

N -Fractional Calculus of Some Logarithmic Functions and Some Identities

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Abstract

In this article, N-fractional calculus of the logarithmic function in title is discussed.

A theorem is presented as follows for example.

Theorem 1. Let $f = f(z) = (\sqrt{z - b} - c)^2 - d \neq 0, 1$.

We have then;

(i)

$$\begin{aligned} (\log f)_\gamma &= -e^{-i\pi\gamma}(z - b)^{-\gamma} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\frac{1}{2}k + 1)} S^k \right. \\ &+ \left. \sum_{k=1}^{\infty} \frac{1}{k} T^k \sum_{m=0}^{\infty} \frac{[2k]_m \Gamma(\frac{1}{2}m + k + \gamma)}{m! \Gamma(\frac{1}{2}m + k)} S^m \right\} \\ &(|\Gamma(\frac{1}{2}k + \gamma)|, |\Gamma(\frac{1}{2}m + k + \gamma)| < \infty) \end{aligned}$$

and

(ii)

$$(\log f)_n = (-1)^{n+1}(z - b)^{-n} \left\{ \sum_{k=0}^{\infty} \left[\frac{1}{2}k + 1 \right]_{n-1} S^k + \sum_{k=1}^{\infty} \frac{1}{k} T^k \sum_{m=0}^{\infty} \frac{[2k]_m \left[\frac{1}{2}m + k \right]_n}{m!} S^m \right\} (n \in \mathbb{Z}^+) (n\text{-th derivatives})$$

where $S = \frac{c}{(z - b)^{1/2}}$, $T = \frac{d}{z - b}$, $|S| < 1$, $|T| < 1$,

and $[\lambda]_k (k \in \mathbb{Z}_0^+)$; Notation of Pochhammer.

§0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol.1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$

C_- be a curve along the cut joining two points z and $-\infty + i \operatorname{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i \operatorname{Im}(z)$,

D_- be a domain surrounded by C_- , D_+ be a domain surrounded by C_+

(Here D contains the points over the curve C)

Moreover, let $f = f(z)$ be a regular function in $D (z \in D)$,

$$f_\nu = (f)_\nu =_C (f)_\nu = \frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi - z)^{\nu+1}} d\xi \quad (\nu \notin \mathbb{Z}^-), \quad (0.1)$$

$$(f)_{-m} = \lim_{\nu \rightarrow -m} (f)_\nu \quad (m \in \mathbb{Z}^+), \quad (0.2)$$

where $-\pi \leq \arg(\xi - z) \leq \pi$ for C_- , $0 \leq \arg(\xi - z) \leq 2\pi$ for C_+ ,

$\xi \neq z$, $z \in C$, $\nu \in \mathbb{R}$, Γ : Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z , of the function f , if $|(f)_\nu| < \infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A Let fractional calculus operator (Nishimoto's operator) N^ν be

$$N^\nu = \left(\frac{\Gamma(\nu + 1)}{2\pi i} \int_C \frac{d\xi}{(\xi - z)^{\nu+1}} \right) \quad (\nu \notin \mathbb{Z}^-), \quad [\text{Refer to (1.1)}] \quad (0.3)$$

with

$$N^{-m} = \lim_{\nu \rightarrow -m} N^\nu \quad (m \in \mathbf{Z}^+), \quad (0.4)$$

and define the binary operation \circ as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta(N^\alpha f) \quad (\alpha, \beta \in \mathbf{R}), \quad (0.5)$$

then the set

$$\{N^\nu\} = \{N^\mu | \nu \in \mathbf{R}\} \quad (0.6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^\nu)^{-1} = N^{-\nu}$ to the fractional calculus operator N^ν , for the function f such that $f \in F = \{f; 0 \neq |f_\nu| < \infty, \nu \in \mathbf{R}\}$, where $f = f(z)$ and $z \in \mathbf{C}$. (viz. $-\infty < \nu < \infty$).

(For our convenience, we call $N^\beta \circ N^\alpha$ as product of N^β and N^α)

Theorem B The "F.O.G. $\{N^\nu\}$ " is an " Action product group which has continuous index ν " for the set of F . (F.O.G.: Fractional calculus operator group) [3]

Theorem C. Let

$$S := \{\pm N^\nu\} \cup \{0\} = \{N^\nu\} \cup \{-N^\nu\} \cup \{0\} (\nu \in \mathbf{R}) \quad (0.7)$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^\alpha + N^\beta = N^\gamma (N^\alpha, N^\beta, N^\gamma \in S) \quad (0.8)$$

holds. [5]

(III) **Lemma 1** We have [1]

(i)

$$((z-c)^\beta)_\alpha = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty), \quad (0.9)$$

(ii)

$$(\log(z-c))_{\alpha} = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \quad (0.10)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \quad (|\Gamma(\alpha)| < \infty), \quad (0.11)$$

(iv)

$$(u \cdot v)_{\alpha} = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)} u_{\alpha-k} v_k \quad (u = u(z), v = v(z)) \quad (0.12)$$

where $z \neq c$ in (0.9), and $z - c \neq 0, 1$ in (0.10) and (0.11).

