数理解析研究所講究録
第 1729 巻 2011 年 123-145

Normal forms in two normal modal logics *

Katsumi Sasaki (佐々木克巳　南山大学情報理工学部)
sasaki@nanzan-u.ac.jp
Faculty of Information Sciences and Engineering,
Nanzan University

Abstract. Here, we discuss normal modal logics containing the normal modal logic **K4**. To do so, we use normal forms introduced in [Sas10a] and [Sas10b]. For two normal modal logics L_0 and L satisfying $\mathbf{K4} \subseteq L_0 \subseteq L$, we show relation between normal forms in L and normal forms in L_0.

1 Introduction

In the present section, we introduce formulas, sequents, normal modal logics, and normal forms. We use sequents to treat normal modal logics and normal forms.

1.1 Formulas

Formulas are constructed from $\bot$ (contradiction) and the propositional variables $p_1, p_2, \cdots$ by using logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\supset$ (implication), and $\Box$ (necessitation). We use upper case Latin letters, $A, B, C, \cdots$, with or without subscripts, for formulas. Also, we use Greek letters, $\Gamma, \Delta, \cdots$, with or without subscripts, for finite sets of formulas. The expression $\Box\Gamma$ denotes the sets $\{\Box A \mid A \in \Gamma\}$. The *depth* $d(A)$ of a formula A is defined as

$d(p_i) = d(\bot) = 0$,
$d(B \wedge C) = d(B \vee C) = d(B \supset C) = \max\{d(B), d(C)\}$,
$d(\Box B) = d(B) + 1$.

The set of propositional variables $p_1, \cdots, p_m$ $(m \geq 1)$ is denoted by $\mathbf{V}$ and the set of formulas constructed from $\mathbf{V}$ and $\bot$ is denoted by $\mathbf{F}$. Also, for any $n = 0, 1, \cdots$, we define $\mathbf{F}(n)$ as $\mathbf{F}(n) = \{A \in \mathbf{F} \mid d(A) \leq n\}$. In the present paper, we treat the set $\mathbf{F}(n)$.

1.2 Sequents

A *sequent* is the expression $(\Gamma \to \Delta)$. We often refer to $\Gamma \to \Delta$ as $(\Gamma \to \Delta)$ and refer to

$$A_1, \cdots, A_i, \Gamma_1, \cdots, \Gamma_j \to \Delta_1, \cdots, \Delta_k, B_1, \cdots, B_\ell$$

*The work was supported by Nanzan University Pache Research Subsidy I-A-2 for Academic Year 2010.

as

$$\{A_1, \cdots, A_i\} \cup \Gamma_1 \cup \cdots \cup \Gamma_j \to \Delta_1 \cup \cdots \cup \Delta_k \cup \{B_1, \cdots, B_\ell\}.$$

We use upper case Latin letters $X, Y, Z, \cdots$, with or without subscripts, for sequents. The *antecedent* $\mathbf{ant}(\Gamma \to \Delta)$ and the *succedent* $\mathbf{suc}(\Gamma \to \Delta)$ of a sequent $\Gamma \to \Delta$ are defined as

$$\mathbf{ant}(\Gamma \to \Delta) = \Gamma \quad \text{and} \quad \mathbf{suc}(\Gamma \to \Delta) = \Delta,$$

respectively. Also, for a sequent X and a set $\mathcal{S}$ of sequents, we define $\mathbf{for}(X)$ and $\mathbf{for}(\mathcal{S})$ as

$$\mathbf{for}(X) = \begin{cases} \bigwedge \mathbf{ant}(X) \supset \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) \neq \emptyset \\ \bigvee \mathbf{suc}(X) & \text{if } \mathbf{ant}(X) = \emptyset \end{cases}$$

and

$$\mathbf{for}(\mathcal{S}) = \{\mathbf{for}(X) \mid X \in \mathcal{S}\}.$$

1.3 Normal modal logics

A normal modal logic is a set of formulas containing all tautologies and the axiom

$K : \Box(p \supset q) \supset (\Box p \supset \Box q)$

and closed under modus ponens, substitution, and necessitation ($A/\Box A$). By **K4**, we mean the smallest normal modal logic containing the axiom

$4 : \Box p \supset \Box\Box p$.

For a normal modal logic L, we use $A \equiv_L B$ instead of $(A \supset B) \wedge (B \supset A) \in L$.

In order to treat normal modal logics, we use sequent systems obtained by adding axioms or inference rules to the sequent system **LK** given by Gentzen [Gen35]. Here, we do not use $\neg$ as a primary connective, so we use the additional axiom $\bot \to$ instead of the inference rules $(\neg \to)$ and $(\to \neg)$. For a sequent system L and for a sequent X, we write $X \in L$ if X is provable in L. It is known that a sequent system for **K4** is obtained by adding the inference rule

$$\frac{\Gamma, \Box\Gamma \to A}{\Box\Gamma \to \Box A}(\Box)$$

to **LK**. In other words,

$$X \text{ is provable in the above system if and only if } \mathbf{for}(X) \in \mathbf{K4}.$$

Therefore, we can identify the above system with **K4** and call the above system **K4**.

Also, for any sequent system L, the following two conditions are equivalent:

- L is a sequent system for a normal modal logic containing **K4**,
- L satisfies the inference rule $(\Box)$;

and thus, we treat a sequent system satisfying the second conditions above as a normal modal logic containing **K4**. Also, for a set $\mathcal{S}$ of sequents and for a normal modal logic (or a sequent system) L, we refer to $\mathcal{S} - L$ as $\{X \in \mathcal{S} \mid X \notin L\}$ and refer to $\mathcal{S} \cap L$ as $\{X \in \mathcal{S} \mid X \in L\}$.

