Some applications for subordination principle

Kazuo Kuroki and Shigeyoshi Owa

Abstract

By considering some subordinations for a more general linear transformation, an extension of the Briot-Bouquet differential subordination relations given by S. S. Miller and P. T. Mocanu (Pure and Applied Mathematics 225, Marcel Dekker, 2000) for certain linear transformations are discussed.

1 Introduction

Let $\mathcal{H}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1 \}$. For a positive integer $n$ and a complex number $a$, let $\mathcal{H}[a, n]$ be the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = a + \sum_{k=n}^{\infty} a_k z^k.$$

Also, let $\mathcal{A}_n$ denote the class of functions $f(z) \in \mathcal{H}$ of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k$$

with $\mathcal{A}_1 = \mathcal{A}$. If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some real number $\alpha$ with $0 \leq \alpha < 1$, then $f(z)$ is said to be starlike of order $\alpha$ in $U$. This class is denoted by $S^*(\alpha)$. Similarly, we say that $f(z)$ belongs to the class $\mathcal{K}(\alpha)$ of convex functions of order $\alpha$ in $U$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some real number $\alpha$ with $0 \leq \alpha < 1$.

2000 Mathematics Subject Classification: Primary 30C45.

Keywords and Phrases: Differential subordination, Briot-Bouquet differential equation, convex function, linear transformation.
For some real numbers $A$ and $B$ with $-1 \leq B < A \leq 1$, Janowski [1] has investigated the following linear transformation

$$p(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})$$

which is analytic and univalent in $\mathbb{U}$. This function $p(z)$ is called the Janowski function. Moreover, as a generalization of the Janowski functions, Kuroki and Owa [2] have discussed the Janowski functions for some complex parameters $A$ and $B$ which satisfy

$$A \neq B, \quad |B| \leq 1 \quad \text{and} \quad |A - B| + |A + B| \leq 2.$$  

Note that the Janowski function defined by the conditions (1.1) is analytic and univalent in $\mathbb{U}$ and has a positive real part in $\mathbb{U}$ (see [2]).

We next introduce the familiar principle of differential subordinations between analytic functions. Let $p(z)$ and $q(z)$ be members of the class $\mathcal{H}$. Then the function $p(z)$ is said to be subordinate to $q(z)$ in $\mathbb{U}$, written by

(1.2) \quad p(z) \prec q(z) \quad (z \in \mathbb{U}),

if there exists a function $w(z)$ which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1 \quad (z \in \mathbb{U})$, and such that $p(z) = q(w(z)) \quad (z \in \mathbb{U})$. From the definition of the subordinations, it is easy to show that the subordination (1.2) implies that

(1.3) \quad p(0) = q(0) \quad \text{and} \quad p(\mathbb{U}) \subset q(\mathbb{U}).

In particular, if $q(z)$ is univalent in $\mathbb{U}$, then the subordination (1.2) is equivalent to the condition (1.3).

Miller and Mocanu [4] developed the definitive result concerning the Briot-Bouquet differential subordinations as follows.

**Lemma 1.1** Let $n$ be a positive integer, and let $\beta$ and $\gamma$ be complex numbers with $\beta \neq 0$. Also, let $h(z)$ be convex and univalent in $\mathbb{U}$ with $h(0) = a$, and suppose that

(1.4) \quad \text{Re}(\beta h(z) + \gamma) > 0 \quad (z \in \mathbb{U})

with $\text{Re}(\beta a + \gamma) > 0$. If $p(z) \in \mathcal{H}[a, n]$ with $p(z) \neq a$ satisfies the differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \quad (z \in \mathbb{U}),$$

then $p(z) \prec q(z) \prec h(z) \quad (z \in \mathbb{U})$, where $q(z)$ with $q(0) = a$ is the univalent solution of the differential equation

$$q(z) + \frac{nq'(z)}{\beta q(z) + \gamma} = h(z) \quad (z \in \mathbb{U}).$$

As applications of Lemma 1.1, Miller and Mocanu [4] derived some subordination relation for certain linear transformations.
Lemma 1.2  Let $n$ be a positive integer. Also, let $\beta$, $\gamma$ and $A$ be complex numbers with $\text{Re}(\beta + \gamma) > 0$, and let $B$ be a real number with $-1 \leq B \leq 0$. If $\beta$, $\gamma$, $A$ and $B$ satisfy either