§1. Preliminary

Theorem D. below for the fractional calculus of a logarithmic function is reported by K. Nishimoto (cf. J. Frac. Calc. Vol. 29, May (2006), pp. 35-44)[12].

Theorem D. We have

(i)

$$\begin{aligned} & (\log((z-b)^{\beta} - c))_{\gamma} = -e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\Gamma(\gamma) \\ & \times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\gamma)\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^{\beta}} \right)^k \quad \left(\left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \right| < \infty \right) \quad (1.1) \end{aligned}$$

and

(ii)

$$\begin{aligned} & (\log((z-b)^{\beta} - c))_m = (-1)^{m+1}\beta(z-b)^{-m}\Gamma(m) \\ & \times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + m)}{\Gamma(m)\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^{\beta}} \right)^k \quad (m \in \mathbf{Z}^+) \end{aligned}$$

where $(z-b)^{\beta} - c \neq 0, 1$, and $|c/(z-b)^{\beta}| < 1$,

and $[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$ with $[\lambda]_0 = 1$,

(Notation of Pochhammer). (1.2)

And the Theorem E below is reported by K. Nishimoto already (cf. JFC Vol. 32, Nov. (2007), pp. 17-28)[13].

Theorem E. We have

(i)

$$\begin{aligned}
 & (\log(((z-b)^\beta - c)^\alpha - d))_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma} \\
 & \times \left[\alpha\beta \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \left(\frac{c}{(z-b)^\beta} \right)^k \right. \\
 & \left. + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^k \sum_{m=0}^{\infty} \frac{[\alpha k]_m \Gamma(\beta m + \alpha\beta k + \gamma)}{m! \Gamma(\beta m + \alpha\beta k)} \left(\frac{c}{(z-b)^\beta} \right)^m \right] \\
 & \left(\left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \right|, \left| \frac{\Gamma(\beta m + \alpha\beta k + \gamma)}{\Gamma(\beta m + \alpha\beta k)} \right| < \infty \right) \quad (1.3)
 \end{aligned}$$

and

(ii)

$$\begin{aligned}
 & (\log(((z-b)^\beta - c)^\alpha - d))_n \\
 & = (-1)^{n+1} (z-b)^{-n} \left[\alpha\beta \sum_{k=0}^{\infty} [\beta k + 1]_{n-1} \left(\frac{c}{(z-b)^\beta} \right)^k \right. \\
 & \left. + \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{d}{(z-b)^{\alpha\beta}} \right)^k \sum_{m=0}^{\infty} \frac{[\alpha k]_m [\beta m + \alpha\beta k]_n}{m!} \left(\frac{c}{(z-b)^\beta} \right)^m \right] \\
 & \text{where } ((z-b)^\beta - c)^\alpha - d \neq 0, 1. (n \in \mathbb{Z}^+) \\
 & \text{and } \left| \frac{d}{((z-b)^\beta - c)^\alpha} \right|, \left| \frac{c}{(z-b)^\beta} \right|, \left| \frac{d}{(z-b)^{\alpha\beta}} \right| < 1. \quad (1.4)
 \end{aligned}$$

§2. N-Fractional Calculus of A Logarithmic Function

Theorem 1. Let

$$f = f(z) = (\sqrt{z-b} - c)^2 - d \neq 0, 1 \quad (2.1)$$

We have then;

