sequence is empty. Hence, by Theorem 1.4, the formal degrees of all discrete series representations of $GL_N(F)$ are computed.

2. AN APPLICATION TO UNRAMIFIED p-ADIC UNITARY GROUPS

2.1. Unramified unitary groups. Let $F$ be a non-Archimedean local field of odd residual characteristic, with a non-trivial galois involution $x \mapsto \overline{x}$ with fixed field $F_0$. Let $N$ be an even integer $\geq 4$, and $V$ an $N$-dimensional $F$-vector space equipped with a non-degenerate $F/F_0$-Hermitian form $h$ with anisotropic part $(0)$. Let $G = U(V, h)$ be the unitary group of $(V, h)$. Hereafter, we assume that $F/F_0$ is unramified.

Recently, in [9], we defined a self-dual simple type $(J, \lambda)$ in $G$ associated with a certain skew simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, and, in [11], we proved that the Hecke algebra $\mathcal{H}(G, \lambda)$ is the affine Hecke algebra of type $\tilde{C}_m$ for some positive integer $m \geq 2$, and determined the parameters of the Hecke algebra completely. Thanks to these results on $G$, we can apply the improved method of the previous section for $GL_N(F)$ to the unramified unitary group $G$, and we obtain analogous results for the group $G$. Here we present a part of these results without proofs.

2.2. Self-dual simple types. We also denote by $x \mapsto \overline{x}$ the adjoint (anti-)involution on $A = \text{End}_F(V)$ induced by the Hermitian form $h$.

A simple stratum $[\mathfrak{A}, n, 0, \beta]$ in $A$, defined in Section 1.1, is called skew if $\mathfrak{A}$ is defined by a self-dual strict $\mathfrak{o}_F$-lattice sequence $\Lambda$ in $V$ (cf. [14, 1.2]) and $\beta$ is skew in $A$, i.e., $\overline{\beta} = -\beta$. Assume that $[\mathfrak{A}, n, 0, \beta]$ is a skew simple stratum in $A$ with $E = F[\beta]$. Write $E_0 = \{x \in E : \overline{x} = x\}$. Then there exists a non-degenerate $E/E_0$-Hermitian form $h_E$ on the $E$-vector space $V$ such that, setting $L^\# = \{v \in V : h_E(v, L) \subset \mathfrak{p}_E\}$ for an $\mathfrak{o}_F$-lattice $L$ in $V$, we have $L^\# = \{v \in V : h(v, L) \subset \mathfrak{p}_F\}$ (cf. [15, Section 2]).
Definition 2.1. Following [9], we say that a skew simple stratum \( \mathfrak{A}, n, 0, \beta \) in \( A \) is good if the following conditions are satisfied:

1. \( E/E_0 \) is an unramified quadratic extension;
2. \( R = \dim_E(V) \) is even;
3. there exists an \( \sigma_E \)-lattice \( L \) in \( \{ \Lambda(n) : n \in \mathbb{Z} \} \) such that \( L^\# = \omega_EL \). where \( \Lambda \) is as above and \( \omega_E \) is a uniformizer of \( E \).

Hereafter, we assume that \( \mathfrak{A}, n, 0, \beta \) is a good skew simple stratum in \( A \) with \( E = F[\beta] \). Let \( B \) be the \( A \)-centralizer of \( \beta \), \( \mathfrak{B} = B \cap \mathfrak{A} \), 
\( e_1 = e(\mathfrak{B}|0_E) \), and \( f_1 = N/([E : F]e_1) \), as before.

