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Failure of the uniqueness of eigen-distribution on random assignments for game trees

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1 Introduction

An AND-OR tree is a complete binary tree such that its root is an AND-gate, layers of AND-gates and those of OR-gates alternate and that leaves are assigned Boolean values. A tree of this type with k alternations is denoted by T_2^k. In this context, a probabilistic algorithm denotes a probability distribution on a set of deterministic algorithms. We restrict ourselves to deterministic algorithms of a certain type. We assume that a deterministic algorithm finds truth value of the root by making queries to leaves, and that a deterministic algorithm obeys alpha-beta pruning method, an efficient version of Min-Max method. In general, a deterministic algorithm of this type find the value of root without making queries to all the leaves. In this context, the cost of computation is the number of leaves queried during a computation. For a probabilistic algorithm, the cost is defined as the expected value of cost.

Then, the value

$$\min_{\mathcal{A}} \max_t \mathrm{cost}(\mathcal{A}, t),$$

where $\mathcal{A}$ runs over probabilistic algorithms and t runs over truth assignments, is called *randomized complexity* [1, chapter 12].

A probability distribution on the truth assignments is easier to handle than a probability distribution on the algorithms. Fortunately, Yao shows that the randomized complexity equals to *distributional complexity*, that is,

$$\max_d \min_A \mathrm{cost}(A, d),$$

where A runs over deterministic algorithms and d runs over probability distributions of truth assignments. This equality is a variation of Von-Neumann's Min-Max theorem, and is known as Yao's principle. A distribution d_0 on the truth assignments is called *eigen-distribution* if it achieves the distributional complexity, that is, if it satisfies the following.

$$\min_A \mathrm{cost}(A, d_0) = \max_d \min_A \mathrm{cost}(A, d),$$

C.-G. Liu and K. Tanaka [2] ("Eigen-distribution on random assignments for game trees", *Inform. Process. Lett.*, **104** pp.73-77 (2007)) define the concept of 1-set (0-set) as the set of all assignments such that the root has value 1 (0, respectively) and cost is forced to be high in a certain sense. They define E_1-distribution (E_0-distribution) as a distribution on the 1-set (0-set) such that all the deterministic algorithm has the same cost. They prove the following Assertion 1, 2.

Assertion 1 [2, Theorem 8] *For any tree T_2^k, we have, in the E^1-distribution (E^0-distribution), the probability of each assignment of* 1*-set (or* 0*-set) is equal to* $1/(4^{(4^k-1)/3})$.

Assertion 2 [2, Theorem 9] *For any tree T_2^k, the E^1-distribution is the unique eigen-distribution in the global distribution.*

In general, an algorithm can change its priority of searching leaves throughout a computation. For example, an algorithm can decide the next leaf in such a way that, if beta-cut happens at the current leaf then the next leaf is x, otherwise the next leaf is y. In this note, we consider the case where an algorithm does not change the priority of searching leaves throughout a computation. We show that, under this interpretation, (a counterpart of) Assertion 1 fails. In addition, we give remarks to Assertion 2. We show that, under the above interpretation, there are uncountably many E^1-distributions that are not the uniform distribution on the 1-set, and that there are uncountably many eigen-distributions.

2 Notation

The number of alternations of AND-layers and OR-layers in a given tree is called the *round.* A 1-round uniform binary AND-OR tree T_2^1 denotes a tree of the form in Figure 1. For an integer $k \geq 2$, a k-round tree T_2^k is defined by replacing four leaves of T_2^1 by T_2^{k-1} trees.

Definition 1 In the current note, we label each vertex of T_2^k by a bit string in the canonical way. A node corresponding to a bit string s is denoted by v_s. Figure 2 is an example.

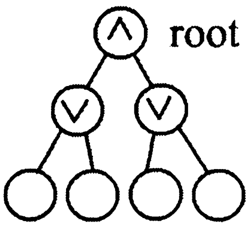

Figure 1: T_2^1

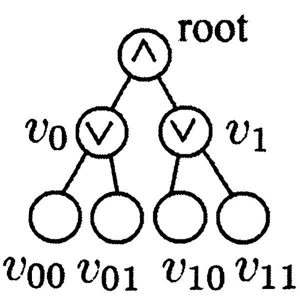

Figure 2: Labels for vertices

Suppose that k is a positive integer. $\mathcal{A}_D^k$ denotes the class of all deterministic alpha-beta pruning algorithms calculating the root-value of T_2^k.