(i)

$$\begin{aligned}
 (\log f)_\gamma &= -e^{-i\pi\gamma}(z-b)^{-\gamma} \left\{ \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\frac{1}{2}k + 1)} S^k \right. \\
 &+ \left. \sum_{k=1}^{\infty} \frac{1}{k} T^k \sum_{m=0}^{\infty} \frac{[2k]_m \Gamma(\frac{1}{2}m + k + \gamma)}{m! \Gamma(\frac{1}{2}m + k)} S^m \right\} \\
 &\quad \left(|\Gamma(\frac{1}{2}k + \gamma)|, |\Gamma(\frac{1}{2}m + k + \gamma)| < \infty \right) \tag{2.2}
 \end{aligned}$$

and

(ii)

$$\begin{aligned}
 (\log f)_n &= -(-1)^n (z-b)^{-n} \left\{ \sum_{k=0}^{\infty} [\frac{1}{2}k + 1]_{n-1} S^k \right. \\
 &+ \left. \sum_{k=1}^{\infty} \frac{1}{k} T^k \sum_{m=0}^{\infty} \frac{[2k]_m [\frac{1}{2}m + k]_n}{m!} S^m \right\} \quad (n \in \mathcal{Z}^+) \text{ (n-th derivatives)} \\
 \text{where } S &= \frac{c}{(z-b)^{1/2}}, T = \frac{d}{z-b}, |S| < 1, |T| < 1, \tag{2.3}
 \end{aligned}$$

Proof of (i). Set $\beta = 1/2$ and $\alpha = 2$ in Theorem E (i), We obtain (2.2) under the conditions stated before.

Theorem 2. Let $f = f(z)$ be (2.1). We have then

(i)

$$\begin{aligned}
 (\log f)_\gamma &= -e^{-i\pi\gamma} \frac{\Gamma(\gamma)}{2} (z-b)^{-\gamma} \\
 &\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k + 1)} \left\{ \left(\frac{c + \sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c - \sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} \\
 &\quad \left(|\Gamma(\frac{1}{2}k + \gamma)| < \infty \right) \tag{2.4}
 \end{aligned}$$

and

(ii)

$$\begin{aligned}
(\log f)_n &= -(-1)^n \frac{\Gamma(n)}{2} (z-b)^{-n} \\
&\times \sum_{k=0}^{\infty} \frac{[\frac{1}{2}k+1]_{n-1}}{\Gamma(n)} \left\{ \left(\frac{c+\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} (n\text{-derivatives}) (n \in \mathbb{Z}+) \\
\text{where } & \left| \frac{c+\sqrt{d}}{(z-b)^{1/2}} \right|, \left| \frac{c-\sqrt{d}}{(z-b)^{1/2}} \right| < 1. \tag{2.5}
\end{aligned}$$

Proof of (i). We have

$$\begin{aligned}
\log f &= \log\{\sqrt{z-b} - c - \sqrt{d}\} \{\sqrt{z-b} - c + \sqrt{d}\} \\
&= \log((z-b)^{1/2} - (c + \sqrt{d})) + \log((z-b)^{1/2} - (c - \sqrt{d})) \tag{2.6}
\end{aligned}$$

Operate N -fractional calculus operator N^γ to the both side of (2.6), we have then

$$(\log f)_\gamma = (\log((z-b)^{1/2} - (c + \sqrt{d})))_\gamma + (\log((z-b)^{1/2} - (c - \sqrt{d})))_\gamma \tag{2.7}$$

Now we have

$$\begin{aligned}
(\log((z-b)^{1/2} - (c + \sqrt{d})))_\gamma &= -e^{-i\pi\gamma} \frac{1}{2} (z-b)^{-\gamma} \Gamma(\gamma) \\
&\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k + 1)} \left(\frac{c + \sqrt{d}}{(z-b)^{1/2}} \right)^k \\
& \quad \left(|\Gamma(\frac{1}{2}k + \gamma)| < \infty \right) \tag{2.8}
\end{aligned}$$

and

$$\begin{aligned}
(\log((z-b)^{1/2} - (c - \sqrt{d})))_\gamma &= -e^{-i\pi\gamma} \frac{1}{2} (z-b)^{-\gamma} \Gamma(\gamma) \\
&\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k + 1)} \left(\frac{c - \sqrt{d}}{(z-b)^{1/2}} \right)^k \\
& \quad \left(|\Gamma(\frac{1}{2}k + \gamma)| < \infty \right) \tag{2.9}
\end{aligned}$$

Therefore, we obtain (2.4) from (2.7), (2.8), and (2.9) clearly.

Proof of(ii). Set $\gamma = n$ in (2.4).

