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Substitutions from Rauzy induction on 4-interval exchange transformations and Quasi-periodic tilings

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This is a survey that focuses on the 2-dimensional quasi-periodic tilings by using the non-Pisot hyperbolic substitution generated by the Rauzy induction on exchanges of four intervals.

Key word: Rauzy induction, 4-interval exchange transformation, Quasi-periodic tiling

1 Introduction

The dynamics of interval exchange transformations (denoted by IET for short in the rest of the paper) has been an object of intensive studies for more than 30 years. One of the main tools of this study is the Rauzy induction, which associates with an IET its first return map on the largest admissible map containing 0. To any minimal IET, one can associate an infinite paths in the so-called Rauzy induction diagram (henceforth denoted by RID for short; it is sometimes also called the Rauzy graph on permutations), whose vertices are given by the permutation on the order of the intervals, and whose edges are given by the induction.

The natural partition in intervals allows us to define a symbolic system associated with the given map, and the properties of these symbolic systems are well-known. In particular, they are examples of S-adic systems, that is, symbolic systems defined by an infinite sequence of substitutions belonging to a finite set of substitutions.

The special case of periodic induction is very interesting for a number of reasons (they correspond to invariant foliations of pseudo-Anosov automorphisms); the corresponding symbolic system is generated by a primitive substitution. Much energy has been devoted to the study of substitution dynamical systems (see [Q], [Fo], [L]), and among them to the so-called Pisot substitutions, where all eigenvalues except one have modulus strictly smaller than 1. Substitutions associated with IETs are in some sense the opposite: for

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geometric reasons, the characteristic polynomial of the matrix is reciprocal, hence the set of eigenvalues is stable by inversion. In particular, there is, by Perron-Frobenius, a largest eigenvalue which is real, hence, by the reciprocity property, there is a smallest eigenvalue in modulus which is also real.

One easily proves that the original IET can be recovered from the substitution by the following method. Consider any infinite periodic point of the substitution (the set of such periodic points is finite and nonempty). Next, consider the subset of $\mathbb{Z}^d$ (where d is the number of continuity intervals of the map, and the number of letters of the substitution) defined as the abelianizations of the prefixes of this periodic point. Project these points on the eigenline corresponding to the smallest eigenvalue, along the hyperplane generated by all the other eigenvectors. This is obviously an bounded set which satisfies a self-similarity relation given by the substitution; one can show that the closure of this set is an interval with a natural partition corresponding to the final letter of the prefix, and the shift on the infinite word induces a map which is (conjugate to) the initial IET.

In this paper, we enlarge this picture, by showing on an example that we can associate to the substitution not only an IET on a subset of the contracting eigenline, but also a self-similar tiling of the contracting plane (and of the expanding plane). In this way, we obtain an $\mathbb{R}^k$-action on a tiling space, where k is the dimension of the contracting (or expanding) space. It is well-known that Pisot substitutions are finite extensions of rotations on a compact group, and it is conjectured that this finite extension is almost everywhere 1-to-1; there is a natural way to build a suspension to obtain a flow on the torus of dimension d. The flow on a tiling space that we exhibit here could be the counterpart of this toral flow for non-Pisot examples.

The construction of the tiling is given in the last section of the paper. We follow the construction given in [S-A-I], [E], [F-I-Rob], but there is something subtle here. Indeed, the matrix we are led to consider (antisymmetric tensor square of the matrix of the substitution) is positive in a suitable basis, but the geometric map corresponding to the substitution is not, its image containing negatively oriented parallelogram. We are led to change the images of the basic parallelogram by a retiling method to define the invariant tiling.The general condition under which this is possible is unclear at the moment, and it would be interesting to know.

The paper is organized as follows: In Section 2, we fix the notations and recall the basic facts about the Rauzy induction and the related substitutions. In Section 3, we define the particular example under study and give its main properties (substitution, matrix, eigenvalues and invariant spaces), and we study the bounded sets obtained by projection of the contracting space (see Figure 12). In Section 4, we give the construction of the polygonal quasi-self-similar tiling and of the limit self-similar tiling associated with the substitution.

2 A Rauzy induction on 4-IETs

In the following, we will use the Kerckhoff coding for IET (see [K]); this coding, by two permutations instead of one, allows to give the same name to intervals which are not changed by induction, even if their relative position in the interval is changed. This coding might look more complicated than the usual one by the ordering of intervals, but as we will see, the resulting permutations and matrices are obtained in this way are elementary, hence simpler.

Let $\mathcal{A}$ be the alphabet given by $\mathcal{A} = \{A, B, C, D\}$ and let us consider seven 2×4 matrices as follows:

$$\mathrm{I} = \begin{bmatrix} A & B & C & D \\ D & C & B & A \end{bmatrix}, \quad \mathrm{II} = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}, \quad \mathrm{III} = \begin{bmatrix} A & D & B & C \\ D & C & B & A \end{bmatrix},$$

$$\mathrm{IV} = \begin{bmatrix} A & D & B & C \\ D & C & A & B \end{bmatrix}, \quad \mathrm{V} = \begin{bmatrix} A & B & C & D \\ D & B & A & C \end{bmatrix}, \quad \mathrm{VI} = \begin{bmatrix} A & B & C & D \\ D & A & C & B \end{bmatrix},$$

$$\mathrm{VII} = \begin{bmatrix} A & B & D & C \\ D & A & C & B \end{bmatrix}.$$

For each $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \cdots, \mathrm{VII}\}$, let us define the two bijections ${}_{\mathrm{J}}\pi_0 : \mathcal{A} \to \{1, 2, 3, 4\}$ and ${}_{\mathrm{J}}\pi_1 : \mathcal{A} \to \{1, 2, 3, 4\}$ by

$$\begin{aligned} {}_{\mathrm{J}}\pi_0 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the first row vector of J}, \\ {}_{\mathrm{J}}\pi_1 &= \text{the location of } \alpha \in \mathcal{A} \text{ in the second row vector of J}. \end{aligned}$$

For example, if $\mathrm{J} = \mathrm{II} = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}$, then we obtain

$$\begin{aligned} ({}_{\mathrm{J}}\pi_0(A), {}_{\mathrm{J}}\pi_0(B), {}_{\mathrm{J}}\pi_0(C), {}_{\mathrm{J}}\pi_0(D)) &= (1, 4, 2, 3), \\ ({}_{\mathrm{J}}\pi_1(A), {}_{\mathrm{J}}\pi_1(B), {}_{\mathrm{J}}\pi_1(C), {}_{\mathrm{J}}\pi_1(D)) &= (4, 3, 2, 1), \\ ({}_{\mathrm{J}}\pi_0^{-1}(1), {}_{\mathrm{J}}\pi_0^{-1}(2), {}_{\mathrm{J}}\pi_0^{-1}(3), {}_{\mathrm{J}}\pi_0^{-1}(4)) &= (A, C, D, B), \\ ({}_{\mathrm{J}}\pi_1^{-1}(1), {}_{\mathrm{J}}\pi_1^{-1}(2), {}_{\mathrm{J}}\pi_1^{-1}(3), {}_{\mathrm{J}}\pi_1^{-1}(4)) &= (D, C, B, A). \end{aligned}$$

For each J, let us consider the 4-*IET* R_{J}, $\mathrm{J} \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$ with the subintervals $\{I_\alpha\}_{\alpha \in \mathcal{A}}$ of $[0, 1)$ as follows [Y] (see Figure 2).

Let $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$ be the *length data* of the intervals I_α satisfying $\sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1$. Then, the transformation $R_J : [0, 1) \to [0, 1)$ is explicitly given by

$$R_{\mathrm{J}}(x) := x - \sum_{\substack{\beta : \\ {}_{\mathrm{J}}\pi_0(\beta) <_{\mathrm{J}} \pi_0(\alpha)}} \lambda_{{}_{\mathrm{J}}\pi_0(\beta)} + \sum_{\substack{\beta : \\ {}_{\mathrm{J}}\pi_1(\beta) <_{\mathrm{J}} \pi_1(\alpha)}} \lambda_{{}_{\mathrm{J}}\pi_1(\beta)} \quad \text{if } x \in I_\alpha.$$

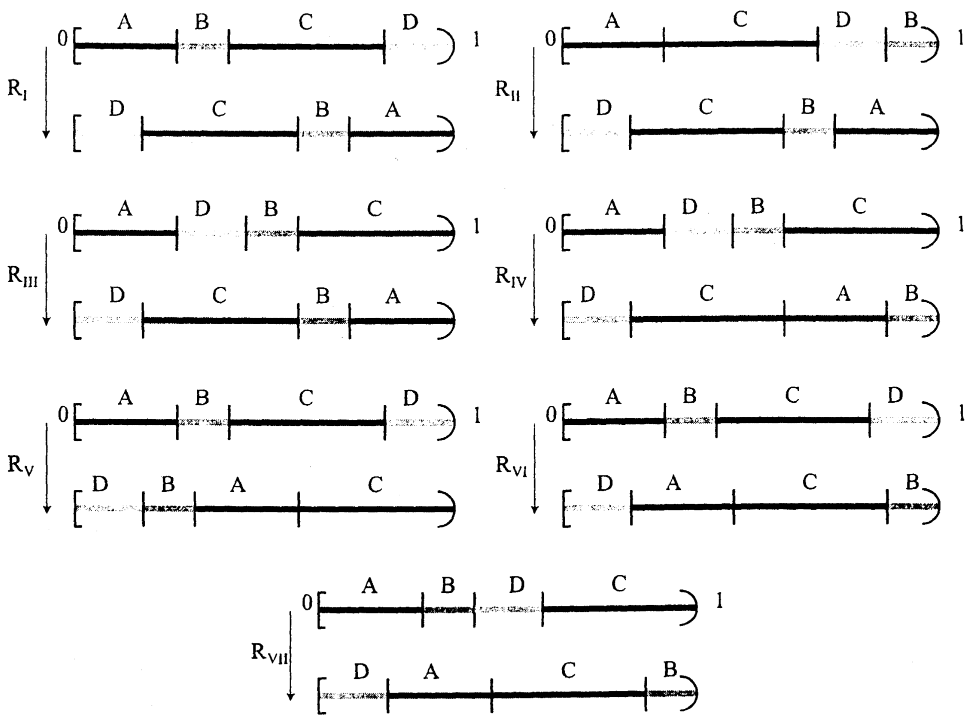


Figure 1: The 4-IETs R_{J}.