Definition 2 By $\mathcal{A}_{\mathrm{FIX}}^k$, we denote the family of all $A \in \mathcal{A}_D^k$ such that A does not change the priority of searching leaves throughout a computation.

We do not write k of $\mathcal{A}_D^k$ or $\mathcal{A}_{\mathrm{FIX}}^k$ when k is clear from context.

$\mathcal{W}$ is the class of all assignments to leaves of T_2^k. For $A \in \mathcal{A}_D$ and $\omega \in \mathcal{W}$, by $C(A_D,\omega)$, we denote the number of leaves examined in the computation by A_D. By the phrase "the computational complexity of A with respect to ω", we denote (not time-complexity but) $C(A_D,\omega)$. If d is a probability distribution on $\mathcal{W}$ then $C(A_D, d)$ denotes the expected value of the complexity with respect to d.

Liu and Tanaka define i-set as follows, where i is 0 or 1. Suppose that k, the round of a tree, is given. We consider a truth assignment that make root $= i$. For each AND-gate whose value is defined as to be 0, we randomly choose a child-node and assign 1 to it, and assign 0 to the other child-node. For each OR-gate whose value is defined as to be 1, we randomly choose a child-node and assign 1 to it, and assign 0 to the other child-node. Then, the set of all such assignments is the i-set. The method of making such assignments is called reverse assignment technique of Liu and Tanaka.

Definition 3 Suppose that $\mathcal{A}$ is a subset of $\mathcal{A}_D^k$.

(1) A distribution d on the truth assignments is called *eigen-distribution with respect to* $\mathcal{A}$ if the following holds.

$$\min_{A_D \in \mathcal{A}} C(A_D, d) = \max_{d'} \min_{A_D \in \mathcal{A}} C(A_D, d'),$$

where d' runs over all distributions on the truth assignments.

(2) Let $i \in \{0,1\}$. A distribution d on the i-set is called E^i*-distribution with respect to* $\mathcal{A}$ if there exists a real number c such that for every $A_D \in \mathcal{A}$, the following holds.

$$C(A_D, d) = c.$$

(3) [2] If a distribution d on the truth assignments is eigen (respectively, E^0, E^1) if it is so with respect to $\mathcal{A}_D^k$.

3 The case where $k = 1$ and root $= 1$

Throughout sections 3–5, let $k = 1$. Table 1 shows the values of $C(A_D, \omega)$ in the case where ω is an element of the 1-set.

		A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
		1234	4312	3421	2143	3412	1243	2143	4321
ω_1	1010	2	3	3	4	2	3	3	4
ω_2	1001	3	2	4	3	3	2	4	3
ω_3	0110	3	4	2	3	3	4	2	3
ω_4	0101	4	3	3	2	4	3	3	2

Table 1: $C(A_D, \omega)$ for $k = 1$ ($\omega \in$ the 1-set).

In the table, each ω_i is a name of an assignment. A bit string $abcd$ denotes the assignment such that a, b, c, d, are assigned to $v_{00}, v_{01}, v_{10}, v_{11}$, respectively.

In the table, each A_j is a name of an element of $\mathcal{A}_{\mathrm{FIX}}$. In a permutation $xyzw$ of $\{1,2,3,4\}$, each of x, y, z, w denotes the search-priority of $v_{00}, v_{01}, v_{10}, v_{11}$, respectively. Since we restrict ourselves to alpha-beta pruning algorithms, only the eight permutations are considered.

Definition 4 Suppose that ε is a real number such that $0 \le \varepsilon \le 1/2$. By $d(\varepsilon)$, we denote the distribution on the 1-set such that the probabilities of $\omega_1, \omega_2, \omega_3, \omega_4$ are $\varepsilon, 1/2 - \varepsilon, 1/2 - \varepsilon, \varepsilon$, respectively.

