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A System of Fifth-Order Nonlinear Partial Differential Equations and a Surface Which Contains Many Circles

Dedicated to Professor Naoto Kumano-go and Professor Susumu Yamazaki

By

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Abstract

Let $z = f(x, y)$ be a germ of a C^5-surface at the origin in $\mathbb{R}^3$ containing several continuous families of circles. Indeed, we have a usual torus with 4 such families and R. Blum's cyclide with 6 such families. Then, we get a system of fifth-order nonlinear partial differential equations for f. As an application, we obtain the analyticity of f, and the finite dimensionality of the solution space of such system of differential equations.

§ 1. A Surface Containing Several Continuous Families of Circles

In 1848, Yvon Villarceau [1] found that a usual torus includes 4 continuous families of circles passing through every point of the surface; of course, only two of them are new. These new circles, so called Villarceau circles, are slanted against the rotation axis and are not perpendicular to this axis. Further in 1980, Richard Blum [2] found that some special cyclides include 4~6 continuous families of circles passing through every point of them. Here, a general cyclide is defined by a quartic equation

$$\alpha(x_1^2 + x_2^2 + x_3^2)^2 + 2(x_1^2 + x_2^2 + x_3^2)\sum_{i=1}^{3}\beta_i x_i + \sum_{i,j=1}^{3}\gamma_{ij}x_i x_j + 2\sum_{i=1}^{3}\delta_i x_i + \epsilon = 0 \tag{1.1}$$

with real numbers $\alpha \neq 0, \beta_i, \gamma_{ij}, \delta_i, \epsilon$ (Darboux [3]). Then a usual torus and a 6-circle Blum cyclide correspond to the case $\alpha = 1, \beta_* = 0$, $\delta_* = 0$, $\gamma_{ij} = -2a_i\delta_{ij}$, $\epsilon = \ell^2$ with $0 < \ell < a_1 =$

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$a_2, a_3 = -\ell$, and to that with $0 < \ell < a_2 < a_1$, $-\ell \neq a_3 < \ell$, respectively. At the same time, Blum gave the following conjecture in [2]:

Conjecture (R.Blum) *A closed C^∞-surface in $\mathbb{R}^3$ which contains seven circles through each point is a sphere.*

N. Takeuchi [4] solved this conjecture affirmatively for closed surfaces with genus ≤ 1 by using the intersection number theory for 1-dimensional homotopy groups. Further, replacing 1-dimensional homotopy groups by 1-dimensional homology groups with $\mathbb{Z}_2$-coefficients, we can easily get the following extension:

Theorem 1.1. *We have some positive integer N_g ($\leq 2^{2g+1} - 1$) for any $g = 1,2,3,\ldots$ such that, for $\forall g \geq 1$, there is no closed surface with genus g in E^3 which contains N_g circles through each point. In particular, we can take $N_2 = 11$.*

Proof. Let M be a closed surface in $\mathbb{R}^3$ with genus $g \geq 1$. Then it is well-known that the intersection number $\mathrm{Int}(C_1, C_2)$ for closed curves $C_1, C_2 \subset M$ is a skew-symmetric bi-additive form

$$H_1(M,\mathbb{Z}) \otimes H_1(M,\mathbb{Z}) \ni ([C_1],[C_2]) \mapsto \mathrm{Int}(C_1,C_2) \in \mathbb{Z}.$$

Further the 1-dimensional homology group of M is given by

$$H_1(M,\mathbb{Z}) = \mathbb{Z}[\alpha_1] \oplus \mathbb{Z}[\beta_1] \oplus \cdots \oplus \mathbb{Z}[\alpha_g] \oplus \mathbb{Z}[\beta_g] \simeq \mathbb{Z}^{2g}$$

with generators $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g$ satisfying

$$\mathrm{Int}(\alpha_i,\alpha_j) = \mathrm{Int}(\beta_i,\beta_j) = 0, \quad \mathrm{Int}(\alpha_i,\beta_j) = -\mathrm{Int}(\beta_j,\alpha_i) = \delta_{ij} \quad (i,j = 1,\ldots,g).$$