For example, if J = I,

$$R_{\mathrm{I}}(x) := \begin{cases} x + \lambda_D + \lambda_C + \lambda_B & \text{if} \quad x \in I_A \\ x - \lambda_A + \lambda_D + \lambda_C & \text{if} \quad x \in I_B \\ x - (\lambda_A + \lambda_B) + \lambda_D & \text{if} \quad x \in I_C \\ x - (\lambda_A + \lambda_B + \lambda_C) & \text{if} \quad x \in I_D \end{cases}$$

(see Figure 2).

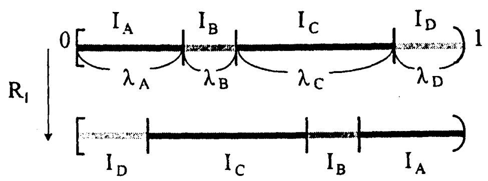


Figure 2: The 4-IET R_I given by the length data $(\lambda_\alpha)_{\alpha \in \mathcal{A}}$.

For each J $\in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$, let us consider the *induced transformation* $(R_{\mathrm{J}})_{[0, \lambda^*_\varepsilon(\mathrm{J}))}$ of R_{J} where ε is given by

$$\varepsilon := \begin{cases} 0 & \text{if} \quad \lambda_{{}_{\mathrm{J}}\pi_0^{-1}(4)} > \lambda_{{}_{\mathrm{J}}\pi_1^{-1}(4)} \\ 1 & \text{if} \quad \lambda_{{}_{\mathrm{J}}\pi_0^{-1}(4)} < \lambda_{{}_{\mathrm{J}}\pi_1^{-1}(4)} \end{cases}$$

and $\lambda^*_\varepsilon\,(\mathrm{J})$ is given by

$$\lambda^*_\varepsilon\,(\mathrm{J}) := 1 - \min\left\{\lambda_{_{\mathrm{J}\pi_0^{-1}(4)}}, \lambda_{_{\mathrm{J}\pi_1^{-1}(4)}}\right\}.$$

Then, for J and ε, there exists J$'$ such that the induced transformation $(R_{\mathrm{J}})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$ is *isomorphic* to $R_{\mathrm{J}'}$ by the isomorphism $\varphi_{\binom{\mathrm{J}}{\varepsilon}}\,(x) = \frac{x}{\lambda^*_\varepsilon(\mathrm{J})}$ from $[0, \lambda^*_\varepsilon\,(\mathrm{J}))$ to $[0, 1)$. For example, if J = I, the induced transformations $(R_{\mathrm{I}})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$, $\varepsilon \in \{0, 1\}$ are following (see Figure 3):

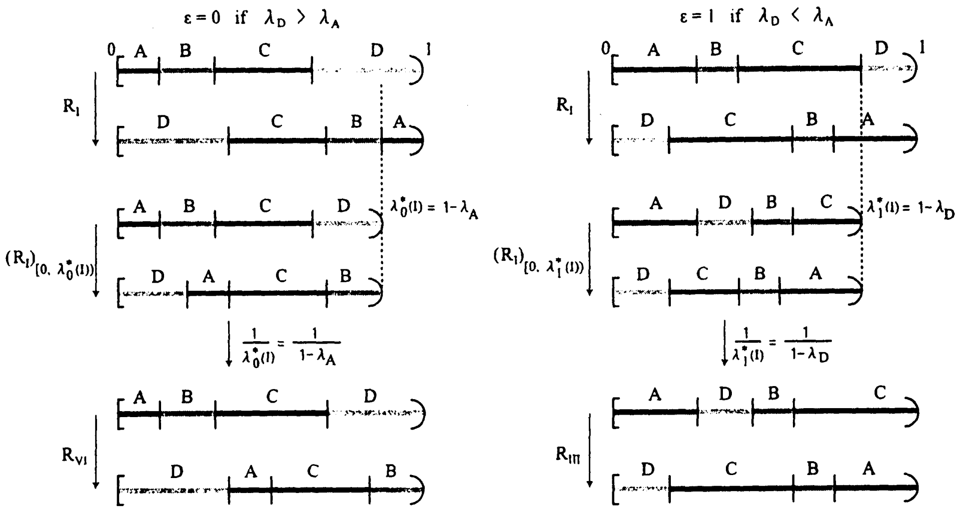


Figure 3: The induced transformations $(R_{\mathrm{I}})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$ of R_{I}, $\varepsilon = 0, 1$ and the renormalized transformations R_{VI} and R_{III} of $(R_{\mathrm{I}})_{[0,\lambda^*_\varepsilon(\mathrm{J}))}$, $\varepsilon = 0, 1$.

The other cases of J = II, III, ..., VII are defined analogoulsy.

By the length $\lambda_{_{\mathrm{J}\pi_0^{-1}(4)}}$ and $\lambda_{_{\mathrm{J}\pi_1^{-1}(4)}}$ of the subintervals $I_{_{\mathrm{J}\pi_0^{-1}(4)}}$ and $I_{_{\mathrm{J}\pi_1^{-1}(4)}}$ respectively, we have a part of the *directed graph* with the vertices $\{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\}$ and the labels $\varepsilon \in \{0, 1\}$. For example, if J = I, see Figure 4.

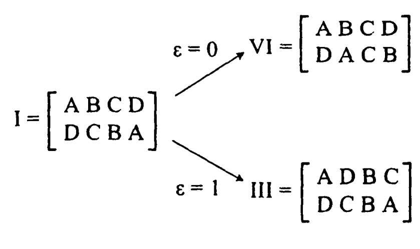


Figure 4: The directed graph that starting vertex is I.

The other cases are defined analogously.

Then we have the following *RID* from the 4-IETs (see [Y]).

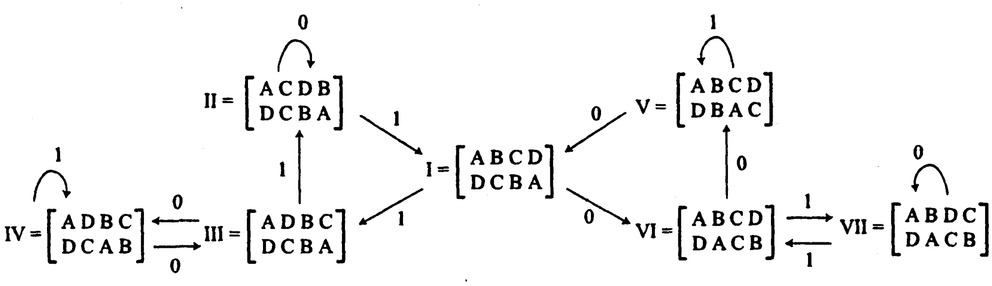


Figure 5: RID.

Proposition 2.1 (RID). *We have the following RID (see Figure 5):*

Using RID, we obtain the *RID-admissible path* $\left(\binom{J_1}{\varepsilon_1}\binom{J_2}{\varepsilon_2}\cdots\binom{J_i}{\varepsilon_i}\cdots\right)$ of $\binom{J_i}{\varepsilon_i} \in \{\mathrm{I}, \mathrm{II}, \ldots, \mathrm{VII}\} \times \{0,1\}$.

Now let us introduce the family of the substitutions $\sigma_{\binom{\mathrm{J}}{\varepsilon}}$ on $\mathcal{A}^*$ related to the induced transformation $(R_{\mathrm{J}})_{[0,\lambda_\varepsilon^*(\mathrm{J}))}$ as follows:

$$\sigma_{\binom{\mathrm{I}}{0}} : \begin{array}{lcl} A & \to & AD \\ B & \to & B \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{I}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & C \\ D & \to & AD \end{array} \quad \sigma_{\binom{\mathrm{II}}{0}} : \begin{array}{lcl} A & \to & AB \\ B & \to & B \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{II}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & AB \\ C & \to & C \\ D & \to & D \end{array}$$

$$\sigma_{\binom{\mathrm{III}}{0}} : \begin{array}{lcl} A & \to & AC \\ B & \to & B \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{III}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & AC \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{IV}}{0}} : \begin{array}{lcl} A & \to & A \\ B & \to & BC \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{IV}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & BC \\ D & \to & D \end{array}$$

$$\sigma_{\binom{\mathrm{V}}{0}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & CD \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{V}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & C \\ D & \to & CD \end{array} \quad \sigma_{\binom{\mathrm{VI}}{0}} : \begin{array}{lcl} A & \to & A \\ B & \to & BD \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{VI}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & C \\ D & \to & BD \end{array}$$

$$\sigma_{\binom{\mathrm{VII}}{0}} : \begin{array}{lcl} A & \to & A \\ B & \to & BC \\ C & \to & C \\ D & \to & D \end{array} \quad \sigma_{\binom{\mathrm{VII}}{1}} : \begin{array}{lcl} A & \to & A \\ B & \to & B \\ C & \to & BC \\ D & \to & D \end{array}.$$

We write the *incidence* matrices of the above substitutions $\sigma_{\binom{\mathrm{J}_i}{\varepsilon_i}}$ as M_i.

Then, we have the following RID with the substitutions.

