Geometric properties of certain analytic functions with real coefficients

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Abstract

Let $T$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$ and $a_k$ are real numbers. For a function $f(z) \in T$, some sufficient conditions for starlikeness and convexity are discussed.

1 Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

(1.1)

which are analytic in the open unit disk $U = \{z : |z| < 1\}$, and let $S$ be the subclass of $A$ of the univalent functions in $U$. By $S^*$ and $K$, we denote the subclasses of $A$ whose members map $U$ onto the domain which are starlike and convex.

Further, the function $f(z) \in A$ is said to be starlike of order $\alpha$ ($\alpha < 1$) in $U$ if and only if

$$\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha \quad (z \in U).$$

(1.2)

Similarly, $f(z) \in A$ is said to be convex of order $\alpha$ ($\alpha < 1$) in $U$ if and only if

$$1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \alpha \quad (z \in U).$$

(1.3)

We shall denote by $S^*(\alpha)$ and $K(\alpha)$ the subclasses of $A$ whose members satisfy (1.2) and (1.3), respectively.

It is known that for $0 \leq \alpha < 1$, $S^*(\alpha) \subset S^*$, $K(\alpha) \subset K$ and that $S^*(0) \equiv S^*$, $K(0) \equiv K$.

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Furthermore, we define $T$ the class of analytic functions with real coefficients, that is,

$$
T := \left\{ f(z) \in A : f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \ a_k \in \mathbb{R} \right\}, \tag{1.4}
$$

where $\mathbb{R}$ is the set of real numbers.

According to Silverman, we introduce $\mathcal{N}$ the class of analytic functions with negative coefficients, that is,

$$
\mathcal{N} := \left\{ f(z) \in A : f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \ a_k \geq 0 \right\}. \tag{1.5}
$$

We note that

$$
\mathcal{N} \subset T \subset A.
$$

Next, we define the Hadamard product or convolution by

$$
(f * g)(z) = f(z) * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k, \tag{1.6}
$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$.

With a view to introducing the Srivastava-Attiya convolution operator $J_{s,b}$, we begin by recalling a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined by

$$
\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{1.7}
$$

$(a \in \mathbb{C} \setminus \mathbb{Z}_{0}^{-}; s \in \mathbb{C}$ when $|z| < 1$; $\text{Re}(s) > 1$ when $|z| = 1$).

Srivastava and Attiya [3] introduced the linear operator

$$
J_{s,b}(f) : A \rightarrow A
$$

defined; in term of the Hadamard product (or convolution), by

$$
J_{s,b}(f)(z) := G_{s,b}(z) * f(z) \quad (z \in \mathcal{U}; \ b \in \mathbb{C} \setminus \mathbb{Z}_{0}^{-}; \ s \in \mathbb{C}), \tag{1.8}
$$

where for convenience,

$$
G_{s,b}(z) := (1+b)^s \left[ \Phi(z, s, b) - b^{-s} \right] \quad (z \in \mathcal{U}). \tag{1.9}
$$

It is easy to observe from (1.1) and the definition (1.7) and (1.8) that

$$
J_{s,b}(f)(z) = z + \sum_{k=2}^{\infty} \left( \frac{1+b}{k+b} \right)^s a_k z^k. \tag{1.10}
$$

For $f(z) \in A$, we define the class $S^{*}_{s,b}(\alpha)$ by

$$
f(z) \in S^{*}_{s,b}(\alpha) \iff \text{Re} \left( \frac{zJ'_{s,b}(f)(z)}{J_{s,b}(f)(z)} \right) > \alpha, \tag{1.11}
$$

that is, $J_{s,b}(f)(z)$ is in $S^{*}(\alpha) \ (z \in \mathcal{U}; \ 0 \leq \alpha < 1; \ b \in \mathbb{C} \setminus \mathbb{Z}_{0}^{-}; \ s \in \mathbb{C})$. 
Remark 1 For $f(z) \in A$, we put
\[ G(z) = \sum_{n=1}^{\infty} \frac{1+c}{n+c} z^n \]
is convex ($\text{Re}(c) > -1$). So we have
\[ \Phi_c(f(z)) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \]
\[ = (f * G)(z) \]
\[ = z + \sum_{n=2}^{\infty} \frac{1+c}{n+c} a_n z^n \]
\[ = J_{1,c}(f)(z). \]

2 Preliminaries

We introduce the following lemmas for our results.