According to [4], for two circles C_1, C_2 on M passing through a point p, there are only two possibilities ; i) $C_1 \cap C_2 = \{p\}$ and they cross each other transversally at p, ii) C_1 and C_2 have two points in common, or C_1 is tangent to C_2 at p. Then the value of $\mathrm{Int}(C_1, C_2)$ is equal to ± 1 in case i), and to 0 in case ii). Therefore, the value $\widetilde{\mathrm{Int}}(C_1,C_2) := [\mathrm{Int}(C_1,C_2)]$ in $\mathbb{Z}_2$ is sufficient to know this difference. Hence we can consider homology groups with $\mathbb{Z}_2$-coefficients:

$$\widetilde{\mathrm{Int}}\colon H_1(M,\mathbb{Z}_2) \otimes H_1(M,\mathbb{Z}_2) \ni ([C_1],[C_2]) \mapsto [\mathrm{Int}(C_1,C_2)] \in \mathbb{Z}_2.$$

On the other hand, since $G := H_1(M,\mathbb{Z}_2) = \mathbb{Z}_2^{2g}$ is a finite group of order 2^{2g}, we can divide G into a direct sum $G = S_1 \cup \cdots \cup S_{n_g}$ of subsets $S_j \subset G$ ($j = 1, \ldots, n_g$) satisfying the following conditions:

1. $S_i \cap S_j = \emptyset$ for $i, j = 1, \ldots, n_g$ ($i \neq j$),
2. for any $i = 1, \ldots, n_g$, $\widetilde{\mathrm{Int}}(c,c') = 0$ on $S_i \times S_i$.

In fact, we can take $n_g \leq 2^{2g} - 1$ for any $g \geq 1$ because we have a trivial decomposition $G = S_1 \cup \cdots \cup S_{2^{2g}-1}$ satisfying the above-conditions; S_1 is chosen as a two-element subset containing 0 of G, and $S_2, \ldots, S_{2^{2g}-1}$ as a one-element subset of G. Further for $g = 2$, we can take $n_2 = 5 (< 2^4 - 1)$ by setting

$$S_1 = \{OO, OA, AO, AA\}, S_2 = \{OB, BO, BB\}, S_3 = \{OC, CO, CC\},$$

$$S_4 = \{AB, BC, CA\}, \quad S_5 = \{BA, CB, AC\}.$$

Here, we use the following notation: $O = 0, A = [\alpha], B = [\beta], C = [\alpha] + [\beta]$ for single letters O, A, B, C, and double letters AB, AC mean that

$$AB := [\alpha_1] + [\beta_2], \quad AC = [\alpha_1] + ([\alpha_2] + [\beta_2]).$$

Indeed, since $\widetilde{\mathrm{Int}}(*, *)$ is calculated as $\widetilde{\mathrm{Int}}(AB, BC) = A \cdot B + B \cdot C \pmod 2$ with an inner product between O, A, B, C:

$$O \cdot A = A \cdot O = O \cdot B = B \cdot O = O \cdot C = C \cdot O = A \cdot A = B \cdot B = C \cdot C = 0,$$
$$A \cdot B = B \cdot C = C \cdot A = B \cdot A = C \cdot B = A \cdot C = 1,$$

we easily verify the condition 2 for S_i. Suppose that M contains $N_g := 2n_g + 1$ circles through each point of M. Since M is not a sphere, nor a plane, there is a non-umbilical point $p \in M$; that is, two principal curvatures at p is different from each other. Take $N_g = 2n_g + 1$ circles $C_1, ..., C_{2n_g+1}$ through p. Since $G = S_1 \cup \cdots \cup S_{n_g}$, some three circles $C_{i_1}, C_{i_2}, C_{i_3}$ $(i_1 < i_2 < i_3)$ must belong to some S_j. By the condition 2 for $\{S_i\}_i$ we know that any two of $C_{i_1}, C_{i_2}, C_{i_3}$ have two points in common, or are tangent to each other at p. Therefore by Theorem 1 of [7] we conclude that p is an umbilical point. This contradicts that p is a non-umbilical point. This completes the proof. □