Similarly, we have compact open subgroups \( H^1(\beta, \mathfrak{A}) \subset J^1(\beta, \mathfrak{A}) \subset J(\beta, \mathfrak{A}) \) of \( G \). Denote simply by \( H^1 \subset J^1 \subset J \) these subgroups. As in Section 1.2, we begin with a skew simple character \( \theta \) of \( H^1 \), and there exists a unique irreducible representation \( \eta \) of \( J^1 \) containing \( \theta \). Moreover, we also have a \( \beta \)-extension \( \kappa \) of \( \eta \) to \( J \) by [15, 4.2]. Set

\[ m = [e_1/2]. \]

It follows from the conditions of Definition 2.1 that the quotient group \( J/J^1 \) is isomorphic to

\[ U(\mathfrak{B})/U^1(\mathfrak{B}) \simeq \begin{cases} GL_{f_1}(k_E)^m & \text{if } e_1 \text{ is even}, \\ GL_{f_1}(k_E)^m \times U_{f_1}(k_{E_0}) & \text{if } e_1 \text{ is odd}, \end{cases} \]

where by Definition 2.1(2), \( f_1 \) is even and \( U_{f_1}(k_E/k_{E_0}) \) denotes the unitary group of a non-degenerate \( k_E/k_{E_0} \)-Hermitian form on an \( f_1 \)-dimensional \( k_E \)-vector space. Let \( \sigma_0 \) and \( \sigma_1 \) be irreducible cuspidal representations of \( GL_{f_1}(k_E) \) and \( U_{f_1}(k_E/k_{E_0}) \), respectively. Let \( \sigma \) be the inflation of \( \sigma_0^{\otimes m} \) to \( J \) if \( e_1 \) is even, and that of \( \sigma_0^{\otimes m} \otimes \sigma_1 \) if \( e_1 \) is odd. A simple type \( (J, \lambda) \) in \( G \) (of positive level) is defined similarly by \( \lambda = \kappa \otimes \sigma \). By [15, p.334], on each factor \( GL_{f_1}(k_E) \) of \( U(\mathfrak{B})/U^1(\mathfrak{B}) \), a certain Weyl group element of \( G_E = B \cap G \) induces an involution \( g \mapsto \overline{g} \).

Definition 2.2. ([11, Definition 5.2]) A simple type \( (J, \lambda) \) in \( G \) is called self-dual if \( \sigma_0 \) is equivalent to the representation \( g \mapsto \sigma_0(\overline{g}) \).

2.3. A formal degree formula for unramified \( U(V, h) \). Recently, by [10] and [11], we determined the structure of the Hecke algebra \( \mathcal{H}(G, \lambda) \) for such a self-dual simple type \( (J, \lambda) \) above as follows:

Proposition 2.3. Let \( \mathfrak{A}, n, 0, \beta \) be a good skew simple stratum in \( A \) with \( E = F[\beta] \), and \( (J, \lambda) \) a self-dual simple type in \( G \) associated with \( \mathfrak{A}, n, 0, \beta \) in \( A \). Let \( B \) be the \( A \)-centralizer of \( \beta \), \( e_1 = e(\mathfrak{B}|0_E) \), and \( m = [e_1/2] \). Assume that \( m \geq 2 \). Then the Hecke algebra \( \mathcal{H}(G, \lambda) \) for \( (J, \lambda) \) is an affine Hecke algebra of type \( \widetilde{C}_m \) with parameter \( (q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0}) \), where \( e_0 = e(\mathfrak{A}|0_F) \) as in Section 1.1.

Let \( K/E_0 \) be a field extension such that \( K \supset E, K \supset K_0 \supset E_0, e(K|E_0) = 1 \), and \( [K : E] = [K_0 : E_0] = f_1 \). Let \( C^\times \) be the unitary
group of type $C_m$ defined by a non-degenerate $K/K_0$-Hermitian form, and $I$ an Iwahori subgroup of $C^\times$. Immediately, $|k_K| = |k_E|^{1/2} = q^{N/e_0}$, and so the Iwahori-Hecke algebra $\mathcal{H}(C^\times, 1_I)$ turns out to be also the affine Hecke algebra of type $\tilde{C}_m$ with parameter $(q_1, q_2, q_3) = (q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})$ provided that $m \geq 2$.