Lemma 1 [4] Let $f(z) \in T$ and $\text{Re}\{f'(z)\} > 0$, then the function
\[ F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt \quad (c > -1) \]
belongs to $\mathcal{K}(-c)$ for all $c$ ($0 \leq -c < 1$).

Lemma 2 [1, Carathéodory] Let $\varphi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in $U$ and $\text{Re}\{\varphi(z)\} > 0$ ($z \in U$). Then,
\[ |c_n| \leq 2 \quad (n = 1, 2, 3, \ldots). \]

Lemma 3 [2] Let $f(z) \in T$ and suppose that
\[ \text{Re}\{f'(z) + \alpha z f''(z)\} > 0 \quad (z \in U) \]
where $\alpha \geq 1$. Then, we have
\[ 1 + \text{Re} \left\{ \frac{zf''(z)}{f'(z)} \right\} > \frac{\alpha - 1}{\alpha} \quad (z \in U), \]
or $f(z)$ is convex of order $\frac{\alpha - 1}{\alpha}$.
3 Main results

Theorem 1 Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in T \) and \( 0 \leqq \alpha < 1 \).

(i) If \( |zf''(z) + (1 - \alpha)(f'(z) - 1)| \leqq 1 - \alpha \), then \( f(z) \in \mathcal{K}(\alpha) \) \( (z \in U) \).

(ii) If \( |f'(z) + \alpha \left(1 - \frac{f(z)}{z}\right) - 1| \leqq 1 - \alpha \), then \( f(z) \in S^*(\alpha) \) \( (z \in U) \).

Proof. Using Lemma 3, we have (i) and (ii). \( \square \)

Remark 2. From Theorem 1, we have the following results given by H. Silverman [5].

(i) \( \sum_{n=2}^{\infty} n(n - \alpha)|a_n| \leqq 1 - \alpha \Rightarrow f(z) \in \mathcal{K}(\alpha) \).

(ii) \( \sum_{n=2}^{\infty} (n - \alpha)|a_n| \leqq 1 - \alpha \Rightarrow f(z) \in S^*(\alpha) \).

Next, we prove the following theorem.

Theorem 2 Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A \). If

\[ \Re \{(1 - \alpha)f'(z) + zf''(z)\} > 0 \quad (0 \leqq \alpha < 1), \]

then \( |a_n| \leqq \frac{2(1 - \alpha)}{n(n - \alpha)} \). The result is sharp.

Proof. The coefficient bounds are maximized at the extreme point. Now the extreme point of (3.1) may be expressed as

\[ f(z) = z + \sum_{n=2}^{\infty} \frac{2(1 - \alpha)x^{n-1}}{n(n - \alpha)} z^n, \quad |x| = 1 \]

and the result follows. \( \square \)

Remark 3 If \( f(z) \in T \) and \( \alpha = 0 \), then \( |a_n| \leqq \frac{2}{n^2} \). So, we have \( \sum_{n=2}^{\infty} |a_n| \leqq \frac{\pi^2 - 6}{3} = 1.289 \cdots \).

Moreover, in the case of \( f(z) \in T \), we have \( f(z) \in \mathcal{K}(\alpha) \).

Theorem 3 Let \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A \). If

\[ \Re \left\{f'(z) - \alpha \frac{f(z)}{z}\right\} > 0 \quad (0 \leqq \alpha < 1), \]

then \( |a_n| \leqq \frac{2(1 - \alpha)}{n - \alpha} \). The result is sharp.
Proof. The coefficient bounds are maximized at the extreme point. The extreme point of (3.3) is
\[
f(z) = z + \sum_{n=2}^{\infty} \frac{2(1-\alpha)x^{n-1}}{n-\alpha}z^n, \quad |x| = 1
\] (3.4)
and the result follows. \(\square\)

Remark 4 In the case of \(f(z) \in \mathcal{T}\), we have \(f(z) \in S^*(\alpha)\).