James A. Montaldi [8], by using infinitesimal analysis, proved that there is an open dense set of immersions $g : X \longrightarrow \mathbb{R}^3$, such that at any point $x \in X$, there will be at most 10 circles with at least 5-point contact with the surface at x. Here, "a circle has 5-point contact to X at x" means that $g(X)$ includes a circle infinitesimally up to $(5 - 1)$-th derivatives at x. On the other hand, Takeuchi [5] proved the following theorem concerning general cyclides by using conformal transformations; that is, translations, rotations, and inversions (for example, $\vec{x} = \vec{y}/|\vec{y}|^2$), which transform a circle into a circle or a line in $\mathbb{R}^3$.

Theorem 1.2. *A non-singular cyclide is conformally equivalent to a cyclide of Blum's type which is homeomorphic to a torus, a sphere or two spheres; that is, a cyclide with parameters* $\alpha = 1, \beta_* = 0, \delta_* = 0, \gamma_{ij} = -2a_i\delta_{ij}, \epsilon \neq 0$ *for some* $a_1, a_2, a_3 \in \mathbb{R}$. *Further, A cyclide contains* n *circles through each non-umbilical point and* $n - 1$ *circles through each isolated umbilical point unless it is a sphere or a pair of two spheres, where* $n = 1, 2, 3, 4, 5$ *or* 6.

In [6], Takeuchi gave some explicit expressions of continuous families of circles for 5-circle Blum cyclides. Our purpose in this paper is to find all the surface germs $z = f(x, y)$ at the origin for given integer n (≥ 2) which contain n continuous families of circles.

§ 2. The System of Fifth-Order Non-Linear Partial Differential Equations and the Results

Let $z = f(x, y)$ be a C^4-class function defined in a neighborhood of $(0, 0) \in \mathbb{R}^2$. For a surface germ $z = f(x, y)$ at $(0, 0)$, we can normalize it by suitable translations and rotations as follows:

(2.1) $$f(0,0) = 0, f_x(0,0) = 0, f_y(0,0) = 0, f_{xy}(0,0) = 0.$$

Then, if $(x,y)=(0,0)$ is not an umbilical point (that is, $f_{xx}(0,0)\neq f_{yy}(0,0)$), we can define polynomials in T and t of degree 10, *the key polynomial* $Z(T)$, and *the characteristic polynomial* $P(t)$ for the germ $z=f(x,y)$ at $(0,0)$. As we see in Theorem 2.3, any non-zero simple real root of $P(t)=0$ corresponds to a continuous family of circles contained in $\{z=f(x,y)\}$.

Definition 2.1. (*the key polynomial* $Z(T)$ and *the characteristic polynomial* $P(t)$) Let $z=f(x,y)$ be a C^4-class function defined in a neighborhood of $(0,0)\in\mathbb{R}^2$ satisfying condition (2.1). Put the Taylor coefficients $a,b,c_0,c_1,c_2,d_0,d_1,d_2,d_3,e_0,e_1,e_2,e_3,e_4$ at (x,y) of f as follows:

$$(2.2)\quad \begin{cases} a:=f_x(x,y), & b:=f_y(x,y), \\ c_0:=f_{xx}(x,y)/2, & c_1:=f_{xy}(x,y), \qquad c_2:=f_{yy}(x,y)/2, \\ d_0:=f_{xxx}(x,y)/3!, & d_1:=f_{xxy}(x,y)/2!, \\ d_2:=f_{xyy}(x,y)/2!, & d_3:=f_{yyy}(x,y)/3!, \\ e_0:=f_{xxxx}(x,y)/4!, & e_1:=f_{xxxy}(x,y)/3!, \qquad e_2:=f_{xxyy}(x,y)/2!^2, \\ e_3:=f_{xyyy}(x,y)/3!, & e_4:=f_{yyyy}(x,y)/4!. \end{cases}$$