When $m = [e_1/2] = 1$, again by [11], $\mathcal{H}(G, \lambda)$ is isomorphic to the affine Hecke algebra with parameter $|k_{K_0}| = q^{N/2e_0}$ over the infinite dihedral group, and so is the Iwahori-Hecke algebra $\mathcal{H}(C^\times, 1_I)$ of unramified $C^\times = U(1,1)(K_0)$ relative to $1_I$ as well (cf. [2, 3.d]). Hence, we obtain the following:

**Proposition 2.4.** Let notations and assumptions be as above. In particular, let $m \geq 1$. Then there exists a canonical isomorphism

$$\Psi : \mathcal{H}(G, \lambda) \simeq \mathcal{H}(C^\times, 1_I),$$

that is support-preserving.

**Theorem 2.5.** Via the Hecke isomorphism $\Psi$ in Proposition 2.4, the Steinberg representation $St_{C^\times}$ of $C^\times$ corresponds to the equivalence class of an irreducible square-integrable representation, say $(\pi, \mathcal{V})$, of $G$ that contains the self-dual simple type $(J, \lambda)$ in $G$ as above.

**Proof.** This is the analogue of [5, (7.7.1)] for $GL_N(F)$. The method of proof remains valid for unramified $G$.

**Corollary 2.6.** Let notations and assumptions be as in Theorem 2.5. Then

$$\text{vol}(J, dx) \frac{\deg(\pi, dx)}{\dim(\lambda)} = \text{vol}(I, dy) \deg(St_{C^\times}, dy)$$

for any Haar measures $dx$ on $G$ and $dy$ on $C^\times$.

### 2.4. A formal degree formula for unramified $G$.

We normalize $dx$ on $G$ so that the formal degree of the Steinberg representation $St_G$ of $G$ is equal to 1 relative to $dx$. Then, as in Section 1.5, the formal degree $\deg(\pi, dx)$ is rewritten as

$$\frac{\overline{W}_{C_{N/2}}(q^{-1}, q^{-1/2}, q^{-1/2})}{\overline{W}_{C_m}(q^{-N/e_0}, q^{-N/2e_0}, q^{-N/2e_0})} \left( U(A_m) : U^1(\mathfrak{A}) \right) \left( \frac{\dim(\sigma)}{(U(\mathfrak{B}) : U^1(\mathfrak{B}))} \right) \left( (U^1(\mathfrak{A}) : J^1) \dim(\eta) \right),$$

where $U(A_m)$ is an Iwahori subgroup of $G$ that is contained in $U(\mathfrak{A})$, and, for example, $\overline{W}_{C_m}(t_1, t_2, t_3)$ denotes the Poincaré series of type $\tilde{C}_m$ (see [12, Section 3]). We note that this also holds in the case of $m = 1$. For, if we formally set $m = 1$ in the Poincaré series $\overline{W}_{C_m}(t_1, t_2, t_3)$, we have

$$\frac{(1-t_1)(1+t_2)(1+t_3)}{(1-t_1)(1-t_2t_3)} = \frac{(1+t_2)(1+t_3)}{1-t_2t_3}.$$
This is nothing but the Poincaré series for the infinite dihedral group.

Similarly, we can easily compute the factors except for the last $(U^{1} awful : J^{1}) \dim(\eta)$ as in Section 1.6.

The calculation of this last factor is rather more laborious than that for $GL_{N}$ in Section 1.7. We also have a defining sequence for the pure stratum $[\mathfrak{U}, n, r, \beta]$ in $A$ with $r = -k_{0}(\beta, \mathfrak{U})$, and get a family $\{(r_{i}, \gamma_{i})|0 \leq i \leq s\}$, together with $(r_{-1}, \gamma_{-1}) = (1, \beta)$, where $\gamma_{i}$ is skew and simple in $A$, by [14, Section 3], as in Section 1.7. Set $d_{i} = ((r_{i+1} + 1)/2) - [r_{i}/2]$ and $d'_{i} = [r_{i+1}/2] - [r_{i}/2]$, for $-1 \leq i \leq s$, as before.