§3. Some Identities

Theorem 3. We have the identities below;

$$(i) \quad \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{[2k]_m \Gamma(\frac{1}{2}m + k + \gamma)}{k \cdot m! \Gamma(\frac{1}{2}m + k)} S^m T^k \\ = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\frac{1}{2}k + 1)} \left\{ \left(\frac{c + \sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c - \sqrt{d}}{(z-b)^{1/2}} \right)^k - 2S^k \right\}, \quad (3.1)$$

$$(|\Gamma(\frac{1}{2}k + \gamma)|, |\Gamma(\frac{1}{2}m + k + \gamma)| < \infty)$$

and

$$(ii) \quad \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} \frac{[2k]_m [\frac{1}{2}m + k]_n}{k \cdot m!} S^m T^k \\ = 2 \sum_{k=0}^{\infty} [\frac{1}{2}k + 1]_{n-1} \left\{ \left(\frac{c + \sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c - \sqrt{d}}{(z-b)^{1/2}} \right)^k - 2S^k \right\} \quad (n \in \mathbb{Z}^+) \quad (3.2)$$

where

$$S = c/(z-b)^{1/2}, \quad T = d/(z-b), \quad |S| < 1, \quad |T| < 1, \quad |(c \pm \sqrt{d})/(z-b)^{1/2}| < 1.$$

Proof of (i). It is clear from Theorem 1 (i) and Theorem 2 (i).

Proof of (ii). Set $\gamma = n$ in (i).

Corollary 1. We have the following identities;

(i)

$$(z-b-d)^{-\gamma} = \frac{1}{2} (z-b)^{-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma) \Gamma(\frac{1}{2}k + 1)} \\ \times \left\{ \left(\frac{\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} \\ (|\Gamma(\frac{1}{2}k + \gamma)| < \infty) \quad (3.3)$$

and
(ii)

$$(z - b - d)^{-n} = \frac{1}{2}(z - b)^{-n} \sum_{k=0}^{\infty} \frac{[\frac{1}{2}k + 1]_{n-1}}{\Gamma(n)} \left\{ \left(\frac{\sqrt{d}}{(z - b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z - b)^{1/2}} \right)^k \right\}$$

where $\left| \frac{\pm\sqrt{d}}{(z - b)^{1/2}} \right| < 1, (n \in \mathbb{Z}^+)$. (3.4)

Proof of (i). Set $c = 0$ in (1).
Indeed we have

$$(\log(z - b - d))_{\gamma} = -e^{-i\pi\gamma} \Gamma(\gamma) (z - b - d)^{-\gamma} (|\Gamma(\gamma)| < \infty) \quad (3.5)$$

from (2.2), setting $c = 0$.
Next we have

$$\begin{aligned} (\log(z - b - d))_{\gamma} &= -e^{-i\pi\gamma} \frac{\Gamma(\gamma)}{2} (z - b)^{-\gamma} \\ &\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k + 1)} \left\{ \left(\frac{\sqrt{d}}{(z - b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z - b)^{1/2}} \right)^k \right\} \\ &\quad (|\Gamma(\frac{1}{2}k + \gamma)| < \infty) \end{aligned} \quad (3.6)$$

from (2.4), setting $c = 0$.
Therefore we have

$$\begin{aligned} (z - b - d)^{-\gamma} &= \frac{1}{2}(z - b)^{-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k + 1)} \\ &\times \left\{ \left(\frac{\sqrt{d}}{(z - b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z - b)^{1/2}} \right)^k \right\} \end{aligned} \quad (3.7)$$

from (3.5) and (3.6).

Proof of (ii). Set $\gamma = n$ in (3.3).

§4. Semi Derivatives

I We have

$$\begin{aligned}
 & (\log((\sqrt{z-b} - c)^2 - d))_{1/2} = i(z-b)^{-1/2} \\
 & \times \left[\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \frac{1}{2})}{\Gamma(\frac{1}{2}k + 1)} S^k + \sum_{k=1}^{\infty} \frac{1}{k} T^k \sum_{m=0}^{\infty} \frac{[2k]_m \Gamma(\frac{1}{2}m + k + \frac{1}{2})}{m! \Gamma(\frac{1}{2}m + k)} S^m \right]
 \end{aligned} \tag{4.1}$$

(S is the one shown in Theorem 1) (semi derivatives) from Theorem 1.(i), setting $\gamma = 1/2$.

II We have

$$\begin{aligned}
 & (\log((\sqrt{z-b} - c)^2 - d))_{1/2} = i \frac{\Gamma(\frac{1}{2})}{2} (z-b)^{-1/2} \\
 & \times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}k + 1)} \left\{ \left(\frac{c + \sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c - \sqrt{d}}{(z-b)^{1/2}} \right)^k \right\}
 \end{aligned} \tag{4.2}$$

from Theorem 2. (i).

III

$$(\log(\sqrt{z-b} - c))_{1/2} = i(z-b)^{-1/2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k + \frac{1}{2})}{\Gamma(\frac{1}{2}k + 1)} S^k. \tag{4.3}$$

(semi derivatives)

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