We define polynomials $C(T)$, $D(T)$, $E(T)$, $R(T)$, $S(T)$, $K(T)$, $W(T)$ in T in the following way, where $C'(T)=\partial_T C(T)$, $R'(T)=\partial_T R(T)$ etc.:

$$(2.3)\qquad C(T)=c_0+c_1T+c_2T^2,$$

$$(2.4)\qquad D(T)=d_0+d_1T+d_2T^2+d_3T^3,$$

$$(2.5)\qquad E(T)=e_0+e_1T+e_2T^2+e_3T^3+e_4T^4,$$

$$(2.6)\qquad R(T)=(b^2+1)T^2+2abT+a^2+1,$$

$$(2.7)\qquad S(T)=D(T)R(T)-2(bT+a)C(T)^2,$$

$$(2.8)\qquad \begin{aligned} K(T)&=R'(T)C(T)-R(T)C'(T) \\ &=\big((b^2+1)c_1-2abc_2\big)T^2+2\big((b^2+1)c_0-(a^2+1)c_2\big)T \\ &\quad +2abc_0-(a^2+1)c_1, \end{aligned}$$

$$(2.9)\qquad W(T)=bS(T)+C(T)K(T).$$

Then *the key polynomial* $Z(T)$ for f is defined by

$$(2.10)\quad \begin{aligned} Z(T)&=K(T)^2\big(R(T)E(T)-C(T)^3\big)+R(T)K(T)D(T)\big(D'(T)R(T)-3(b^2+1)TD(T)\big) \\ &\quad +D(T)^2R(T)[-ab\big(2K(T)+TK'(T)\big)-2(a^2+1)(b^2+1)C(T) \\ &\quad +\big((a^2+1)c_2+(b^2+1)c_0\big)R(T)]+2R(T)C(T)[(bT+a)\{D(T)K'(T)C(T) \\ &\quad +D(T)K(T)C'(T)-D'(T)K(T)C(T)\}-bD(T)C(T)K(T)]+4C(T)^4(bT+a) \\ &\quad \times\{\big((a^2-1)c_2+(b^2+1)c_0\big)(bT+a)-\frac{1}{2}ac_1R'(T)+2a(c_2-c_0)-bc_1\}. \end{aligned}$$

It is easy to verify that (the degree of $Z(T)$ in T) ≤ 10. Further, *the characteristic polynomial* $P(t)$ at $(0,0)$ is defined by

$$(2.11)\qquad P(t):=\left[\frac{Z(t)}{c_0-c_2}\right]_{x=y=0}$$

$$= \Big[(t^2+1)D(t)\{2t(t^2+1)D'(t)-(5t^2+1)D(t)\}$$
$$+4(c_0-c_2)t^2\{(t^2+1)E(t)-C(t)^3\}\Big]_{x=y=0}$$
$$= \Big[-d_0^2+\big(-4c_0^3(c_0-c_2)-6d_0^2+d_1^2+2d_0d_2+4(c_0-c_2)e_0\big)t^2$$
$$+\big(-8d_0d_1+4d_1d_2+4d_0d_3+4(c_0-c_2)e_1\big)t^3$$
$$+\big(-12c_0^2(c_0-c_2)c_2-5d_0^2-2d_1^2$$
$$-4d_0d_2+3d_2^2+6d_1d_3+4(c_0-c_2)e_0+4(c_0-c_2)e_2\big)t^4$$
$$+\big(-8d_0d_1+8d_2d_3+4(c_0-c_2)e_1+4(c_0-c_2)e_3\big)t^5$$
$$+\big(-12c_0(c_0-c_2)c_2^2-3d_1^2-6d_0d_2$$
$$+2d_2^2+4d_1d_3+5d_3^2+4(c_0-c_2)e_2+4(c_0-c_2)e_4\big)t^6$$
$$+\big(-4d_1d_2-4d_0d_3+8d_2d_3+4(c_0-c_2)e_3\big)t^7$$
$$+\big(-4(c_0-c_2)c_2^3-d_2^2-2d_1d_3+6d_3^2+4(c_0-c_2)e_4\big)t^8+d_3^2t^{10}\Big]_{x=y=0},$$

when the origin is not an umbilical point of $z=f(x,y)$.