Next, in Theorem 4 below, we present the coefficient inequalities for functions in the class \(\mathcal{K}(\alpha)\).

Theorem 4 Let \(0 \leq \alpha < 1\). If \(f(z) \in \mathcal{A}\) satisfies the following inequality
\[
\sum_{n=2}^{\infty} n(n-\alpha) \left| \left(\frac{1+b}{n+b}\right)^{\alpha} \right| |a_n| \leq 1-\alpha,
\] (3.5)
then \(f(z) \in \mathcal{K}(\alpha)\).

Proof. Using Silverman’s result (Remark 2 (i)), we can prove this theorem. \(\square\)

Letting \(\alpha = 0\) in Theorem 4, we have

Corollary 1 If \(f(z) \in \mathcal{A}\) satisfies the following inequality
\[
\sum_{n=2}^{\infty} n^{2} \left| \left(\frac{1+b}{n+b}\right)^{\alpha} \right| |a_n| \leq 1,
\] (3.6)
then \(f(z)\) is convex.

Furthermore, we can have

Theorem 5 Let \(0 \leq \alpha < 1\). If \(f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{K}(\alpha)\), then
\[
|a_n| \leq \frac{2(1-\alpha)}{n(n-1)} \left| \left(\frac{n+b}{1+b}\right)^{\alpha} \right| \cdot \prod_{j=2}^{n-1} \left(1 + \frac{2(1-\alpha)}{j-1}\right) \quad (n \in \mathbb{N} \setminus \{1\}).
\] (3.7)

Proof. We set
\[
p(z) := \frac{1 + z \mathcal{J}_{a,b}^n(f)(z)}{\mathcal{J}_{a,b}^n(f)(z)} \frac{\mathcal{J}_{a,b}^n(f)(z)}{1-\alpha} = 1 + \sum_{n=2}^{\infty} c_n z^n.
\]
Then $p(z)$ is analytic with
\[ p(0) = 1 \quad \text{and} \quad \text{Re}\{p(z)\} > 0 \quad (z \in U). \]

Since
\[ z\mathcal{J}_{s,b}''(f)(z) = [(1 - \alpha)(p(z) - 1)] \mathcal{J}_{s,b}^{f}(f)(z), \]
by virtue of equation
\[ \mathcal{J}_{s,b}(f)(z) = z + \sum_{n=2}^{\infty} \left( \frac{1 + b}{n + b} \right)^{s} a_{n} z^{n}, \tag{3.8} \]
we have
\[ n(n - 1) \left( \frac{1 + b}{n + b} \right)^{s} a_{n} = (1 - \alpha) \left[ c_{n-1} + \sum_{m=2}^{n-1} m \left( \frac{1 + b}{m + b} \right)^{s} a_{m} c_{n-m} \right] \quad (n \in \mathbb{N} \setminus \{1\}). \tag{3.9} \]

By applying Lemma 2, we obtain
\[ n(n - 1) \left| \left( \frac{1 + b}{n + b} \right)^{s} \right| a_{n} \leq 2(1 - \alpha) \left[ 1 + \sum_{n=2}^{\infty} m \left( \frac{1 + b}{m + b} \right)^{s} \right] |a_{n}|. \tag{3.10} \]

We shall prove, by using the principle of mathematical induction, that the inequality (3.7) is satisfied for $n \in \mathbb{N} \setminus \{1\}$. \hfill \Box

Putting $\alpha = 0$ in Theorem 5, we have

**Corollary 2** If $f(z) = z + \sum_{n=2}^{\infty} a_{n} z^{n} \in \mathcal{K}$, then
\[ |a_{n}| \leq \left| \left( \frac{n + b}{1 + b} \right)^{s} \right|. \]

**References**