Theorem 1 *There are uncountably many (the cardinality of the continuum) E^1-distributions with respect to $\mathcal{A}_{\mathrm{FIX}}^1$ that are not the uniform distribution on the 1-set. Hence, Assertion 1 fails (under this interpretation) with respect to $\mathcal{A}_{\mathrm{FIX}}^1$.*

Proof. Suppose that ε is a real number such that $0 \le \varepsilon \le 1/2$. Let $j \in \{1, 2, \cdots, 8\}$. By Table 1, it holds that

$$C(A_j, d(\varepsilon)) = \begin{cases} \varepsilon(2+4) + (1/2 - \varepsilon)(3+3) = 3, \text{ or} \\ \varepsilon(3+3) + (1/2 - \varepsilon)(2+4) = 3. \end{cases} \tag{3.1}$$

Therefore, for every ε such that $0 \leq \varepsilon \leq 1/2$, $d(\varepsilon)$ is a distribution on the 1-set such that for each $j \in \{1, 2, \cdots, 8\}$, the value $C(A_j, d)$ is the same. $d(\varepsilon)$ is not the uniform distribution on the 1-set unless $\varepsilon = 1/4$. □

Note that $d(1/4)$ is the E^1-distribution (for $k = 1$) discussed in [2].

4 Eigen-distribution for $k = 1$

Assertion 2 suggests that, in the case where $k = 1$, $d(1/4)$ is the unique eigen-distribution. In this section, we show that $d(1/4)$ is not the unique eigen-distribution under the interpretation that an algorithm does not change the priority of searching leaves throughout a computation.

The following Lemma 2 is implicitly shown in [2, p.76]. We explicitly prove it.

Lemma 2 *Let $k = 1$. Suppose that d is a distribution on $\mathcal{W}$.*
(1) If $\min_{1 \leq j \leq 8} C(A_j, d) \geq 3$ then d is a distribution on the 1-set.
(2) If d is a distribution on the 1-set then $\min_{1 \leq j \leq 8} C(A_j, d) \leq 3$.
(3) "d is eigen with respect to $\mathcal{A}^1_D$ " if and only if "d is a distribution on the 1-set and $\min_{1 \leq j \leq 8} C(A_j, d) = 3$".

Proof. For each i such that $1 \leq i \leq 16$, let p_i denote $\mathrm{Prob}[d = \omega_i]$.

(1) Tables 2, 3 show the values of $C(A_D, \omega)$ in the case where ω is not an element of the 1-set.

		A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
		1234	4312	3421	2143	3412	1243	2143	4321
ω_5	1000	3	2	2	4	2	3	4	2
ω_6	0100	4	2	2	3	2	4	3	2
ω_7	0010	2	4	3	2	3	2	2	4
ω_8	0001	2	3	4	2	4	2	2	3

Table 2: $C(A_D, \omega)$ for $k = 1$ ($\omega \in$ the 0-set).

		A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
		1234	4312	3421	2143	3412	1243	2143	4321
ω_9	1110	2	2	3	3	2	3	2	3
ω_{10}	1101	3	3	2	2	3	2	3	2
ω_{11}	1011	2	3	2	3	2	2	3	3
ω_{12}	0111	3	2	3	2	3	3	2	2
ω_{13}	1100	3	2	2	3	2	3	3	2
ω_{14}	0011	2	3	3	2	3	2	2	3
ω_{15}	1111	2	2	2	2	2	2	2	2
ω_{16}	0000	2	2	2	2	2	2	2	2

Table 3: $C(A_D, \omega)$ for $k = 1$ ($\omega \notin$ the 1-set $\cup$ the 0-set).

Suppose that $\min_{1 \leq j \leq 8} C(A_j, d) \geq 3$. Since $C(A_1, d) \geq 3$, by Tables 2 and 3, we get the following..

$$p_1 + p_7 + p_8 + p_9 + p_{11} + p_{14} + p_{15} + p_{16} \leq p_4 + p_6. \tag{4.1}$$

For each $j \in \{2, \cdots, 8\}$, we get similar inequalities. By using these inequalities, it is shown that $5 \leq \forall i \leq 16$ $p_i = 0$. Hence, d is a distribution on the 1-set.

(2) Suppose that d is a distribution on the 1-set. Assume for a contradiction that we have $\min_{1\le j\le 8} C(A_j,d) > 3$. Since $C(A_1,d) = 2p_1 + 3p_2 + 3p_3 + 4p_4 > 3$, we have $p_1 < p_4$. Since $C(A_4,d) = 4p_1 + 3p_2 + 3p_3 + 2p_4 > 3$ we have $p_1 > p_4$, a contradiction.