Example 2.2. We consider the Blum 6-circle cyclide (the roles of y,z are interchanged with each other, and a_3 is replaced by $-a_3$):

$$S:\ (x^2+y^2+z^2)^2-2(a_1x^2+a_2z^2)+2a_3y^2+\ell^2=0,$$

where $a_1>a_2>\ell>0, a_3>\ell$. Then the characteristic polynomial at $(0,0,\sqrt{a_2\pm m})$ $(m=\sqrt{a_2^2-\ell^2})$ for S reduces to the following polynomial of degree 8:

$$P(t)=4(c_0-c_2)t^2\{(a_2+a_3)t^2-(a_1-a_2)\}\{(a_3+\ell)t^2-(a_1-\ell)\}\{(a_3-\ell)t^2-(a_1+\ell)\}.$$

Therefore $P(t)=0$ has 6 non-zero simple real roots

$$t=\pm\sqrt{\frac{a_1-a_2}{a_2+a_3}},\quad \pm\sqrt{\frac{a_1-\ell}{a_3+\ell}},\quad \pm\sqrt{\frac{a_1+\ell}{a_3-\ell}}.$$

In the degenerate case $(a_1=a_2>a_3=\ell>0$: a usual torus), Villarceau circles correspond to two simple roots $\pm\sqrt{\dfrac{a_1-\ell}{2\ell}}$.

The first main result is the following theorem concerning the equivalency of one-circle property and a fifth-order partial differential equation.

Theorem 2.3. *Let $z=f(x,y)$ be a C^5-function defined in a neighborhood $U_{\delta_0}=\{x^2+y^2<\delta_0^2\}$ $(\delta_0>0)$ satisfying the normalization condition* (2.1) *at* $(0,0)$. *Assume that the origin is not an umbilical point of $M:=\{z=f(x,y),(x,y)\in U_{\delta_0}\}$, that is, $c_2-c_0=\frac{1}{2}(f_{yy}(0,0)-f_{xx}(0,0))\neq 0$. Let $P(t)$ be the characteristic polynomial* (2.11) *at the origin for M, and $t_0\neq 0$ be a real number such that $P(t_0)=0, P'(t_0)\neq 0$. Then we have the following* (i), (ii).

(i) *Let $t(x,y), s(x,y)$ be real-valued continuous functions defined in a neighborhood of* $(0,0)$ *such that, for any $\delta>0$ and any $(x_0,y_0)\in U_\delta$, the set*

(2.12) $$M\cap\{y-y_0=t(x_0,y_0)(x-x_0)+s(x_0,y_0)(z-f(x_0,y_0))\}$$

coincides with a circle in a neighborhood of $(x_0,y_0,f(x_0,y_0))$*. Then,* $t(0,0)$ *is a real root of* $P(t)=0$*. In particular, when* $t(0,0)=t_0$*; that is,* $t(0,0)$ *is a non-zero real simple root of* $P(t)=0$*, we consider a continuous function*

$$T(x,y):=\frac{t(x,y)+f_x(x,y)s(x,y)}{1-f_y(x,y)s(x,y)} \tag{2.13}$$

defined in a neighborhood of $(0,0)$*. Then,* $T(x,y)$ *is a* C^1*-function satisfying*

$$Z(T(x,y))=0. \tag{2.14}$$

Moreover, if $t(x,y)$, $s(x,y)$ *are constant on each circle* (2.12)*, we have the following equation:*

$$\begin{aligned}(\partial_x+T(x,y)\partial_y)T(x,y)&=\frac{2s(x,y)C(T(x,y))}{1-b(x,y)s(x,y)}=\frac{2S(T)}{K(T)}\\&=\frac{2D(T)R(T)-4(bT+a)C(T)^2}{2((b^2+1)T+ab)c_0+((b^2+1)T^2-a^2-1)c_1-2T(abT+a^2+1)c_2}.\end{aligned} \tag{2.15}$$