Proposition 2.2 (RID with the substitutions). *We have the following RID with the substitutions (see Figure 6):*

For any RID-admissible periodic path $\overline{\left(\binom{\mathrm{J}_0}{\varepsilon_0}\binom{\mathrm{J}_1}{\varepsilon_1}\cdots\binom{\mathrm{J}_i}{\varepsilon_i}\cdots\binom{\mathrm{J}_{k-1}}{\varepsilon_{k-1}}\right)}$ with period k, we have the substitution σ_i as follows:

$$\sigma_i = \sigma_{\binom{\mathrm{J}_i}{\varepsilon_i}} \circ \sigma_{\binom{\mathrm{J}_{i+1}}{\varepsilon_{i+1}}} \circ \cdots \circ \sigma_{\binom{\mathrm{J}_{k-1}}{\varepsilon_{k-1}}} \circ \sigma_{\binom{\mathrm{J}_0}{\varepsilon_0}} \circ \cdots \circ \sigma_{\binom{\mathrm{J}_{i-1}}{\varepsilon_{i-1}}}$$

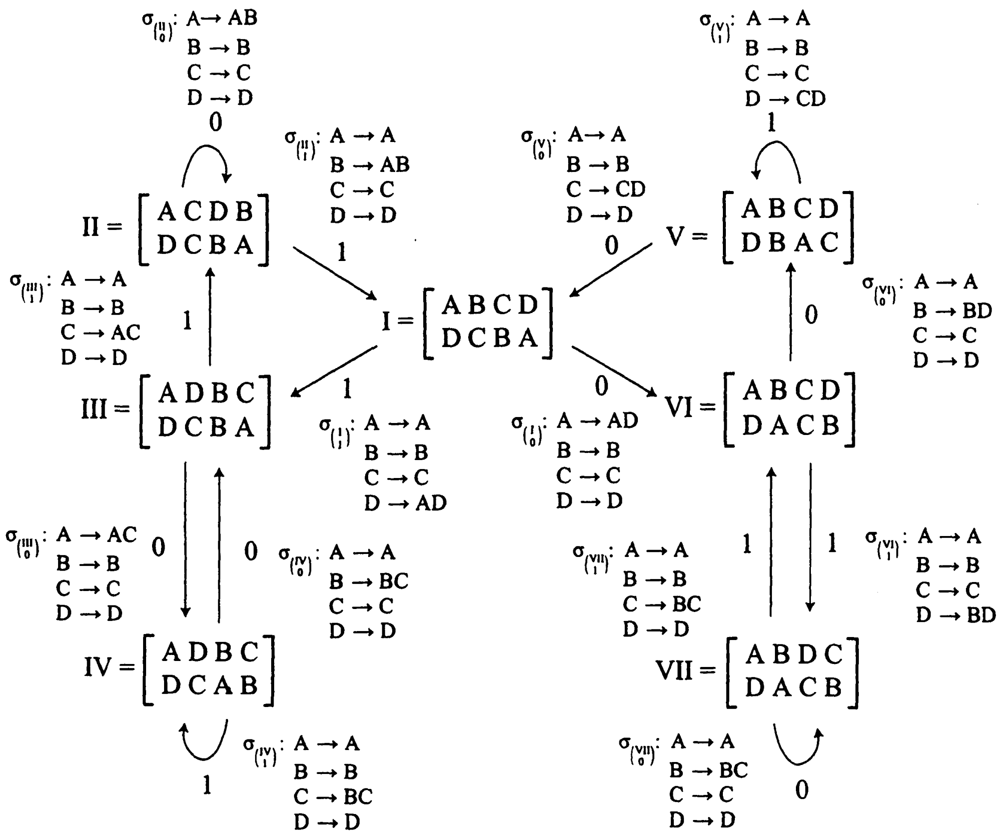


Figure 6: RID with the substitutions.

on $\mathcal{A}^*$. In this survey, we only consider the following RID-admissible periodic path:

$$\overline{\left(\binom{\mathrm{J}_0}{\varepsilon_0}\binom{\mathrm{J}_1}{\varepsilon_1}\cdots\binom{\mathrm{J}_i}{\varepsilon_i}\cdots\binom{\mathrm{J}_7}{\varepsilon_7}\right)} = \overline{\left(\binom{\mathrm{II}}{0}\binom{\mathrm{II}}{1}\binom{\mathrm{I}}{0}\binom{\mathrm{VI}}{0}\binom{\mathrm{V}}{1}\binom{\mathrm{V}}{0}\binom{\mathrm{I}}{1}\binom{\mathrm{III}}{1}\right)}$$

with period 8.

The substitution σ will be sometimes written by

$$\sigma(\alpha) = W_1^{(\alpha)} W_2^{(\alpha)} \cdots W_{l_\alpha}^{(\alpha)} = P_k^{(\alpha)} W_k^{(\alpha)} S_k^{(\alpha)}$$

where $P_k^{(\alpha)}$ (resp. $S_k^{(\alpha)}$) is the prefix (resp. suffix) of the letter $W_k^{(\alpha)}$.

3 On an example

Let us consider the following substitution σ as an example:

$$\sigma = \sigma_{\binom{\mathrm{II}}{0}} \circ \sigma_{\binom{\mathrm{II}}{1}} \circ \sigma_{\binom{\mathrm{I}}{0}} \circ \sigma_{\binom{\mathrm{VI}}{0}} \circ \sigma_{\binom{\mathrm{V}}{1}} \circ \sigma_{\binom{\mathrm{V}}{0}} \circ \sigma_{\binom{\mathrm{I}}{1}} \circ \sigma_{\binom{\mathrm{III}}{1}}$$

generated by a RID-admissible periodic path with period 8 (see Figure 7).

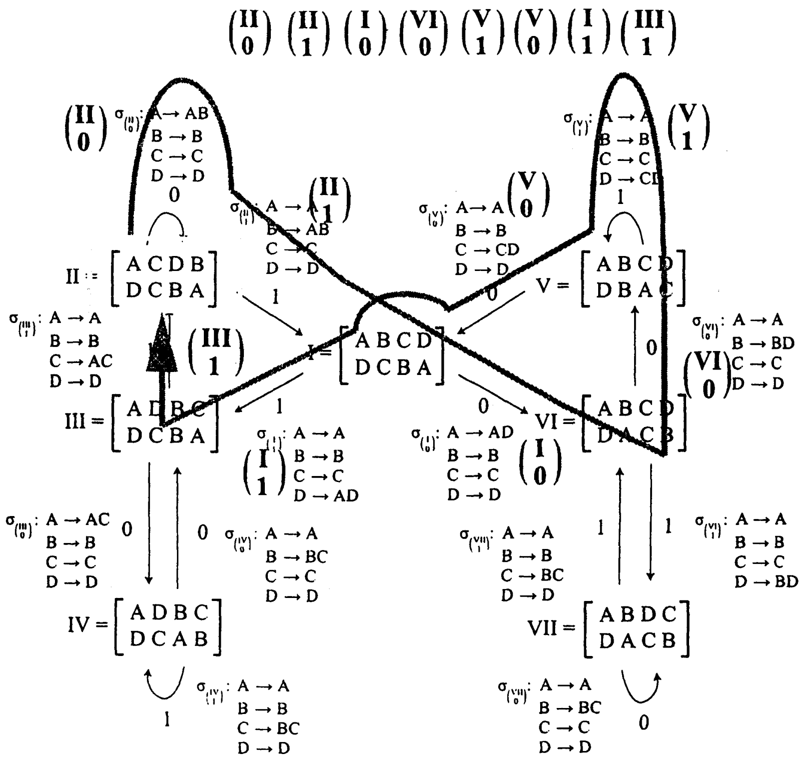


Figure 7: An example of a RID-admissible periodic path with period 8.

The substitution σ is explicitly given by

$$\sigma : \begin{array}{lcl} A & \to & ABD \\ B & \to & ABBD \\ C & \to & ABDCCD \\ D & \to & ABDCD \end{array},$$

and its incidence matrix M_σ and its characteristic polynomial $\Phi_\sigma(x)$ are given by

$$M_\sigma = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \\ 1 & 1 & 2 & 2 \end{bmatrix}, \quad \Phi_\sigma(x) = x^4 - 7x^3 + 13x^2 - 7x + 1$$

respectively. Then, we see that the root of $\Phi_\sigma(x)$ is distributed by Figure 8.

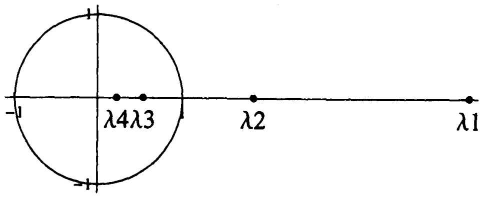


Figure 8: The distribution of the roots of $\Phi_\sigma(x)$.

Therefore we have the Perron-Frobenius eigenvector v_1 satisfying

$$v_1 = {}^t[\lambda_A, \lambda_B, \lambda_C, \lambda_D], \ \lambda_\alpha > 0, \ \text{and} \ \sum_{\alpha \in \mathcal{A}} \lambda_\alpha = 1$$

where ${}^tM$ means the transpose of the matrix M.

Starting from σ, we obtian the following 4-IET R_{II}; let us define the partition $\{I_\alpha \mid \alpha \in \mathcal{A}\}$ of $[0,1)$ by

$$I_A = [0, \lambda_A), \ I_B = [\lambda_A, \lambda_A + \lambda_C), \ I_C = [\lambda_A + \lambda_C, \lambda_A + \lambda_C + \lambda_D), \\ I_D = [\lambda_A + \lambda_C + \lambda_D, 1)$$

(see Figure 9).

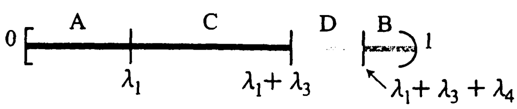


Figure 9: The partition of $[0, 1)$.

From the definition, R_{II} is explicitly given by

$$R_{\mathrm{II}}(x)=\begin{cases} x+\lambda_D+\lambda_C+\lambda_B & \text{if} \quad x\in I_A \\ x-\lambda_A+\lambda_D & \text{if} \quad x\in I_C \\ x-(\lambda_A+\lambda_C) & \text{if} \quad x\in I_D \\ x-\lambda_A & \text{if} \quad x\in I_B \end{cases}.$$

Then, $R_{\mathrm{II}}(x)$ by $\lambda_B>\lambda_A$ and the induced transformation $(R_{\mathrm{II}})_{[0,\lambda_0^*(\mathrm{II}))}$ of R_{II} is isomorphic to R_{II} by the isomorphism $\varphi_{\binom{\mathrm{II}}{0}}(x)=\frac{x}{\lambda_0^*(\mathrm{II})}$ from $[0,\lambda_0^*(\mathrm{II}))$ to $[0,1)$ (see Figure 10).

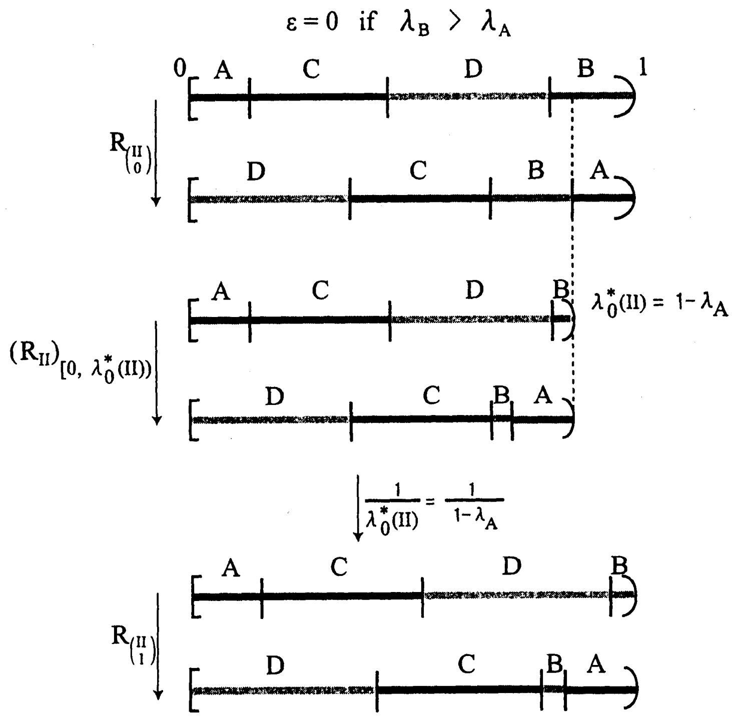


Figure 10: The induced transformation $(R_{\mathrm{II}})_{[0,\lambda_0^*(\mathrm{II}))}$ of R_{II}.