(3) Note that $\min_{1\le j\le 8} C(A_j, d(1/4)) = 3$. By this fact and the above (1) and (2), the equivalence holds. □

The following is a remark to Assertion 2.

Corollary 3 *(to Theorem 1 and Lemma 2) There are uncountably many (the cardinality of the continuum) eigen-distributions with respect to $\mathcal{A}^1_{\mathrm{FIX}}$.*

Proof. Suppose that ε is a real number such that $0 \le \varepsilon \le 1/2$. Then, for each $j \in \{1,2,\cdots,8\}$, it holds that $C(A_j, d(\varepsilon)) = 3$, and hence $d(\varepsilon)$ is an eigen-distribution. □

5 The case where $k = 1$ and root $= 0$

In this section, we discuss a special case where the statement of Assertion 1 holds.

Proposition 4 *Let $k = 1$. Suppose that d is an E^0-distribution with respect to $\mathcal{A}^1_{\mathrm{FIX}}$. Then, d is the uniform distribution on the 0-set. Hence, d is the E^0-distribution of [2].*

Proof. For each $i \in \{5,6,7,8\}$, let $p_i = \mathrm{Prob}[d = \omega_i]$. Then we have $2p_5 + 2p_6 + 2p_7 + 2p_8 = 1$. Hence, we get the following.

$$C(A_j,d) = \sum_{r=0}^{1} C_{\mathrm{leaf}=r}(A_j,d) = \begin{cases} 2+p_5+2p_6 & \text{If } j \in \{1,6\}, \\ 2+2p_7+p_8 & \text{If } j \in \{2,8\}, \\ 2+p_7+2p_8 & \text{If } j \in \{3,5\}, \\ 2+2p_5+p_6 & \text{If } j \in \{4,7\}. \end{cases} \tag{5.1}$$

Since d is an E^0-distribution, the right-hand side of (5.1) does not depend on i. Thus it holds that $p_5 = p_6 = p_7 = p_8 = 1/4$. □

Now, the following assertion in [2] is justified.

Proposition 5 *[2, p.76] Let $k = 1$. Let E^0 denote the (unique) E^0-distribution for $k = 1$. If a distribution d on $\mathcal{W}$ is such that* Prob[root $= 0$] $= 1$ *and that d is not E^0, then it holds that*

$$\min_{A_D \in \mathcal{A}_D} C(A_D, d) < \min_{A_D \in \mathcal{A}_D} C(A_D, E^0) = 2.75. \tag{5.2}$$

Proof. (sketch) The 0 value of root is achieved in the case where $i \in$ (the 0-set) $\cup \{13,14,16\}$. By using Tables 2 and 3, we can show the proposition. □

6 Eigen-distribution for $k \ge 2$

Corollary 3 holds without assumption of $k = 1$.

Theorem 6 *Consider the case where an algorithm does not change the priority of searching leaves throughout a computation. Under this interpretation, for every positive integer k, there are uncountably many (the cardinality of the continuum) eigen-distributions.*

Proof. (sketch) By means of Lemma 2 and Proposition 5, we can show the following (*).

(*) Suppose n is a positive integer. Then, for every E^1-distribution d_1 for $k = 1$, there is an eigen-distribution d_n for $k = n$ such that the first round (the subtree consisting of the root, its child nodes and their child nodes) of d_n is d_1.

By Corollary 3, the current theorem holds. □

7 Appendix

Suppose that $A \in \mathcal{A}_D$ and $\omega \in \mathcal{W}$. For each $r \in \{0, 1\}$, let $C_{\mathsf{leaf}=r}(A_D, \omega)$ denote the number of leaves v satisfying the following two conditions: (1) v is examined in the computation by A_D; (2) the value of v under ω is r. If d is a distribution on $\mathcal{W}$ then $C_{\mathsf{leaf}=r}(A_D, d)$ is defined in a natural way.

Table 4 is a list of $C_{\mathsf{leaf}=1}(A_D, \omega)$.

i	$j \in \{1,4,6,7\}$	$j \in \{2,3,5,8\}$	root value
$i \in \{5,6,13\}$	1	0	0
$i \in \{7,8,14\}$	0	1	0
$i = 16$	0	0	0
otherwise	2	2	1

Table 4: $C_{\mathsf{leaf}=1}(A_D, \omega)$ for $k = 1$.

References

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