1.4 Normal forms

Let L be a normal modal logic containing **K4**. Let **ED** be a finite set of sequents satisfying the following two conditions:

(I) $\mathbf{F}(n)/\equiv_L \quad = \quad \{[\bigwedge \mathbf{for}(\mathcal{S})] \mid \mathcal{S} \subseteq \mathbf{ED}\}$,

(II) for any subsets $\mathcal{S}_1$ and $\mathcal{S}_2$ of **ED**, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\bigwedge \mathbf{for}(\mathcal{S}_2) \to \bigwedge \mathbf{for}(\mathcal{S}_1) \in L$.

Then we call an element of **ED** a normal form for $\mathbf{F}(n)$ in L.

If $n = 0$, then the set $\mathbf{ED}_0 = \{p_1^* \vee \cdots \vee p_m^* \mid p_i \in \{p_i, \neg p_i\}\}$ satisfies the above two conditions. An element of this $\mathbf{ED}_0$ is called an elementary disjunction. In this sense, each elementary disjunction is a normal form for $\mathbf{F}(0)$ in L.

Below, we define the set $\mathbf{ED}_L(n)$, which was proved to satisfy the above two conditions (I) and (II) in [Sas10b].

Definition 1.1 Let L be a normal modal logic containing **K4**. The sets $\mathbf{G}_L(n)$ and $\mathbf{G}_L^*(n)$ of sequents are defined inductively as follows.

$\mathbf{G}_L(0) = \{(\mathbf{V} - V_1 \to V_1) \mid V_1 \subseteq \mathbf{V}\}$,

$\mathbf{G}_L^*(0) = \emptyset$,

$\mathbf{G}_L(k+1) = \bigcup_{X \in \mathbf{G}(k) - \mathbf{G}^*(k)} \mathbf{next}_L(X)$,

$\mathbf{G}_L^*(k+1) = \{X \in \mathbf{G}_L(k+1) \mid \mathbf{Ant}(X) \subseteq \mathbf{Ant}(Y)$ implies $\mathbf{Ant}(X) = \mathbf{Ant}(Y)$, for any $Y \in \mathbf{G}_L(k+1)\}$,

where for any $X \in \mathbf{G}_L(k)$,

$\mathbf{next}_L^+(X) = \{(\Box\Gamma, \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\Delta) \mid \Gamma \cup \Delta = \mathbf{for}(\mathbf{G}_L(k)), \Gamma \cap \Delta = \emptyset\}$,

$\mathbf{next}_L(X) = \mathbf{next}_L^+(X) - L$,

$$\mathbf{Ant}(X) = \{Y \in \bigcup_{i=0}^{k-1} \mathbf{G}_L(i) \mid \Box\mathbf{for}(Y) \in \mathbf{ant}(X)\},$$

$$\mathbf{Suc}(X) = \{Y \in \bigcup_{i=0}^{k-1} \mathbf{G}_L(i) \mid \Box\mathbf{for}(Y) \in \mathbf{suc}(X)\}.$$

Definition 1.2 We define the sets $\mathbf{ED}_L(n)$, $\mathbf{G}_L^+(n)$, $\mathbf{G}_L^\bullet(n)$, and $\mathbf{G}_L^\circ(n)$, as

$$\mathbf{ED}_L(n) = \mathbf{G}_L(n) \cup \bigcup_{i=0}^{n-1} \mathbf{G}_L^*(i),$$

$$\mathbf{G}_L^+(n) = \begin{cases} \mathbf{G}_L(0) & \text{if } n = 0 \\ \bigcup_{X \in \mathbf{G}_L(n-1) - \mathbf{G}_L^*(n-1)} \mathbf{next}_L^+(X) & \text{if } n > 0, \end{cases}$$

$$\mathbf{G}_L^\bullet(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{Y \in \mathbf{G}_L^* \mid Y_\ominus \in \mathbf{Ant}(Y)\} & \text{if } n > 0, \end{cases}$$

$$\mathbf{G}_L^\circ(n) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{Y \in \mathbf{G}_L^* \mid Y_\ominus \in \mathbf{Suc}(Y)\} & \text{if } n > 0. \end{cases}$$

Let X be a sequent in $\mathbf{G}_L^+(n+1)$. Then there exists only one sequent $Y \in \mathbf{G}_L(n) - \mathbf{G}_L^*(n)$ such that $X \in \mathbf{next}_L^+(Y)$. We refer to $X_\ominus$ as this sequent Y. We note that $X_\ominus \in \mathbf{G}_L(n) - \mathbf{G}_L^*(n)$ and $X \in \mathbf{next}_L^+(X_\ominus)$.

For $X \in \mathbf{G}_L(0)$, we also refer to $X_\ominus$ as X.

By an induction on n, we can easily observe that for any $X \in \mathbf{G}_L^+(n)$,

- $\mathbf{Ant}(X) \cup \mathbf{Suc}(X) = \bigcup_{i=0}^{n-1} \mathbf{G}_L(i)$,
- $\mathbf{ant}(X) \cup \mathbf{suc}(X) = \Box\mathbf{for}(\bigcup_{i=0}^{n-1} \mathbf{G}_L(i)) \cup \mathbf{V} = \Box\mathbf{for}(\mathbf{Ant}(X) \cup \mathbf{Suc}(X)) \cup \mathbf{V}$,
- $\mathbf{Ant}(X) \cap \mathbf{Suc}(X) \neq \emptyset$,
- $\mathbf{ant}(X) \cap \mathbf{suc}(X) \neq \emptyset$.