Let W be the fixed point of σ, that is,

$$W=s_1s_2\ldots s_k\ldots=\lim_{n\to\infty}\sigma^n(A).$$

Let

$$\mathcal{L}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4):=\mathcal{L}(\boldsymbol{v}_1)\oplus\mathcal{L}(\boldsymbol{v}_2)\oplus\mathcal{L}(\boldsymbol{v}_3)\oplus\mathcal{L}(\boldsymbol{v}_4)$$

and let us define the projection π_i and π_{ij} by

$$\begin{array}{rlcl} \pi_i: & \mathcal{L}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4) & \to & \mathcal{L}(\boldsymbol{v}_i) \\ \pi_{ij}: & \mathcal{L}(\boldsymbol{v}_1,\boldsymbol{v}_2,\boldsymbol{v}_3,\boldsymbol{v}_4) & \to & \mathcal{L}(\boldsymbol{v}_i,\boldsymbol{v}_j) \end{array}$$

where v_i, $i = 1, 2, 3, 4$ are the eigenvectors associated to the eigenvalues λ_i, $i = 1, 2, 3, 4$ of M_σ satisfying $\lambda_1 > \lambda_2 > 1 > \lambda_3 > \lambda_4 > 0$ respectively.

Moreover, let us define the homomorphism $f : \mathcal{A}^* \to \mathbb{Z}^4$ by

$$f(A) := e_1,\ f(B) := e_2,\ f(C) := e_3,\ f(D) := e_4,\ f(\emptyset) := \emptyset$$
$$f(W_1 W_2 \ldots W_k) := f(W_1) + f(W_2) + \cdots + f(W_k).$$

On the above notation, we have firstly the following proposition.

Proposition 3.1. *Let us define the set X_α, X'_α, X as follows:*

$$\begin{array}{rcl} X_\alpha & := & \text{the closure of } \pi_4 \{ f(s_1 s_2 \ldots s_{k-1}) \mid s_k = \alpha, k = 1, 2, \ldots \}, \alpha \in \mathcal{A} \\ X'_\alpha & := & \text{the closure of } \pi_4 \{ f(s_1 s_2 \ldots s_k) \mid s_k = \alpha, k = 1, 2, \ldots \}, \alpha \in \mathcal{A} \\ X & := & \text{the closure of } \pi_4 \{ f(s_1 s_2 \ldots s_{k-1}) \mid k = 1, 2, \ldots \}. \end{array}$$

Then, we have the following properties:

(1) X_α is an interval of the line $\mathcal{L}(v_4)$;

(2) $X = \bigcup_{\alpha \in \mathcal{A}} X_\alpha = \bigcup_{\alpha \in \mathcal{A}} X'_\alpha$;

(3) $X_\alpha \cap X_\beta$ $(\alpha \neq \beta)$, $\alpha, \beta \in \mathcal{A}$ are not overlapped;

(4) $\{X_\alpha\}_{\alpha \in \mathcal{A}}$ satisfies the set equation:

$$\lambda_1 X_\alpha \left(= M_\sigma^{-1} X_\alpha\right) = \bigcup_{\beta \in \mathcal{A}} \bigcup_{W_k^{(\beta)} = \alpha} \left(\pi_4 f\left(P_k^{(\beta)}\right) + X_\beta \right);$$

(5) The IET $D : X \to X$ such that $D(X_\alpha) = X'_\alpha$ is isomorphic to $R_{\binom{\text{II}}{0}}$ where $D : X \to X$ such that $D(X_\alpha) = X'_\alpha$ is isomorphic to $R_{\text{II}} = \begin{bmatrix} A & C & D & B \\ D & C & B & A \end{bmatrix}$ (see Figure 11).

Moreover, we have the following theorem.

Theorem 3.2. *(cf. [F-I-Rao]) Let us define*

$$\begin{array}{rcl} \widehat{X}_\alpha & := & \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_{k-1}) \mid s_k = \alpha, k = 1, 2, \ldots \} \\ \widehat{X}'_\alpha & := & \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_k) \mid s_k = \alpha, k = 1, 2, \ldots \} \\ \widehat{X} & := & \text{the closure of } \pi_{34} \{ f(s_1 s_2 \ldots s_{k-1}) \mid k = 1, 2, \ldots \}. \end{array}$$

Then,

(1) $\widehat{X} = \bigcup_{\alpha \in \mathcal{A}} \widehat{X}_\alpha = \bigcup_{\alpha \in \mathcal{A}} \widehat{X}_{\alpha'}$ (non-overlapping);

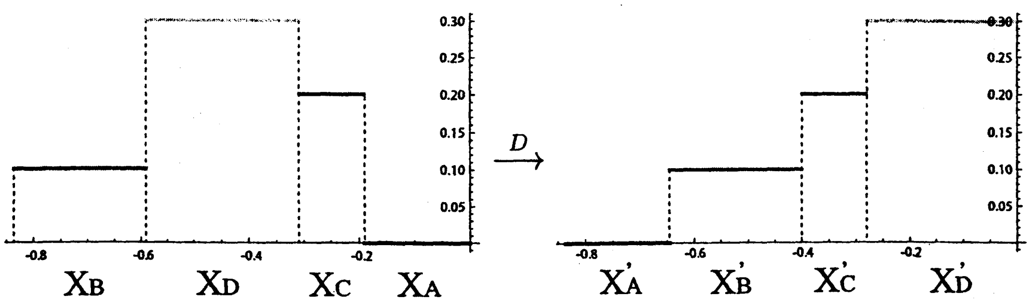


Figure 11: X_α and X'_α.

(2) $\left\{\widehat{X}_\alpha\right\}_{\alpha\in\mathcal{A}}$ *satisfies the set set equation:*

$$M_\sigma^{-1}\widehat{X}_\alpha = \bigcup_{\beta\in\mathcal{A}} \bigcup_{W_k^{(\beta)}=\alpha} \left(\pi_{34}\left(f\left(P_k^{(\beta)}\right)+\widehat{X}_\beta\right)\right);$$

(3) The above set equation satisfies open set condition, that is, there exist a family of open set U_α*,* $\alpha \in \mathcal{A}$ *such that*

$$M_\sigma^{-1}U_\alpha \supset \bigcup_{\beta\in\mathcal{A}} \bigcup_{W_k^{(\beta)}=\alpha} \left(\pi_{34}\left(f\left(P_k^{(\beta)}\right)+U_\beta\right)\right)$$

where the right-hand side is non-overlapping union;

(4) The domain exchange transformation $\widehat{D} : \widehat{X} \to \widehat{X}$ *satisfying* $\widehat{D}\left(\widehat{X}_\alpha\right) = \widehat{X}_{\alpha'}$ *is well-defined (see Figure 12).*

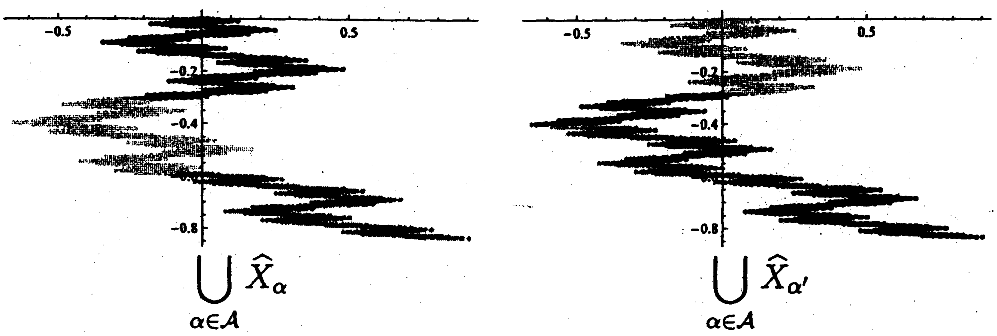


Figure 12: $\bigcup_{\alpha\in\mathcal{A}} \widehat{X}_\alpha$ and $\bigcup_{\alpha\in\mathcal{A}} \widehat{X}_{\alpha'}$.

4 Quasi-periodic tiling

Starting from the hyperbolic and non-Pisot substitutions (automorphism) σ of degree 4, the generating method of quasi-periodic tiling on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ and $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ were discussed in [A-F-H-I], [F-I-Rob], [H-F-I]. In this section, we show the existence of the quasi-periodic polygonal/self-affine tilings generated by substitution σ analogously.

Let us observe the figure of $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$ (see Figure 13). Using the projected basis $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$, we consider the *proto tiles* of parallelograms on $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ (see Figure 14).

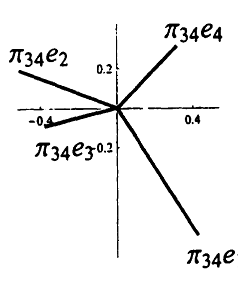


Figure 13: $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$.