Further, $s(x,y),t(x,y)$ *are also* C^1*-functions written by using* $T(x,y)$ *as follows:*

$$s(x,y)=\frac{S(T)}{W(T)}=\frac{D(T)R(T)-2(bT+a)C(T)^2}{2TC(T)^2+(bD(T)-C'(T)C(T))R(T)}, \tag{2.16}$$

$$t(x,y)=(1-b(x,y)s(x,y))T(x,y)-a(x,y)s(x,y)=\frac{TK(T)C(T)-aS(T)}{W(T)}. \tag{2.17}$$

(ii) *Conversely, let* $T(x,y)$ *be a real-valued* C^1*-function defined in a neighborhood of* $(0,0)$ *satisfying* $T(0,0)=t_0$ *and equations* (2.14), (2.15)*. Then,* $t(x,y),s(x,y)$ *defined by* (2.17), (2.16) *belong to* $C^1(U_\delta)$ *for a small* $\delta>0$*, and satisfy that, for any* $(x_0,y_0)\in U_\delta$*, the set*

$$M\cap\{y-y_0=t(x_0,y_0)(x-x_0)+s(x_0,y_0)(z-f(x_0,y_0))\}$$

coincides with a circle in a neighborhood of $(x_0,y_0,f(x_0,y_0))$*, and that* $t(x,y),s(x,y)$ *are constant on this circle.*

Since T is a function of fourth order derivatives of f, the equation (2.15) is a nonlinear partial differential equation of fifth order. Indeed, applying the implicit function theorem to $Z(T)=0$, we have

$$\begin{aligned}(\partial_x+T(x,y)\partial_y)T(x,y)&=-\frac{Z_x(T)+T(x,y)Z_y(T)}{Z'(T)}\\&=\frac{-R(T)K(T)^2\sum_{j=0}^5\binom{5}{j}T(x,y)^j\partial_x^{5-j}\partial_y^jf+G(T)}{24\,Z'(T)}.\end{aligned}$$

Here, $G(T)$ is a polynomial in T of degree 10 whose coefficients are polynomials in a, b, c_*, d_*, e_*. Noting the non-vanishing coefficient $R(T)K(T)^2|_{x=y=0}=(t_0^2+1)(c_0-c_2)t_0\neq 0$ for the fifth order part of $(\partial_x+T\partial_y)T$, we set

$$N(T):=-K(T)\left(Z_x(T)+TZ_y(T)-\frac{K(T)^2R(T)}{24}\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^jf\right)-2S(T)Z'(T). \tag{2.18}$$

Then $N(T)$ is a polynomial in T of degree at most 14 whose coefficients are also polynomials in a,b,c_*,d_*,e_*. Hence the equation (2.15) is rewritten as follows:

$$\sum_{j=0}^{5}\binom{5}{j}T^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T)}{R(T)K(T)^3}. \tag{2.19}$$

Since T satisfying $Z(T)=0$ is an analytic function of a,b,c_*,d_*,e_* in a neighborhood of a simple root $T=t_0$, we have the following proposition:

Proposition 2.4. *Equation* (2.15) *is a quasilinear analytic partial differential equation for f with fifth order principal symbol*

$$(\xi+T(x,y)\eta)^5,$$

where ξ,η are the symbols for ∂_x,∂_y, respectively.