Definition 1.3 We define the sets $\mathbf{G}_L$, $\mathbf{G}_L^*$, and $\mathbf{G}_L^+$ as

$$\mathbf{G}_L = \bigcup_{i=0}^{\infty} \mathbf{G}_L(i), \quad \mathbf{G}_L^* = \bigcup_{i=0}^{\infty} \mathbf{G}_L^*(i), \quad \text{and} \quad \mathbf{G}_L^+ = \bigcup_{i=0}^{\infty} \mathbf{G}_L^+(i).$$

The following lemma was shown in [Sas10b].

Lemma 1.4 Let X and Y be sequents in $\mathbf{G}_L(n)$. Then
(1) $\mathbf{Ant}(X) \not\subseteq \mathbf{Ant}(Y)$ implies $(\rightarrow \mathbf{for}(X), \Box\mathbf{for}(Y)) \in L$,
(2) $\mathbf{Ant}(X) = \mathbf{Ant}(Y)$ and $Y \in \mathbf{G}_L^\circ(n)$ imply $\Box\mathbf{for}(Y) \rightarrow \mathbf{for}(X) \in L$.

2 Main result

Let L_0 and L be normal modal logics satisfying $\mathbf{K4} \subseteq L_0 \subseteq L$. In the present section, we show relation between $\mathbf{G}_{L_0}^+$ and $\mathbf{G}_L^+$. To do so, we define a mapping from $\mathbf{G}_{L_0}^+$ to $\mathbf{G}_L^+$ and a mapping from $\mathbf{G}_L^+$ to $\mathbf{G}_{L_0}^+$; and investigate their properties.

Definition 2.1
(1) For $X \in \mathbf{G}_{L_0}^+(n)$, we define the sets $\mathbf{pclus}(X)$, $\mathbf{Ant}(X,k)$, $\mathbf{Suc}(X,k)$, and $\mathbf{Suc}(X,L)$; and the sequent $X(k)$ as follows:

- $\mathbf{pclus}(X) = \{Z \in \mathbf{G}_{L_0}(n) \mid \mathbf{Ant}(X) = \mathbf{Ant}(Z)\}$,
- $\mathbf{Ant}(X,k) = \mathbf{Ant}(X) \cap \mathbf{G}_{L_0}(k)$,
- $\mathbf{Suc}(X,k) = \mathbf{Suc}(X) \cap \mathbf{G}_{L_0}(k)$,
- $\mathbf{Suc}(X,L) = \begin{cases} \emptyset & \text{if } n = 0 \\ \{Z \in \mathbf{Suc}(X, n-1) \cup \{X_\ominus\} \mid Z \in L\} & \text{if } n > 0, \end{cases}$
- $X(k) = (\mathbf{ant}(X) - \Gamma \rightarrow \mathbf{suc}(X) - \Gamma)$,

where $\Gamma = \{\Box\mathbf{for}(X) \mid X \in \bigcup_{i=k}^{\infty} \mathbf{G}_{L_0}(i)\}$.

(2) For $X \in \mathbf{G}^+_{L_0}(n)$, we define the sequent $f_{L_0\to L}(X)$ as

$$f_{L_0\to L}(X) = \begin{cases} X & \text{if } n = 0 \\ \bot \to \bot & \text{if } n > 0 \text{ and } \mathcal{S} \neq \emptyset \\ X_L & \text{if } n > 0, \mathcal{S} = \emptyset, \text{ and } f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L \\ f_{L_0\to L}(X_\ominus) & \text{if } n > 0, \mathcal{S} = \emptyset, f_{L_0\to L}(X_\ominus) \in \mathbf{G}^*_L, \text{ and } X \notin \mathbf{K4} \\ \bot \to \bot & \text{otherwise,} \end{cases}$$

where

$\mathcal{S} = \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L)$,

$\mathbf{as}(X, L) = \{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L, Z_\ominus \in \mathbf{Suc}(X)\} \cup \{Z \in \mathbf{Suc}(X) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L, Z_\ominus \in \mathbf{Ant}(X)\}$,

$X_L = (\Box\mathbf{for}(\mathcal{S}_{0,a}), \mathbf{ant}(f_{L_0\to L}(X_\ominus)) \to \mathbf{suc}(f_{L_0\to L}(X_\ominus)), \Box\mathbf{for}(\mathcal{S}_{0,s}))$,

$\mathcal{S}_{0,a} = \{f_{L_0\to L}(Z) \mid Z \in \mathbf{Ant}(X, n-1), Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\}$,

$\mathcal{S}_{0,s} = \{f_{L_0\to L}(Z) \mid Z \in \mathbf{Suc}(X, n-1), f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\}$.