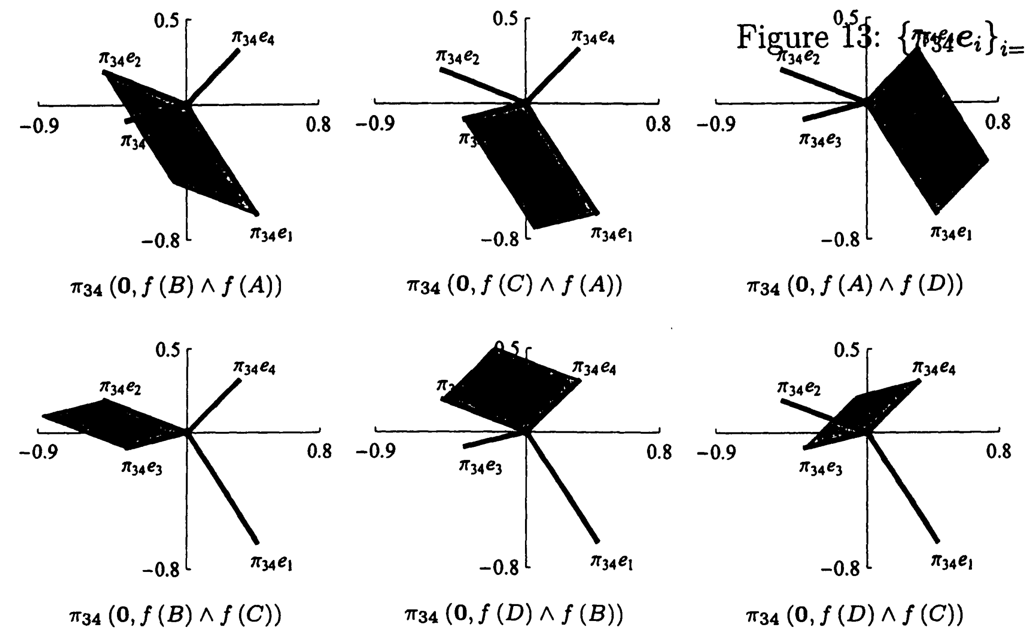


Figure 14: The proto tiles on $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ generated by $f(A) = \boldsymbol{e}_1$, $f(B) = \boldsymbol{e}_2$, $f(C) = \boldsymbol{e}_3$, $f(D) = \boldsymbol{e}_4$.

Using the automorphism $\theta := \overleftarrow{\sigma^{-1}}$ (see [E]):

$$\begin{array}{rcl} \theta(A) & = & AD^{-1}CD^{-1}AB^{-1}A \\ \theta(B) & = & AD^{-1}CD^{-1}BA^{-1}DC^{-1}DA^{-1} \\ \theta(C) & = & AD^{-1}CD^{-1} \\ \theta(D) & = & DC^{-1}DA^{-1} \end{array},$$

we try to consider the 2-dim extension of the automorphisms θ as follows:

$$\begin{array}{rcl} E_2(\theta)(\mathbf{0}, \alpha \wedge \beta) & := & (\mathbf{0}, \theta(\alpha) \wedge \theta(\beta)) \\ & = & \displaystyle\sum_{\substack{1 \le i \le l_\alpha \\ 1 \le j \le l_\beta}} \left(f\left(P_i^{(\alpha)}\right) + f\left(P_j^{(\beta)}\right), W_i^{(\alpha)} \wedge W_j^{(\beta)} \right) \end{array}$$

(see Figure 15). Attention that we find the negative oriented parallelograms in $E_2(\theta)(\mathbf{0}, \alpha \wedge \beta)$ which is characterized as the strong colored parallelograms in Figure 15.

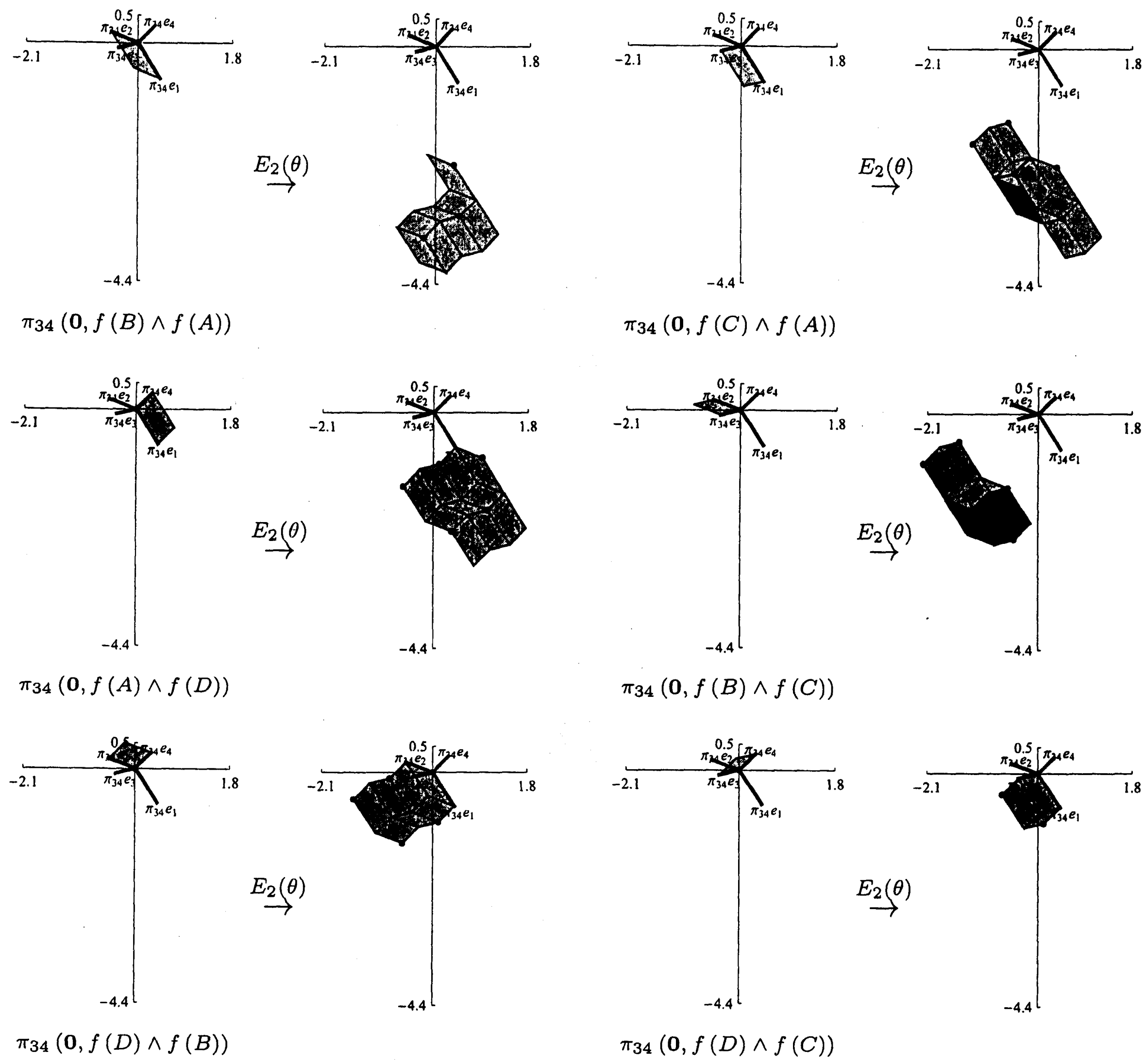


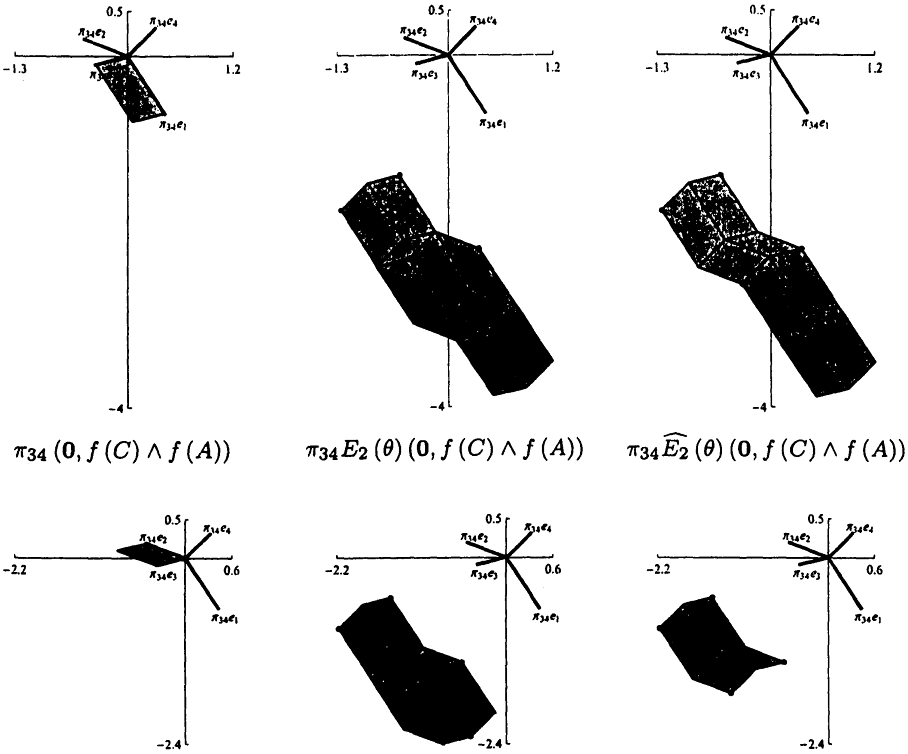
Figure 15: $E_2(\theta)(\mathbf{0}, \alpha \wedge \beta)$.

On the example, we know that A^* is positive, that is,

$$A^* = \begin{array}{c} A \wedge B \\ C \wedge A \\ D \wedge A \\ B \wedge C \\ B \wedge D \\ D \wedge C \end{array} \left[\quad m^*_{i \wedge j, k \wedge l} \quad \right] = \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 4 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 2 & 1 & 2 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \geq O$$

where $m^*_{i\wedge j,k\wedge l} = \det \begin{bmatrix} m_{ik} & m_{il} \\ m_{jk} & m_{jl} \end{bmatrix}$ for $M_\sigma^{-1} = [m_{ij}]_{1\le i,j\le 4}$.

From this fact, we try to find the *tiling substitution* $\widehat{E_2}(\theta)$ by the *retiling method* in [F-I-Rob] (see Figure 16).



$\pi_{34}(\mathbf{0}, f(C)\wedge f(A))$ $\quad$ $\pi_{34}E_2(\theta)(\mathbf{0}, f(C)\wedge f(A))$ $\quad$ $\pi_{34}\widehat{E_2}(\theta)(\mathbf{0}, f(C)\wedge f(A))$

$\pi_{34}(\mathbf{0}, f(B)\wedge f(C))$ $\quad$ $\pi_{34}E_2(\theta)(\mathbf{0}, f(B)\wedge f(C))$ $\quad$ $\pi_{34}\widehat{E_2}(\theta)(\mathbf{0}, f(B)\wedge f(C))$

Figure 16: The retiling method from $E_2(\theta)$ to $\widehat{E_2}(\theta)$.