By this proposition and the well-known elliptic regularity theorem due to Petrowsky and Morrey, we get the following theorem on a surface containing two continuous families of circles:

Theorem 2.5. *Let $M: z=f(x,y)$ be a $C^{5+\theta}$-class surface satisfying the condition* (2.1), *where θ $(0<\theta<1)$ is an exponent for Hölder continuity. Assume that the origin is not an umbilical point of $M:=\{z=f(x,y),(x,y)\in U_{\delta_0}\}$. Let $P(t)$ be its characteristic polynomial at* $(0,0)$. *Suppose that M contains two continuous families of circles in the sense of* (i) *of Theorem* 2.3, *where these families correspond to two distinct non-zero real simple roots t_1,t_2 of $P(t)=0$, respectively. Then, f is analytic at* $(0,0)$.

Hereafter we may assume $z=f(x,y)$ is analytic for surfaces containing several continuous families of circles by this theorem.

Theorem 2.6. *Let $M: z=f(x,y)$ be a $C^{5+\theta}$-class surface satisfying the condition* (2.1), *where θ $(0<\theta<1)$ is an exponent for Hölder continuity. Assume that the origin is not an umbilical point of $M:=\{z=f(x,y),(x,y)\in U_{\delta_0}\}$. Let $P(t)$ be its characteristic polynomial at* $(0,0)$. *Suppose that M contains six continuous families of circles in the sense of (i) of Theorem* 2.3, *where these families correspond to six distinct non-zero real simple roots $t_1,\dots,t_6$ of $P(t)=$* 0, *respectively. Then, f is uniquely determined only by the partial derivatives at* $(0,0)$ *up to fourth-order. In particular, such surfaces are classified by* 11 *real parameters.*

Indeed, let $M: z=f(x,y)$ be such a surface, and $\{T_k(x,y)\}_{k=1}^6$ be the functions T corresponding to $\{t_k\}_{k=1}^6$, respectively. Then, f is a solution of the following system:

$$\begin{cases} Z(T_k(x,y))=0, \\ \displaystyle\sum_{j=0}^{5}\binom{5}{j}T_k(x,y)^j\partial_x^{5-j}\partial_y^j f(x,y)=\frac{24N(T_k(x,y))}{R(T_k(x,y))K(T_k(x,y))^3}, \end{cases} \quad (1\le k\le 6). \tag{2.20}$$

The system (2.20) is solvable with respect to

$$\binom{5}{j}\partial_x^{5-j}\partial_y^j f(x,y) \qquad (j=0,1,\dots,5)$$

by using the formula

$$\det\left(T_k(x,y)^{j-1}\right)_{j,k=1,\dots,6} = \prod_{j<k}(T_j - T_k).$$

Consequently we have an equivalent system to the system (2.20):

$$(2.21)\qquad \begin{cases} Z(T_k(x,y)) = 0 \quad (k = 1,2,\dots,6), \\ \partial_x^{5-j}\partial_y^j f(x,y) = G_j(\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f, T_1, \dots, T_6) \quad (j = 0,1,\dots,5), \end{cases}$$

where G_j $(j = 1,\dots,6)$ are analytic functions of $\nabla f, \nabla^2 f, \nabla^3 f, \nabla^4 f, T_1, \dots, T_6$. However, the complete integrability of this system is not obtained until now. Instead, we have the following theorem concerning the finite dimensionality of surfaces containing two continuous families of circles.

Theorem 2.7. *Let $M : z = f(x,y)$ be a $C^{5+\theta}$-class surface satisfying the condition* (2.1), *where θ $(0 < \theta < 1)$ is an exponent for Hölder continuity. Assume that the origin is not an umbilical point of $M := \{z = f(x,y), (x,y) \in U_{\delta_0}\}$. Let $P(t)$ be its characteristic polynomial at* $(0,0)$. *Suppose that M contains two continuous families of circles in the sense of* (i) *of Theorem* 2.3, *where these families correspond to two distinct non-zero real simple roots t_1, t_2 of $P(t) = 0$, respectively. Then, f is uniquely determined only by the partial derivatives at* $(0,0)$ *up to tenth-order. In particular, such surfaces are classified by* 21 *real parameters.*

The details including calculations and proofs which are not written in this paper will be published elsewhere.

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