(3) For $Y \in \mathbf{G}^+_L(n)$, we define the sequent $f_{L\to L_0}(Y)$ as

$$f_{L\to L_0}(Y) = \begin{cases} Y & \text{if } n = 0 \\ (\Box\mathbf{for}(\mathcal{S}_a), \mathbf{ant}(f_{L\to L_0}(Y_\ominus)) \to \mathbf{suc}(f_{L\to L_0}(Y_\ominus)), \Box\mathbf{for}(\mathcal{S}_s)) & \text{if } n > 0, \end{cases}$$

where

$$\mathcal{S}_a = \begin{cases} \mathcal{S}_{1,a} & \text{if } n = 1 \\ \mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a} & \text{if } n > 1, \end{cases}$$

$$\mathcal{S}_s = \begin{cases} \mathcal{S}_{1,s} & \text{if } n = 1 \\ \mathcal{S}_{1,s} \cup \mathcal{S}_{2,s} & \text{if } n > 1, \end{cases}$$

$\mathcal{S}_{1,a} = \{f_{L\to L_0}(Y') \mid Y' \in \mathbf{Ant}(Y, n-1)\}$,

$\mathcal{S}_{1,s} = \{f_{L\to L_0}(Y') \mid Y' \in \mathbf{Suc}(Y, n-1)\}$,

$\mathcal{S}_{2,a} = \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{Ant}(Y_\ominus) \cap \mathbf{G}^*_L\}$,

$\mathcal{S}_{2,s} = \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{Suc}(Y_\ominus) \cap \mathbf{G}^*_L\}$,

$\mathcal{S}_{3,a} = \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \in L\}$.

(4) For a finite subset $\mathcal{S}_1$ of $\mathbf{G}_{L_0}$ and for a finite subset $\mathcal{S}_2$ of $\mathbf{G}_L$, we define $f_{L_0\to L}(\mathcal{S}_1)$ and $f_{L\to L_0}(\mathcal{S}_2)$ as

$$f_{L_0\to L}(\mathcal{S}_1) = \{f_{L_0\to L}(X) \mid X \in \mathcal{S}_1\},$$

$$f_{L\to L_0}(\mathcal{S}_2) = \{f_{L\to L_0}(X) \mid X \in \mathcal{S}_2\}.$$

(5) We define the set $\mathbf{G}^L_{L_0}(n)$ as

$$\mathbf{G}^L_{L_0}(n) = \{X \in \mathbf{G}^+_{L_0}(n) \mid \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L\}.$$

We note that for $X \in \mathbf{G}^+_{L_0}(n)$ $(n > 0)$,

$$f_{L_0\to L}(X) = X_L \text{ if and only if } X \in \mathbf{G}^L_{L_0}(n),$$

where X_L is as in the above definition.

Lemma 2.2.

(1) For any $X \in \mathbf{G}_{L_0}(0)$, $f_{L_0 \to L}(X) = X$ and $f_{L \to L_0}(f_{L_0 \to L}(X)) = X$.

(2) For any $Y \in \mathbf{G}_L(0)$, $f_{L \to L_0}(Y) = Y$ and $f_{L_0 \to L}(f_{L \to L_0}(Y)) = Y$.

(3) $\mathbf{G}_L(0) = \mathbf{G}_{L_0}(0) = \{f_{L_0 \to L}(X) \mid X \in \mathbf{G}_{L_0}(0)\} = \{f_{L \to L_0}(Y) \mid Y \in \mathbf{G}_L(0)\}$.

(4) For any $X \in \mathbf{G}^+_{L_0}(1)$, $f_{L_0 \to L}(X) = X$ and $f_{L \to L_0}(f_{L_0 \to L}(X)) = X$.

(5) For any $Y \in \mathbf{G}^+_L(1)$, $f_{L \to L_0}(Y) = Y$ and $f_{L_0 \to L}(f_{L \to L_0}(Y)) = Y$.

(6) $\mathbf{G}^+_L(1) = \mathbf{G}^+_{L_0}(1) = \{f_{L_0 \to L}(X) \mid X \in \mathbf{G}^+_{L_0}(1)\} = \{f_{L \to L_0}(Y) \mid Y \in \mathbf{G}^+_L(1)\}$.

Lemma 2.3. Let X be a sequent in $\mathbf{G}^*_L(n)$ and let $X_\oplus$ be a sequent

$$X_\oplus = (\Box\mathbf{for}(\mathcal{S}_1), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\mathcal{S}_2)),$$

where $\mathcal{S}_1$ and $\mathcal{S}_2$ are subsets of $\mathbf{G}_L$. Then either $X_\oplus \in \mathbf{K4}$ or $\mathbf{for}(X) \equiv_{\mathbf{K4}} \mathbf{for}(X_\oplus)$.

Proof. If $\Box\mathbf{for}(\mathcal{S}_1) \cap \mathbf{suc}(X) \neq \emptyset$, then clearly, $X_\oplus \in \mathbf{K4}$. Also, we have

$$\mathbf{for}(X_\oplus) \equiv_{\mathbf{K4}} (\Box\mathbf{for}(\mathcal{S}_1) - \mathbf{ant}(X), \mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(\mathcal{S}_2)).$$

Therefore, we can assume that

$$\mathcal{S}_1 \subseteq \bigcup_{i=n}^{\infty} \mathbf{G}_L(i). \tag{1}$$

Similarly, we can assume that

$$\mathcal{S}_2 \subseteq \bigcup_{i=n}^{\infty} \mathbf{G}_L(i). \tag{2}$$

By $X \in \mathbf{G}^*_L(n)$, we have

$$\mathbf{G}_L(n) = \{Y \in \mathbf{G}_L(n) \mid \mathbf{Ant}(X) \not\subseteq \mathbf{Ant}(Y)\} \cup \mathbf{pclus}(X).$$

Also, by $X \in \mathbf{G}^*_L(n)$, we have $\mathbf{pclus}(X) \subseteq \mathbf{G}^*_L(n)$, and thus,

$$\bigcup_{i=n}^{\infty} \mathbf{G}_L(i) = \{Y \in \bigcup_{i=n}^{\infty} \mathbf{G}_L(i) \mid \mathbf{Ant}(X) \not\subseteq \mathbf{Ant}(Y(n))\} \cup \mathbf{pclus}(X).$$