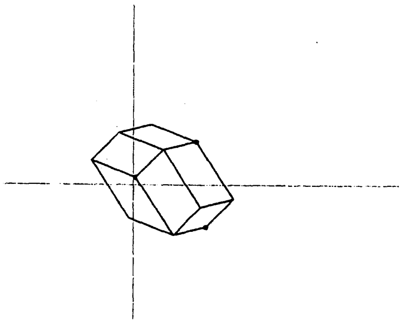
Theorem 4.1. *Let $\mathcal{U}_c$ be a patch generated by the following proto tiles:*

$$\begin{aligned}\mathcal{U}_c &= (\boldsymbol{z}, f(B)\wedge f(A)) + (\boldsymbol{z}+\boldsymbol{e}_4-\boldsymbol{e}_3, f(C)\wedge f(A)) + (\boldsymbol{z}, f(A)\wedge f(D)) \\ &\quad + (\boldsymbol{z}+\boldsymbol{e}_4-\boldsymbol{e}_3, f(B)\wedge f(C)) + (\boldsymbol{z}, f(D)\wedge f(B)) \\ &\quad + (\boldsymbol{z}+\boldsymbol{e}_1-\boldsymbol{e}_3, f(D)\wedge f(C))\end{aligned}$$

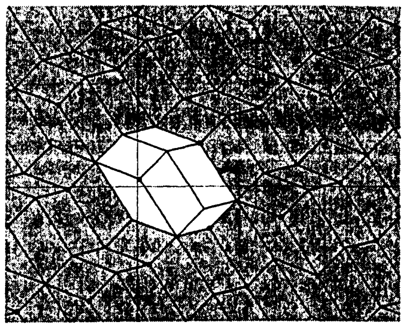
where $\boldsymbol{z} = \left(-\frac{4}{19}, -\frac{5}{19}, -\frac{1}{19}, 0\right)$. Then, we see that $\mathcal{U}_c$ is the seed, that is, $\widehat{E_2}(\theta)^3(\mathcal{U}_c) \succ \mathcal{U}_c$ (see Figure 17).

Moreover,

(1) $\mathcal{T}_{c,1} := \left\{\pi_{34}(\boldsymbol{x}, f(\alpha)\wedge f(\beta)) \;\middle|\; \widehat{E_2}(\theta)^{3n}(\mathcal{U}_c) \ni (\boldsymbol{x}, f(\alpha)\wedge f(\beta))\right\}$ is a quasi-periodic polygonal tiling of $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ (see Figure 18);



$\pi_{34}\mathcal{U}_c$



A part of $\pi_{34}\widehat{E_2}(\theta)^3(\mathcal{U}_c)$

Figure 17: $\pi_{34}\mathcal{U}_c$ and a part of $\pi_{34}\widehat{E_2}(\theta)^3(\mathcal{U}_c)$.

(2) Put

$$X_{\alpha\wedge\beta} := \lim_{n\to\infty} \pi_{34} M_\sigma^{3n} \widehat{E_2}(\theta)^{3n}(\mathbf{0}, f(\alpha)\wedge f(\beta)).$$

Then, $\{X_{\alpha\wedge\beta}\}$ *satisfies the set equations:*

$$M_\sigma(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}) = \sum_k \left(\pi_{34}\boldsymbol{x}_k^{(i)} + X_{\gamma_k^{(i)}}\right)$$

where $\mathcal{U}_c = \sum_{i=1}^6 (\boldsymbol{x}_i, \gamma_i)$ *and* $\widehat{E_2}(\theta)(\boldsymbol{x}_i, \gamma_i) = \sum_k \left(\boldsymbol{x}_k^{(i)} + \gamma_k^{(i)}\right)$;

(3) $\mathcal{T}_{c,2} := \{\pi_{34}\boldsymbol{x} + X_{\alpha\wedge\beta} \mid \pi_{34}(\boldsymbol{x}, f(\alpha)\wedge f(\beta)) \in \mathcal{T}_{c,1}\}$. *Then,* $\mathcal{T}_{c,2}$ *is a quasi-periodic self-affine tiling of* $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$ *(see Figure 20).*

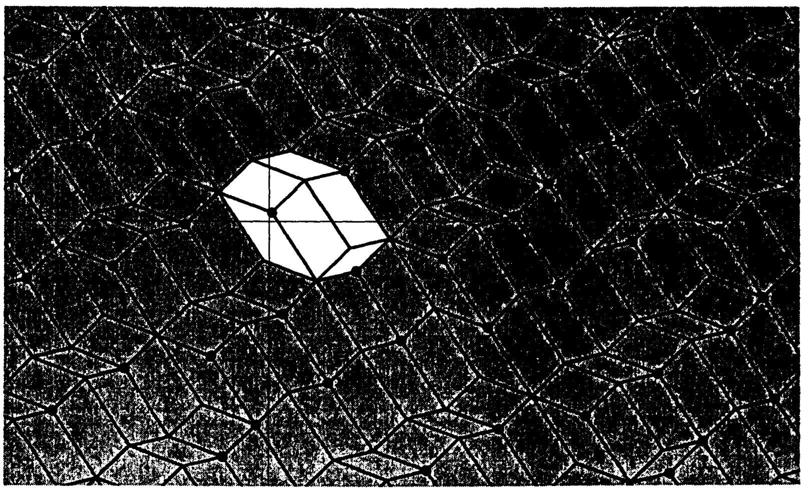
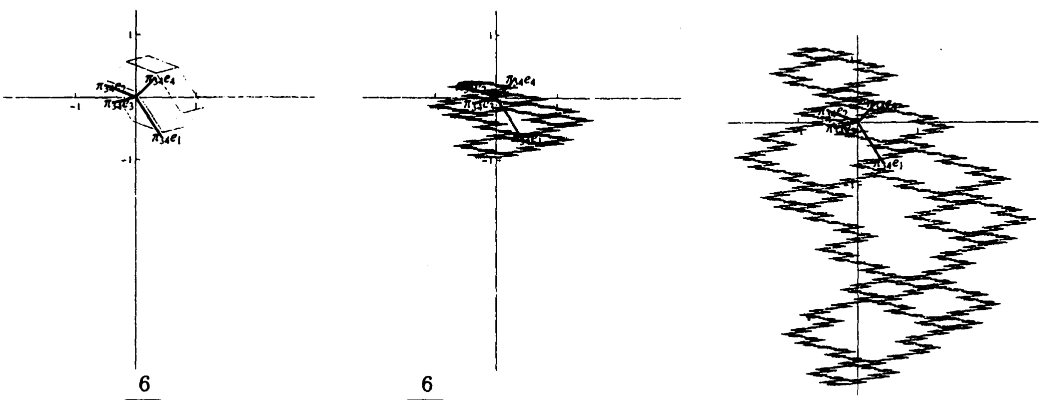


Figure 18: The quasi-periodic polygonal tiling $\mathcal{T}_{c,1}$ of $\mathcal{L}(\boldsymbol{v}_3, \boldsymbol{v}_4)$.



$$\pi_{34}\mathcal{U}_c = \sum_{i=1}^{6} \pi_{34}\left(\boldsymbol{x}_i, \gamma_i\right) \qquad \sum_{i=1}^{6}\left(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}\right) \qquad M_\sigma\left(\pi_{34}\boldsymbol{x}_i + X_{\gamma_i}\right)$$

Figure 19: $\pi_{34}\mathcal{U}_c$ and the proto-tiles of the quasi-periodic self-affine tiling $\mathcal{T}_{c,2}$.

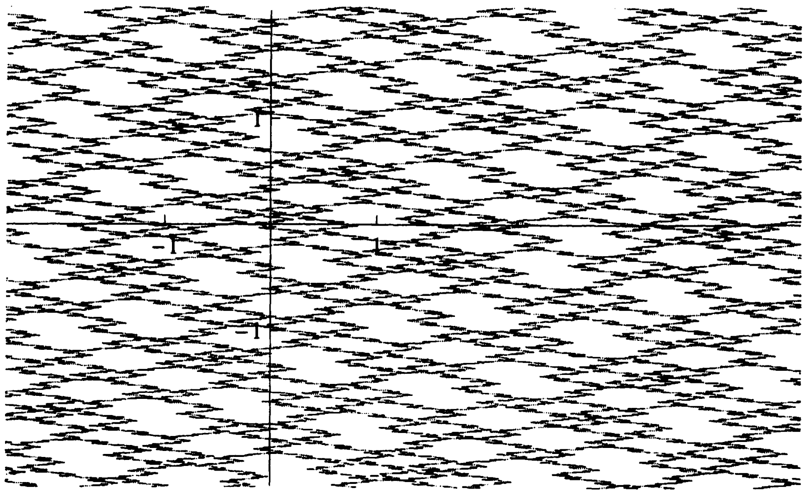


Figure 20: The quasi-periodic self-affine tiling $\mathcal{T}_{c,2}$ of $\mathcal{L}\left(\boldsymbol{v}_3, \boldsymbol{v}_4\right)$.

By the analogous discussion, we can construct the quasi-periodic polygonal/self-affine tiling from the "tiling substitution $\widehat{E_2}(\sigma)$" on the expanding plane $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

Let us observe the figure $\{\pi_{12}\boldsymbol{e}_i\}_{i=1,2,3,4}$ (see Figure 21) and we consider the *proto tiles* of parallelograms on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ (see Figure 22).

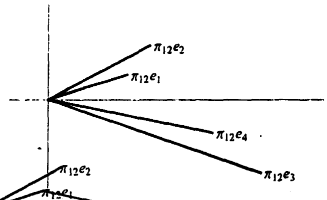


Figure 21: $\{\pi_{34}\boldsymbol{e}_i\}_{i=1,2,3,4}$.