To present the main theorem, we need to define positive integers $\delta_{i}$, for $-1 \leq i \leq s$, as follows: set $\delta_{i} = 0$ if $e_{0}/e = e(\mathfrak{B}|0_{E})$ is even, and otherwise,

\[
\delta_{i} = \frac{d_{i}}{2} + \frac{(d'_{i} + 1)/2}{2} \quad \text{if} \quad r_{i} \equiv 0 \text{ mod 4,}
\]

\[
\delta_{i} = \frac{(d_{i} + 1)/2}{2} + \frac{d'_{i}}{2} \quad \text{if} \quad r_{i} \equiv 1 \text{ mod 4,}
\]

\[
\delta_{i} = \frac{(d_{i} + 1)/2}{2} + \frac{(d'_{i} + 1)/2}{2} \quad \text{if} \quad r_{i} \equiv 2 \text{ mod 4,}
\]

\[
\delta_{i} = \frac{d_{i}}{2} + \frac{d'_{i}}{2} \quad \text{if} \quad r_{i} \equiv 3 \text{ mod 4.}
\]

**Theorem 2.7.** Let $(\pi, \mathcal{V})$ be an irreducible square-integrable representation of $G$ containing a self-dual simple type $(J, \lambda)$ in $G$ as in Theorem 2.5. Let $\{(r_{i}, \gamma_{i})|0 \leq i \leq s\}$ be a family defined as above. Let $e_{0} = e(\mathfrak{U}|0_{F})$, $e = e(E|F)$, and $e_{i}^{0} = e(F[\gamma_{i}]/F[\gamma_{i}])$, for $0 \leq i \leq s$, where each $F[\gamma_{i}]$ is the fixed field of $F[\gamma_{i}]$ under the involution induced by the adjoint one $x \mapsto \overline{x}$ on $A$ in Section 2.2. Set

\[
\Delta = \frac{1}{e_{0}} \sum_{i=-1}^{s} \left(1 - \frac{1}{[F[\gamma_{i}] : F]}\right)(r_{i+1} - r_{i}),
\]

\[
\Delta' = \frac{1}{e_{0}} \sum_{i=-1}^{s} (1 - e_{i}^{0})\delta_{i}
\]

where each $\delta_{i}$ is the integer defined above and we reset $(r_{-1}, \gamma_{-1}) = (0, \beta)$ and $r_{s+1} = n$. Then these $\Delta$ and $\Delta'$ do not depend on the choice of defining sequence, and the formal degree $\deg(\pi, dx)$ is given by

(1) when $e_{0}/e = e(\mathfrak{B}|0_{E})$ is even,

\[
q^{1/[N^{2}\Delta - N(1-1/e) + N\Delta']} \times \prod_{i=0}^{e_{0}/2e-1} \frac{q^{N(i+e_{0}/2e)/e_{0} - 1}}{q^{N(i+1)/e_{0}} - 1} \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1} - 1)(q^{i+1} + 1)^{2}}{q^{N/2+i-1} - 1},
\]

(2) when $e_{0}/e = e(\mathfrak{B}|0_{E})$ is odd,
\[
\frac{q^{\frac{1}{4}\left[N^2\Delta - N(1-1/e) + N\Delta'\right]}}{q^{N/2e_0 - 1}^{(e_0/e-3)/2} \times \prod_{i=0}^{N/2-1} \frac{(q^{i+1} - 1)(q^{i+1} + 1)^2}{q^{N/2 + i - 1} - 1}}\]

 Remarks 2.8. (1) We also obtained analogous results for \(Sp_N(F)\). This formal degree formula is more complicated than that in Theorem 2.7 for the unramified unitary group \(G\).

(2) In [11], it is proved that Theorem 2.5 holds for a self-dual simple type \((J, \lambda)\) in \(G\) associated with not only a skew simple stratum \([\mathfrak{A}, n, 0, \beta]\) in \(A\) which is not good (see Definition 2.1), but also a skew (non-simple) semisimple stratum \([\Lambda, n, 0, \beta]\) in \(A\), that is, \(\beta\) is a skew semisimple element of \(A\) (cf. [14]).

(3) We constructed a special skew semisimple stratum \([\Lambda, n, 0, \beta]\) in \(A\), and computed the corresponding formal degree as well.

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