For brevity, we define $\mathcal{S}_3$ as

$$\mathcal{S}_3 = \{Y \in \bigcup_{i=n}^{\infty} \mathbf{G}_L(i) \mid \mathbf{Ant}(X) \not\subseteq \mathbf{Ant}(Y(n))\}.$$

Then

$$\bigcup_{i=n}^{\infty} \mathbf{G}_L(i) = \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n)) \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)),$$

Using (1) and (2), we have

$$\mathcal{S}_1 \subseteq \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n)) \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)) \tag{3}$$

and

$$\mathcal{S}_2 \subseteq \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n)) \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)). \tag{4}$$

On the other hand, by Lemma 1.4(2),

$$Y \in \mathbf{pclus}(X) \cap \mathbf{G}^\circ(n) \text{ implies } (\Box\mathbf{for}(Y), \mathbf{ant}(X) \to \mathbf{suc}(X)) \in \mathbf{K4}. \tag{5}$$

By Lemma 1.4(1), and $\Box\mathbf{for}(Y(n)) \to \Box\mathbf{for}(Y) \in \mathbf{K4}$, we have

$$Y \in \mathcal{S}_3 \text{ implies } (\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y)) \in \mathbf{K4}. \tag{6}$$

Also, if $Y \in \mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n)$, then $\Box\mathbf{for}(Y_\ominus) \in \mathbf{Ant}(Y) = \mathbf{Ant}(X)$. Therefore,

$$Y \in \mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n) \text{ implies } (\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y)) \in \mathbf{K4}. \tag{7}$$

We divide the cases.

The case that $\mathcal{S}_1 \not\subseteq \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n))$. There exists a sequent

$$Y \in \mathcal{S}_1 - (\mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n))).$$

Using (3), we have

$$Y \in \mathcal{S}_1 \cap \mathbf{pclus}(X) \cap \mathbf{G}^\circ(n).$$

By $Y \in \mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)$ and (5), we have $(\Box\mathbf{for}(Y), \mathbf{ant}(X) \to \mathbf{suc}(X)) \in \mathbf{K4}$, and using $Y \in \mathcal{S}_1$, we obtain $X_\oplus \in \mathbf{K4}$.

The case that $\mathcal{S}_2 \not\subseteq \mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)$. There exists a sequent

$$Y \in \mathcal{S}_2 - (\mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)).$$

Using (4), we have

$$Y \in \mathcal{S}_2 \text{ and } Y \in \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n)).$$

By $Y \in \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n))$, (6), and (7), we have $(\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y)) \in \mathbf{K4}$, and using $Y \in \mathcal{S}_2$, we obtain $X_\oplus \in \mathbf{K4}$.

The case that $\mathcal{S}_1 \subseteq \mathcal{S}_3 \cup (\mathbf{pclus}(X) \cap \mathbf{G}^\bullet(n))$ and $\mathcal{S}_2 \subseteq \mathbf{pclus}(X) \cap \mathbf{G}^\circ(n)$. By (5), we have

$$Y \in \mathcal{S}_2 \text{ implies } (\Box\mathbf{for}(Y), \mathbf{ant}(X) \to \mathbf{suc}(X)) \in \mathbf{K4}.$$

By (6) and (7), we have

$$Y \in \mathcal{S}_1 \text{ implies } (\mathbf{ant}(X) \to \mathbf{suc}(X), \Box\mathbf{for}(Y)) \in \mathbf{K4}.$$

Therefore,

$$\mathbf{for}(X_\oplus) \to \mathbf{for}(X) \in \mathbf{K4},$$

and hence,

$$\mathbf{for}(X_\oplus) \equiv_{\mathbf{K4}} \mathbf{for}(X).$$

Lemma 2.4. Let X be a sequent in $\mathbf{G}^{+}_{L_0}(n)$ $(n > 0)$. Then

$$X \in \mathbf{G}^{L}_{L_0}(n) \text{ implies } X_\ominus \in \mathbf{G}^{L}_{L_0}(n-1).$$

Proof. By $X \in \mathbf{G}^{L}_{L_0}(n)$, we have

$$\mathbf{as}(X, L) = \emptyset,\ \mathbf{Suc}(X, L) = \emptyset, \text{ and } f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^{*}_{L}.$$

By $\mathbf{as}(X, L) = \emptyset$, we have

$$\mathbf{as}(X_\ominus, L) = \emptyset. \tag{1}$$

By $\mathbf{Suc}(X, L) = \emptyset$, we have $X_\ominus \notin \mathbf{Suc}(X, L)$, and thus,

$$X_\ominus \notin L \supseteq \mathbf{K4}. \tag{2}$$

Therefore,

$$\mathbf{Suc}(X_\ominus, L) = \emptyset. \tag{3}$$

By (1), (2), and (3), we have

$$f_{L_0 \to L}((X_\ominus)_\ominus) \in \mathbf{G}^{*}_{L} \text{ implies } f_{L_0 \to L}(X_\ominus) = f_{L_0 \to L}((X_\ominus)_\ominus) \in \mathbf{G}^{*}_{L}.$$

Using $f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^{*}_{L}$, we have

$$f_{L_0 \to L}((X_\ominus)_\ominus) \notin \mathbf{G}^{*}_{L}. \tag{4}$$

By (1), (3), and (4), we obtain the lemma.

Lemma 2.5. Let X be a sequent in $\mathbf{G}^{L}_{L_0}(n)$ $(n \geq 0)$. Then

(1) $f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^{*}_{L}\}) = \mathbf{Ant}(f_{L_0 \to L}(X))$,

(2) $f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X) \mid f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^{*}_{L}\}) = \mathbf{Suc}(f_{L_0 \to L}(X))$.