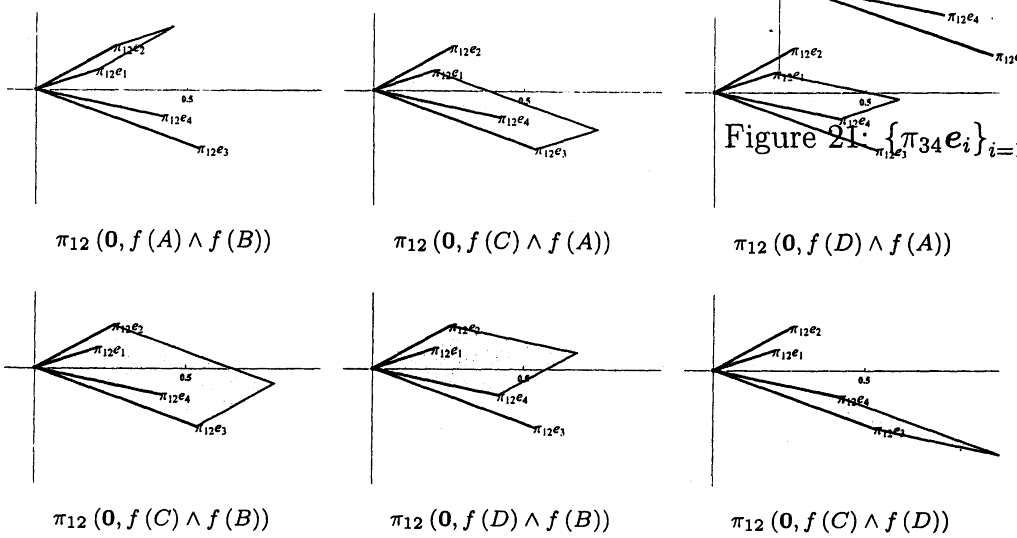


Figure 22: The proto tiles on $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ generated by $f(A) = \boldsymbol{e}_1$, $f(B) = \boldsymbol{e}_2$, $f(C) = \boldsymbol{e}_3$, $f(D) = \boldsymbol{e}_4$.

Using the automorphism σ:

$$\begin{array}{rcl} \sigma(A) & = & ABD \\ \sigma(B) & = & ABBD \\ \sigma(C) & = & ABDCCD \\ \sigma(D) & = & ABDCD \end{array},$$

we try to consider the 2-dimensional extension of the automorphisms (substitutions) σ as follows:

$$\begin{aligned} E_2(\sigma)(\mathbf{0}, \alpha \wedge \beta) &:= (\mathbf{0}, \sigma(\alpha) \wedge \sigma(\beta)) \\ &= \sum_{\substack{1 \le i \le l_\alpha \\ 1 \le j \le l_\beta}} \left(f\left(P_i^{(\alpha)}\right) + f\left(P_j^{(\beta)}\right), W_i^{(\alpha)} \wedge W_j^{(\beta)} \right) \end{aligned}$$

(see Figure 23):

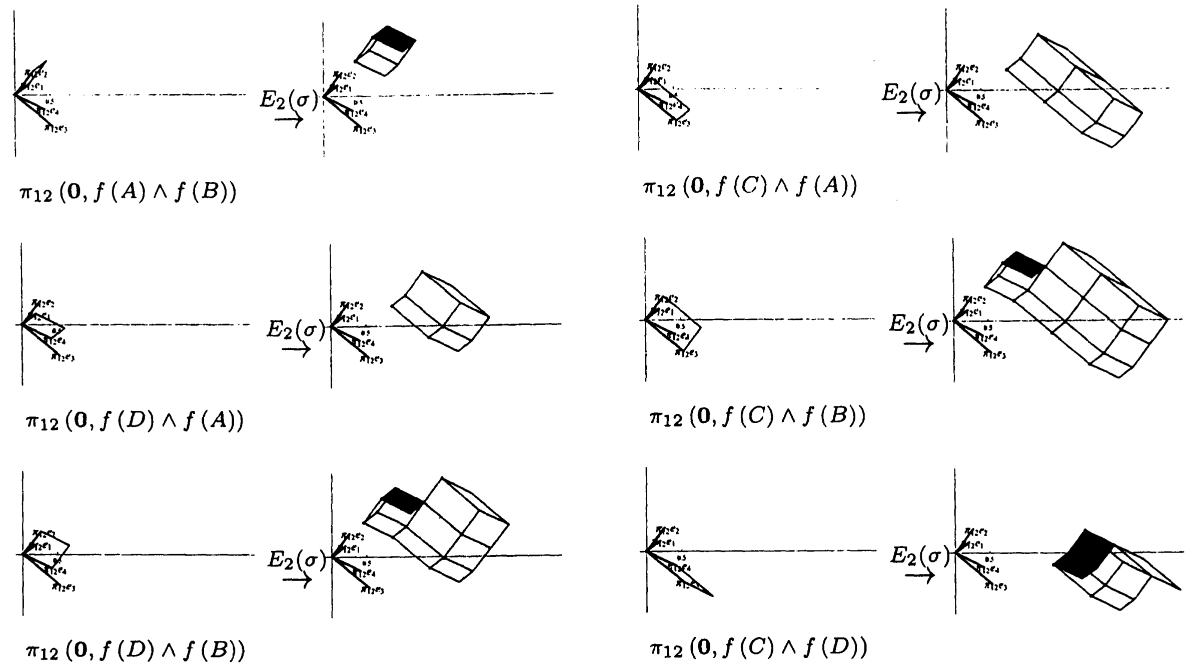


Figure 23: $E_2(\sigma)(\mathbf{0}, \alpha \wedge \beta)$.

On our example, we know that A^* is positive, that is,

$$A^* = \begin{array}{c} A\wedge B \\ C\wedge A \\ D\wedge A \\ C\wedge B \\ D\wedge B \\ C\wedge D \end{array} \left[\begin{array}{c} \\ \\ a^*_{i\wedge j,k\wedge l} \\ \\ \\ \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 2 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 4 & 2 & 1 \\ 1 & 1 & 1 & 3 & 3 & 0 \\ 0 & 2 & 1 & 2 & 1 & 2 \end{bmatrix} \geq O$$

where $a^*_{i\wedge j,k\wedge l} = \det \begin{bmatrix} a_{ik} & a_{il} \\ a_{jk} & a_{jl} \end{bmatrix}$ for $M_\sigma = [a_{ij}]_{1\leq i,j\leq 4}$.

From this fact, we try to find the tiling substitution $\widehat{E_2}(\sigma)$ by the retiling method in [F-I-Rob] analogously (see Figure 24):

Theorem 4.2. *Let $\mathcal{U}_e$ be a patch generated by the following proto tiles:*

$$\begin{aligned} \mathcal{U}_e &= (\boldsymbol{y}+\boldsymbol{e}_4, f(A)\wedge f(B)) + (\boldsymbol{y}+\boldsymbol{e}_4, f(C)\wedge f(A)) + (\boldsymbol{y}+\boldsymbol{e}_2, f(D)\wedge f(A)) \\ &\quad + (\boldsymbol{y}+\boldsymbol{e}_4, +\boldsymbol{e}_1, f(C)\wedge f(B)) + (\boldsymbol{y}, f(D)\wedge f(B)) + (\boldsymbol{y}, f(C)\wedge f(D)), \end{aligned}$$

where $\boldsymbol{y} = \left(\frac{8}{19}, -\frac{14}{19}, -\frac{2}{19}, -\frac{18}{19}\right)$. Then, we see that $\mathcal{U}_e$ is the seed, that is, $\widehat{E_2}(\sigma)^3(\mathcal{U}_e) \succ \mathcal{U}_e$ (see Figure 25).

Moreover,

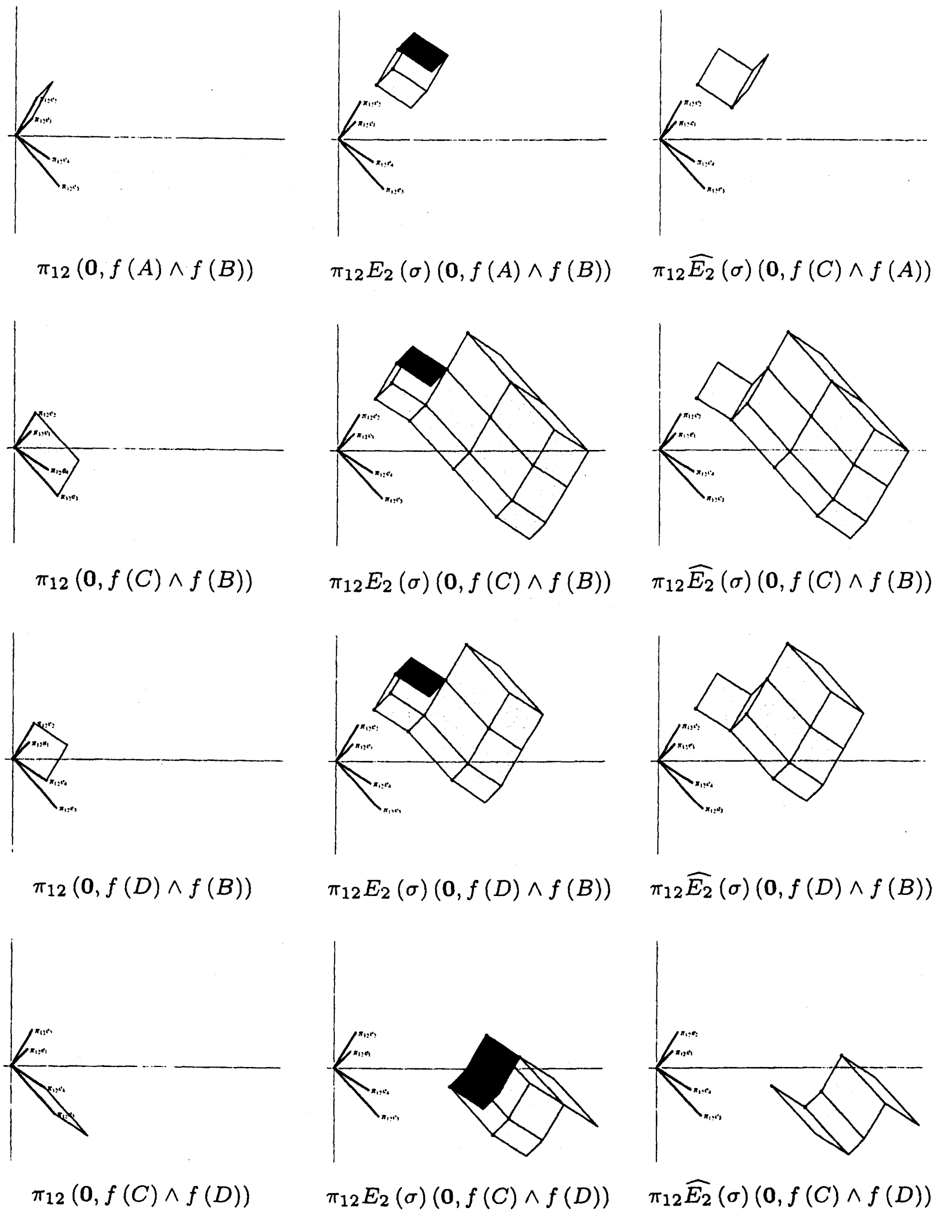
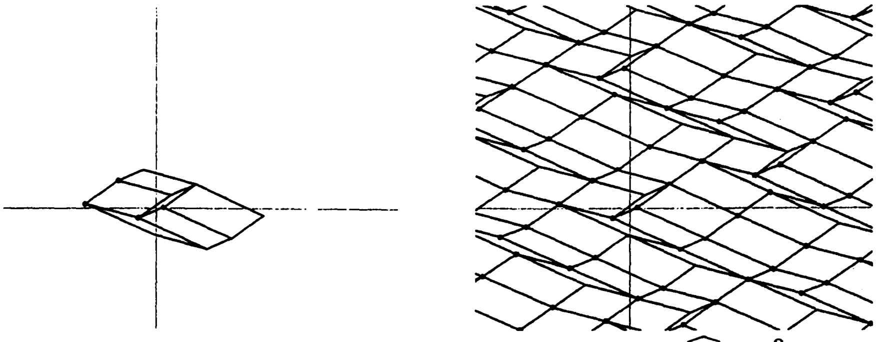


Figure 24: The retiling method from $E_2(\sigma)$ to $\widehat{E_2}(\sigma)$.