$$\text{Re}(\beta(1 + AB) + \gamma(1 + B^2)) \geq |\beta A + \overline{\beta} B + 2B\text{Re}\gamma| \quad (-1 < B \leq 0)$$

or

$$\beta(1 + A) > 0 \quad \text{and} \quad \text{Re}(\beta(1 + A) + 2\gamma) \geq 0 \quad (B = -1),$$

then $p(z) \in \mathcal{H}[1, n]$ with $p(z) \neq 1$ satisfies the following subordination relation

$$(1.5) \quad p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} < \frac{1 + Az}{1 + Bz} \quad \text{implies} \quad p(z) < q(z) < \frac{1 + Az}{1 + Bz}$$

for $z \in U$, where $q(z)$ with $q(0) = a$ is the univalent solution of the differential equation

$$(1.6) \quad q(z) + \frac{nq'(z)}{\beta q(z) + \gamma} = \frac{1 + Az}{1 + Bz} \quad (z \in U).$$

In the present paper, applying the theory of subordinations, we will try to determine the best conditions for complex numbers $\beta$, $\gamma$, $A$ and $B$ to satisfy the condition (1.4) as

$$h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in U)$$

in Lemma 1.1, and deduce an extension of Lemma 1.2.

2  Some subordinations for certain linear transformations

By using the method of a certain generalization of the Janowski functions given by Kuroki and Owa [2], we first consider a certain subordination for a more general linear transformation.

Theorem 2.1  Let $a$, $A$, $B$, $C$ and $D$ be complex numbers with $A \neq aB$ and $C \neq aD$. If $a$, $A$, $B$, $C$ and $D$ satisfy $|B| \leq 1$, $|D| \leq 1$ and

$$(2.1) \quad |A - aB| + |AD - BC| \leq |C - aD|,$$

then

$$(2.2) \quad \frac{a + Az}{1 + Bz} < \frac{a + Cz}{1 + Dz} \quad (z \in U).$$

Proof.  From $A \neq aB$ and the inequality (2.1), it is clear that

$$(2.3) \quad |C - aD| - |AD - BC| > 0.$$
If we define the function $w(z)$ by
\begin{equation}
(2.4) \quad w(z) = \frac{(A-aB)z}{C-aD-(AD-BC)z} \quad (z \in \mathbb{U}),
\end{equation}
then from the inequality (2.3), $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$, and that
\[
\frac{a+Az}{1+Bz} = \frac{a+Cw(z)}{1+Dw(z)} \quad (z \in \mathbb{U}).
\]

Further, noting the inequality (2.3), a simple calculation yields
\[
\left| w(z) - \frac{(A-aB)(\overline{AD-BC})}{|C-aD|^2-|AD-BC|^2} \right| < \frac{|A-aB||C-aD|}{|C-aD|^2-|AD-BC|^2} \quad (z \in \mathbb{U}).
\]

Since the inequality (2.1) shows that
\[
\frac{|A-aB|}{|C-aD|-|AD-BC|} \leqq 1,
\]
we see that $w(z)$ defined by (2.4) satisfies $|w(z)| < 1 \quad (z \in \mathbb{U})$. Therefore, from the definition of the subordinations, we conclude that the subordination (2.2) holds, which completes the proof of Theorem 2.1.

In particular, letting $A = b$, $C = \overline{a}e^{i\theta}$ and $D = -e^{i\theta}$

for a complex number $a$ with $\text{Re} a > 0$ and for some $\theta$ with $0 \leqq \theta < 2\pi$ in Theorem 2.1, we find the following assertion.

**Corollary 2.2** Let $a$ be a complex number with $\text{Re} a > 0$. For some complex numbers $a$, $b$ and $B$ with
\[
b \neq aB, \quad |B| \leqq 1 \quad \text{and} \quad |b-aB| + |b+\overline{a}B| \leqq 2\text{Re} a,
\]
we have
\[
\frac{a+bz}{1+Bz} \prec \frac{a+\overline{a}e^{i\theta}z}{1-e^{i\theta}z} \quad (z \in \mathbb{U}),
\]
where $0 \leqq \theta < 2\pi$. This subordination means the following inequality
\[
\text{Re} \left( \frac{a+bz}{1+Bz} \right) > 0 \quad (z \in \mathbb{U}).
\]

**Remark 2.3** Taking $a = 1$ and $b = A$ in Corollary 2.2, we find the conditions in (1.1) as the conditions for complex numbers $A$ and $B$ to satisfy
\[
\text{Re} \left( \frac{1+Az}{1+Bz} \right) > 0 \quad (z \in \mathbb{U}).
\]
3 The Briot-Bouquet differential subordinations for certain linear transformations

By using the discussion in the previous section, and applying Lemma 1.1, we deduce an improvement of Lemma 1.2 below.