Proof. We use an induction on n.

If $n = 0$, then by

$$\mathbf{Ant}(f_{L_0 \to L}(X)) = \mathbf{Ant}(X) = \emptyset$$

and

$$\mathbf{Suc}(f_{L_0 \to L}(X)) = \mathbf{Suc}(X) = \emptyset,$$

we obtain the lemma. If $n = 1$, then we obtain

$$\mathbf{Ant}(f_{L_0 \to L}(X)) = \mathbf{Ant}(X) = f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^{*}_{L}\})$$

and

$$\mathbf{Suc}(f_{L_0 \to L}(X)) = \mathbf{Suc}(X) = f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X) \mid f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^{*}_{L}\}).$$

We assume that $n > 1$. By Lemma 2.4 and the induction hypothesis, we have the following two conditions:

(3) $f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X_\ominus) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}) = \mathbf{Ant}(f_{L_0 \to L}(X_\ominus))$,
(4) $f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X_\ominus) \mid f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}) = \mathbf{Suc}(f_{L_0 \to L}(X_\ominus))$.

We show (1). Suppose that $Z \in \mathbf{Ant}(X)$, $Z \notin L$, and $f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L$. If $Z \in \mathbf{Ant}(X, n-1)$, then by the definition, we have $f_{L_0 \to L}(Z) \in \mathbf{Ant}(f_{L_0 \to L}(X))$. If $Z \in \mathbf{Ant}(X_\ominus)$, then by (3),

$$f_{L_0 \to L}(Z) \in \mathbf{Ant}(f_{L_0 \to L}(X_\ominus)) \subseteq \mathbf{Ant}(f_{L_0 \to L}(X)).$$

Suppose that $Z' \in \mathbf{Ant}(f_{L_0 \to L}(X))$. Then we have either

$$Z' \in \mathbf{Ant}(f_{L_0 \to L}(X_\ominus))$$

or

$$Z' \in \mathcal{S}_{0,a} = f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}).$$

If $Z' \in \mathbf{Ant}(f_{L_0 \to L}(X_\ominus))$, then by (3) and $\mathbf{Ant}(X_\ominus) \subseteq \mathbf{Ant}(X)$, we have

$$Z' \in f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}). \tag{5}$$

If $Z' \in \mathcal{S}_{0,a}$, then by $\mathbf{Ant}(X, n-1) \subseteq \mathbf{Ant}(X)$, we also have (5).

We show (2). Suppose that $Z \in \mathbf{Suc}(X)$ and $f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L$. If $Z \in \mathbf{Suc}(X, n-1)$, then by the definition, we have $f_{L_0 \to L}(Z) \in \mathbf{Suc}(f_{L_0 \to L}(X))$. If $Z \in \mathbf{Suc}(X_\ominus)$, then by (4),

$$f_{L_0 \to L}(Z) \in \mathbf{Suc}(f_{L_0 \to L}(X_\ominus)) \subseteq \mathbf{Suc}(f_{L_0 \to L}(X)).$$

Suppose that $Z' \in \mathbf{Suc}(f_{L_0 \to L}(X))$. Then we have either

$$Z' \in \mathbf{Suc}(f_{L_0 \to L}(X_\ominus))$$

or

$$Z' \in \mathcal{S}_{0,s} = f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X, n-1) \mid f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}).$$

If $Z' \in \mathbf{Suc}(f_{L_0 \to L}(X_\ominus))$, then by (4) and $\mathbf{Suc}(X_\ominus) \subseteq \mathbf{Suc}(X)$, we have

$$Z' \in f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X) \mid f_{L_0 \to L}(Z_\ominus) \notin \mathbf{G}^*_L\}). \tag{6}$$

If $Z' \in \mathcal{S}_{0,s}$, then by $\mathbf{Suc}(X, n-1) \subseteq \mathbf{Suc}(X)$, we also have (6). ⊣

Lemma 2.6. Let Y be a sequent in $\mathbf{G}^+_L(n)$ $(n \geq 0)$. Then
(1) $\mathbf{Ant}(f_{L \to L_0}(Y)) = f_{L \to L_0}(\mathbf{Ant}(Y)) \cup \mathcal{S}^\star_{2,a} \cup \mathcal{S}^\star_{3,a}$,
(2) $\mathbf{Suc}(f_{L \to L_0}(Y)) = f_{L \to L_0}(\mathbf{Suc}(Y)) \cup \mathcal{S}^\star_{2,s}$,
where

$$\mathcal{S}^\star_{2,a} = \{Z \in \bigcup_{i=1}^{n-1} \mathbf{G}_{L_0}(i) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \in \mathbf{Ant}(Y_\ominus) \cap \mathbf{G}^*_L\},$$

$$\mathcal{S}^\star_{2,s} = \{Z \in \bigcup_{i=1}^{n-1} \mathbf{G}_{L_0}(i) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \in \mathbf{Suc}(Y_\ominus) \cap \mathbf{G}^*_L\},$$

$$\mathcal{S}^{\star}_{3,a} = \{Z \in \bigcup_{i=1}^{n-1} \mathbf{G}_{L_0}(i) \mid Z \in L\}.$$

Proof. We use an induction on n.