$\pi_{12}\mathcal{U}_e$ A part of $\pi_{12}\widehat{E_2}(\sigma)^3(\mathcal{U}_c)$

Figure 25: $\pi_{12}\mathcal{U}_e$ and a part of $\pi_{12}\widehat{E_2}(\sigma)^3(\mathcal{U}_e)$.

(1) $\mathcal{T}_{e,1} := \left\{\pi_{12}(\boldsymbol{x}, f(\alpha)\wedge f(\beta)) \;\middle|\; \widehat{E_2}(\sigma)^{3n}(\mathcal{U}_e) \ni (\boldsymbol{x}, f(\alpha)\wedge f(\beta))\right\}$ *is a quasi-periodic polygonal tiling of* $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ *(see Figure 26);*

(2) Put

$$X_{\alpha\wedge\beta} := \lim_{n\to\infty} \pi_{12} M_\sigma^{-3n} \widehat{E_2}(\sigma)^{3n}(\mathbf{0}, f(\alpha)\wedge f(\beta)).$$

Then, $\{X_{\alpha\wedge\beta}\}$ *satisfies the set equations:*

$$M_\sigma^{-1}(\pi_{12}\boldsymbol{x}_i + X_{\delta_i}) = \sum_k \left(\pi_{12}\boldsymbol{x}_k^{(i)} + X_{\delta_k^{(i)}}\right)$$

where $\mathcal{U}_e = \sum_{i=1}^6 (\boldsymbol{x}_i, \delta_i)$ *and* $\widehat{E_2}(\sigma)(\boldsymbol{x}_i, \delta_i) = \sum_k \left(\boldsymbol{x}_k^{(i)} + \delta_k^{(i)}\right)$*;*

(3) $\mathcal{T}_{e,2} := \{\pi_{12}\boldsymbol{x} + X_{\alpha\wedge\beta} \mid \pi_{12}(\boldsymbol{x}, f(\alpha)\wedge f(\beta)) \in \mathcal{T}_{e,1}\}$. *Then,* $\mathcal{T}_{e,2}$ *is a quasi-periodic self-affine tiling of* $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ *(see Figure 28).*

References

[A-F-H-I] P.Arnoux, M.Furukado, E.Harriss and S.Ito, *Algebraic numbers and automorphisms of free groups*, to appear in Transactions of the AMS.

[E] H.Ei, *Some properties of invertible substitutions of rank d, and higher dimensional substitutions*, Osaka J. Math. **40** (2003), 543–562.

[F-I-Rao] M.Furukado, S.Ito and H.Rao, *Geometric realizations of hyperbolic unimodular substitutions*, Progress in Probability **61** (2009), 251–268.

[F-I-Rob] M.Furukado, S.Ito and E.A.Robinson Jr., *Tilings associated with non-Pisot matrices*, Ann. Inst. Fourier, Grenoble **56** (2006), 2391–2435.

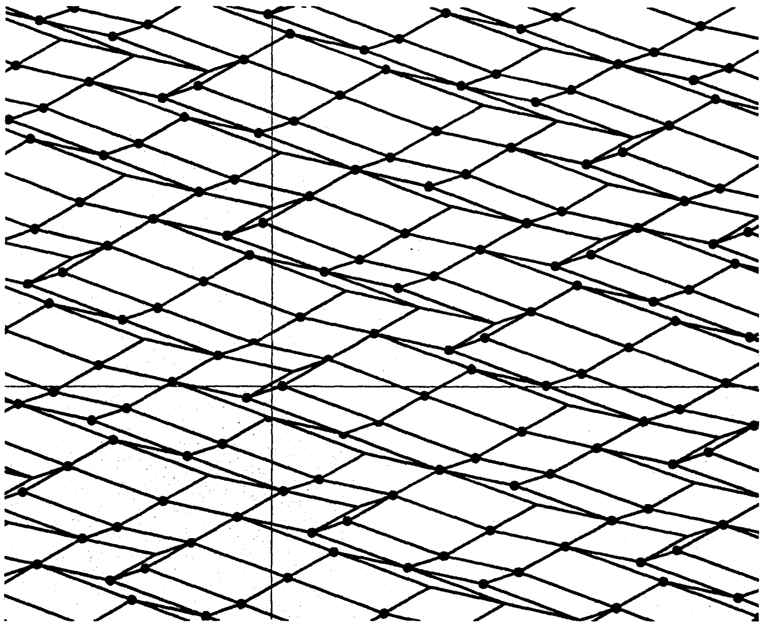
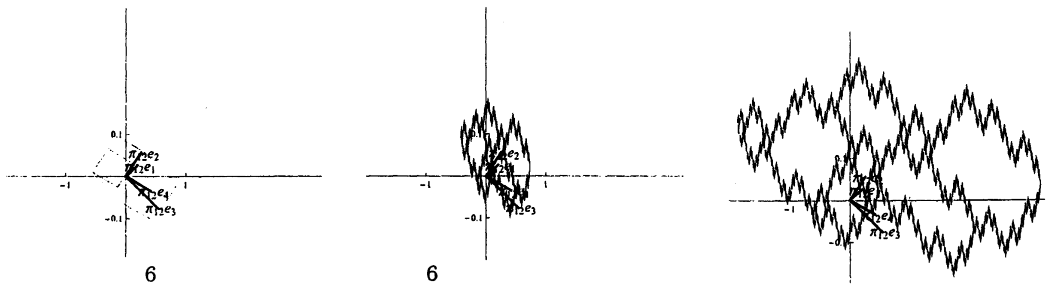


Figure 26: The quasi-periodic polygonal tiling $\mathcal{T}_{e,1}$ of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.



$$\pi_{12}\mathcal{U}_e = \sum_{i=1}^{6} \pi_{12}(\boldsymbol{x}_i, \delta_i) \qquad \sum_{i=1}^{6} (\pi_{12}\boldsymbol{x}_i + X_{\delta_i}) \qquad M_\sigma^{-1}(\pi_{12}\boldsymbol{x}_i + X_{\delta_i})$$

Figure 27: $\pi_{12}\mathcal{U}_e$ and the proto-tiles of the quasi-periodic self-affine tiling $\mathcal{T}_{e,2}$.

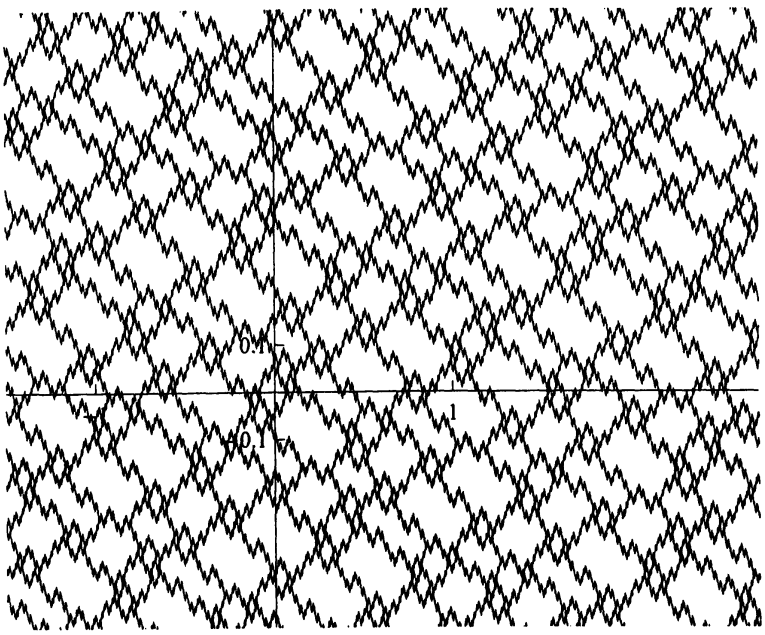


Figure 28: The quasi-periodic self-affine tiling $\mathcal{T}_{e,2}$ of $\mathcal{L}(\boldsymbol{v}_1, \boldsymbol{v}_2)$.

[Fo] N.Pytheas Fogg, *Substitutions in Dynamics, Arithmetics and Combinatorics*, Lect. Notes in Math., Springer-Verlag **1794** (2002).

[H-F-I] M.Hama, M.Furukado and S.Ito, *Complex Pisot Numeration Systems*, Comment. Math. Univ. St. Pauli. **58** (2009), 9–49.

[K] S.P.Kerckhoff, *Simplicial systems for interval exchange maps and measured foliations*, Ergodic Theory Dynam. Systems **5** (1985), no. 2, 257–271.

[L] M.Lothaire, *Algebraic Combinatorics on Words*, Cambridge University Press, Cambridge (2002).

[Q] M.Queffélec, *Substitution dynamical systems. Spectral analysis*, Lect. Notes in Math., Springer-Verlag **1294** (1987).

[S-A-I] Y.Sano, P.Arnoux, and S.Ito *Higher dimensional extensions of substitutions and their dual maps*, J. Anal. Math. **83** (2001), 183–206 .

[Y] J.C.Yoccoz, *Continued fraction algorithms for interval exchange maps: an introduction*, Frontiers in Number Theory, Physics, and Geometry I, Springer Verlag (2006), 403–437.