**Theorem 3.1** Let \( n \) be a positive integer, and let \( \beta, \gamma, A \) and \( B \) be complex numbers with \( \text{Re}(\beta + \gamma) > 0 \), \( A \neq B \) and \( |B| \leq 1 \). If \( \beta, \gamma, A \) and \( B \) satisfy

\[
|\beta(A - B)| + |\beta(A - B) + 2B\text{Re}(\beta + \gamma)| \leq 2\text{Re}(\beta + \gamma),
\]

then \( p(z) \in \mathcal{H}[1, n] \) with \( p(z) \neq 1 \) satisfies the subordination relation (1.5), where \( q(z) \) with \( q(0) = 1 \) is the solution of the differential equation (1.6).

**Proof.** If we let

\[
a = \beta + \gamma, \quad b = \beta A + \gamma B \quad \text{and} \quad h(z) = \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U}),
\]

then, a simple check gives us that

\[
b - aB = (\beta A + \gamma B) - (\beta + \gamma)B = \beta(A - B) \neq 0
\]

and

\[
2\text{Re} a - (|b - aB| + |b - \overline{a}B|) = 2\text{Re} a - (|b - aB| + |b - aB + 2B\text{Re} a|)
\]

\[
= 2\text{Re}(\beta + \gamma) - (|\beta(A - B)| + |\beta(A - B) + 2B\text{Re}(\beta + \gamma)|) \geq 0.
\]

Hence by Corollary 2.2, it is easy to see that

\[
\text{Re}(\beta h(z) + \gamma) = \text{Re}\left(\frac{\beta + \gamma + (\beta A + \gamma B)z}{1 + Bz}\right) = \text{Re}\left(\frac{a + bz}{1 + Bz}\right) > 0 \quad (z \in \mathbb{U}).
\]

Therefore, since the conditions of Lemma 1.1 are satisfied, we conclude the assertion of Theorem 3.1. \( \square \)

By taking \( \beta = 1, \gamma = 0 \) and \( n = 1 \) in Theorem 3.1, and letting

\[
p(z) = \frac{zf''(z)}{f'(z)} \quad (z \in \mathbb{U})
\]

for \( f(z) \in \mathcal{A} \), we obtain the following subordination implication.

**Corollary 3.2** If \( f(z) \in \mathcal{A} \) satisfies

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathbb{U})
\]

for some complex numbers \( A \) and \( B \) which satisfy the conditions in (1.1), then

\[
\frac{zf'(z)}{f(z)} < \begin{cases} 
\frac{Az}{(1 + B)(1 - (1 + Bz)^{-\frac{1}{B}})} & (A \neq 0, B \neq 0) \\
\frac{Bz}{(1 + Bz)\log(1 + Bz)} & (A = 0, B \neq 0) \\
\frac{Aze^{Az}}{e^{Az} - 1} & (A \neq 0, B = 0)
\end{cases}
\]
for $z \in U$.

Moreover, let us consider the case that

$$A = 1 - 2\alpha \quad (0 \leq \alpha < 1) \quad \text{and} \quad B = -1$$

in Corollary 3.2. Then, from the definition of the subordinations, we find the implication that if $f(z) \in \mathcal{K}(\alpha)$, then $f(z) \in S^*(\beta)$, where

$$\beta = \beta(\alpha) = \begin{cases} 
\frac{1-2\alpha}{2^{2-2\alpha}(1-2^{2\alpha-1})} & (\alpha \neq \frac{1}{2}) \\
\frac{1}{2\log 2} & (\alpha = \frac{1}{2})
\end{cases}$$

for each real number $\alpha$ with $0 \leq \alpha < 1$. This relationship for convex and starlike functions was proven by MacGregor [3].

References