If $n \in \{0, 1\}$, then

$$\mathcal{S}^{\star}_{2,a} = \mathcal{S}^{\star}_{2,s} = \mathcal{S}^{\star}_{3,a} = \emptyset.$$

If $n = 0$, then

$$f_{L\to L_0}(\mathbf{Ant}(Y)) = f_{L\to L_0}(\emptyset) = \emptyset = \mathbf{Ant}(Y) = \mathbf{Ant}(f_{L\to L_0}(Y)),$$

$$f_{L\to L_0}(\mathbf{Suc}(Y)) = f_{L\to L_0}(\emptyset) = \emptyset = \mathbf{Suc}(Y) = \mathbf{Suc}(f_{L\to L_0}(Y)).$$

If $n = 1$, then we have

$$f_{L\to L_0}(\mathbf{Ant}(Y)) = \mathbf{Ant}(Y) = \mathbf{Ant}(f_{L\to L_0}(Y)),$$

$$f_{L\to L_0}(\mathbf{Suc}(Y)) = \mathbf{Suc}(Y) = \mathbf{Suc}(f_{L\to L_0}(Y)).$$

If $n > 1$, then by the induction hypothesis,

$$\begin{aligned}
\mathbf{Ant}(f_{L\to L_0}(Y)) &= \mathbf{Ant}(f_{L\to L_0}(Y_\ominus)) \cup \mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a} \\
&= (f_{L\to L_0}(\mathbf{Ant}(Y_\ominus)) \cup \mathcal{S}^{\star\star}_{2,a} \cup \mathcal{S}^{\star\star}_{3,a}) \cup \mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a} \\
&= (f_{L\to L_0}(\mathbf{Ant}(Y_\ominus)) \cup \mathcal{S}_{1,a}) \cup (\mathcal{S}^{\star\star}_{2,a} \cup \mathcal{S}_{2,a}) \cup (\mathcal{S}^{\star\star}_{3} \cup \mathcal{S}_{3,a}) \\
&= f_{L\to L_0}(\mathbf{Ant}(Y)) \cup \mathcal{S}^{\star}_{2,a} \cup \mathcal{S}^{\star}_{3,a},
\end{aligned}$$

$$\begin{aligned}
\mathbf{Suc}(f_{L\to L_0}(Y)) &= \mathbf{Suc}(f_{L\to L_0}(Y_\ominus)) \cup \mathcal{S}_{1,s} \cup \mathcal{S}_{2,s} \\
&= (f_{L\to L_0}(\mathbf{Suc}(Y_\ominus)) \cup \mathcal{S}^{\star\star}_{2,s}) \cup \mathcal{S}_{1,s} \cup \mathcal{S}_{2,s} \\
&= (f_{L\to L_0}(\mathbf{Suc}(Y_\ominus)) \cup \mathcal{S}_{1,s}) \cup (\mathcal{S}^{\star\star}_{2,s} \cup \mathcal{S}_{2,s}) \\
&= f_{L\to L_0}(\mathbf{Suc}(Y)) \cup \mathcal{S}^{\star}_{2,s},
\end{aligned}$$

where $\mathcal{S}_{1,a}, \mathcal{S}_{1,s}, \mathcal{S}_{2,a}, \mathcal{S}_{2,s}$, and $\mathcal{S}_{3,a}$ are as in Definition 2.1; and

$$\mathcal{S}^{\star\star}_{2,a} = \{Z \in \bigcup_{i=1}^{n-2} \mathbf{G}_{L_0}(i) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{Ant}((Y_\ominus)_\ominus) \cap \mathbf{G}^*_L\},$$

$$\mathcal{S}^{\star\star}_{2,s} = \{Z \in \bigcup_{i=1}^{n-2} \mathbf{G}_{L_0}(i) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{Suc}((Y_\ominus)_\ominus) \cap \mathbf{G}^*_L\},$$

$$\mathcal{S}^{\star\star}_{3} = \{Z \in \bigcup_{i=1}^{n-2} \mathbf{G}_{L_0}(i) \mid Z \in L\}.$$

⊣

Theorem 2.7.

(1) For any $X \in \mathbf{G}^L_{L_0}(n)$,

(1a) $f_{L_0\to L}(\{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\}) \subseteq \mathbf{Ant}(f_{L_0\to L}(X))$,

(1b) $f_{L_0\to L}(\{Z \in \mathbf{Suc}(X) \mid f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\}) \subseteq \mathbf{Suc}(f_{L_0\to L}(X))$.

(2) For any $X \in \mathbf{G}^+_{L_0}(n)$,

(2a) $\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) \neq \emptyset$ implies $X \in L$,

(2b) $\mathbf{for}(X) \equiv_L \mathbf{for}(f_{L_0\to L}(X))$, (2c) for any $X' \in \mathbf{G}^+_{L_0}(n)$, $X \neq X'$ and $f_{L_0 \to L}(X) = f_{L_0 \to L}(X')$ imply $X \in L$,

(2d) $X \in \mathbf{G}^L_{L_0}(n)$ implies $f_{L_0 \to L}(X) \in \mathbf{G}^+_L(n)$,

(2e) $X \in \mathbf{G}^L_{L_0}(n)$ implies $f_{L \to L_0}(f_{L_0 \to L}(X)) = X$,

(2f) $X \notin L$ implies $f_{L_0 \to L}(X) \in \bigcup_{i=0}^{n-1} \mathbf{G}_L(i)$.

(3) For any $Y \in \mathbf{G}^+_L(n)$,

(3a) $\mathbf{for}(Y) \equiv_L \mathbf{for}(f_{L \to L_0}(Y))$,

(3b) for any $Y' \in \mathbf{G}^+_L(n)$, $Y \neq Y'$ and $f_{L \to L_0}(Y) = f_{L \to L_0}(Y')$ imply $Y \in L$,

(3c) $f_{L \to L_0}(Y) \in \mathbf{G}^L_{L_0}(n)$,

(3d) $f_{L_0 \to L}(f_{L \to L_0}(Y)) = Y$.

(4) $f_{L_0 \to L}(\mathbf{G}^L_{L_0}(n)) = \mathbf{G}^+_L(n)$.

(5) $f_{L_0 \to L}(\{X \in \mathbf{G}_{L_0}(n) \mid X \notin L, f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L\}) = f_{L_0 \to L}(\mathbf{G}^L_{L_0}(n) - L) = \mathbf{G}_L(n)$.

(6) $\mathbf{G}^L_{L_0}(n) = f_{L \to L_0}(\mathbf{G}^+_L(n))$.

(7) $\{X \in \mathbf{G}_{L_0}(n) \mid X \notin L, f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L\} = \mathbf{G}^L_{L_0}(n) - L = f_{L \to L_0}(\mathbf{G}_L(n))$.

(8) $f_{L_0 \to L}(\{X \in \mathbf{G}^*_{L_0}(n) \mid X \notin L, f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L\}) = f_{L_0 \to L}(\mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L) \subseteq \mathbf{G}^*_L(n)$.

(9) $\{X \in \mathbf{G}^*_{L_0}(n) \mid X \notin L, f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L\} = \mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L \subseteq f_{L \to L_0}(\mathbf{G}^*_L(n))$.

Proof. We use an induction on n. If $n = 0$, then by Lemma 2.1, $\mathbf{Ant}(X) = \mathbf{Suc}(X) = \emptyset$, and $\mathbf{G}^*_{L_0}(0) = \mathbf{G}^*_L(0) = \emptyset$, we obtain the lemma. We assume that $n > 0$.

We show (1). Let X be a sequent in $\mathbf{G}^L_{L_0}(n)$. By Lemma 2.4 and the induction hypothesis of (1), we have the following two conditions:

(1a.1) $f_{L_0 \to L}(\{Z \in \mathbf{Ant}(X_\ominus) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L\}) \subseteq \mathbf{Ant}(f_{L_0 \to L}(X_\ominus))$,

(1b.1) $f_{L_0 \to L}(\{Z \in \mathbf{Suc}(X_\ominus) \mid f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L\}) \subseteq \mathbf{Suc}(f_{L_0 \to L}(X_\ominus))$.

We show (1a). Suppose that

$$Z \in \mathbf{Ant}(X),\ Z \notin L, \text{ and } f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L.$$

By $X \in \mathbf{G}^L_{L_0}(n)$, we have $\mathbf{as}(X, L) = \emptyset$, and thus,

$$Z_\ominus \in \mathbf{Ant}(X_\ominus).$$

By $Z \notin L$, we have $Z_\ominus \notin L$, and thus,

$$Z_\ominus \in \{Z' \in \mathbf{Ant}(X_\ominus) \mid Z' \notin L, f_{L_0 \to L}(Z'_\ominus) \in \mathbf{G}^*_L \text{ or } f_{L_0 \to L}(Z'_\ominus) \notin \mathbf{G}^*_L\}.$$

Using (1a.1) and Lemma 2.5(1), we have

$$f_{L_0 \to L}(Z_\ominus) \in \mathbf{Ant}(f_{L_0 \to L}(X_\ominus)) \subseteq \mathbf{Ant}(f_{L_0 \to L}(X)).$$

Also, by $Z \notin L$ and the induction hypothesis of (2a), we have

$$\mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset,$$

and using $Z \notin L \supseteq \mathbf{K4}$,

$$f_{L_0 \to L}(Z) = f_{L_0 \to L}(Z_\ominus) \in \mathbf{Ant}(f_{L_0 \to L}(X)).$$

We show (1b). Suppose that

$$Z \in \mathbf{Suc}(X) \text{ and } f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L.$$

By $X \in \mathbf{G}^L_{L_0}(n)$, we have $\mathbf{as}(X, L) = \emptyset$, and thus,

$$Z_\ominus \in \mathbf{Suc}(X_\ominus) = \{Z' \in \mathbf{Suc}(X_\ominus) \mid f_{L_0 \to L}(Z'_\ominus) \in \mathbf{G}^*_L \text{ or } f_{L_0 \to L}(Z'_\ominus) \notin \mathbf{G}^*_L\}.$$

Using (1b.1) and Lemma 2.5(2), we have

$$f_{L_0 \to L}(Z_\ominus) \in \mathbf{Suc}(f_{L_0 \to L}(X_\ominus)) \subseteq \mathbf{Suc}(f_{L_0 \to L}(X)).$$

Also, by $X \in \mathbf{G}^L_{L_0}(n)$, we have $\mathbf{Suc}(X, L) = \emptyset$, and thus, $Z \notin L$. Using the induction hypothesis of (2a), we have

$$\mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset,$$

and using $Z \notin L \supseteq \mathbf{K4}$,

$$f_{L_0 \to L}(Z) = f_{L_0 \to L}(Z_\ominus) \in \mathbf{Suc}(f_{L_0 \to L}(X)).$$

We show (2). Let X be a sequent in $\mathbf{G}^+_{L_0}(n)$. We use $\mathcal{S}_{0,a}$, $\mathcal{S}_{0,s}$, and X_L as in Definition 2.1.

We show (2a). Suppose that $\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) \neq \emptyset$. Then we have either one of the following three conditions:

(2a.1) $\mathbf{Suc}(X, L) \neq \emptyset$,

(2a.2) $\{Z \in \mathbf{Ant}(X) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L, Z_\ominus \in \mathbf{Suc}(X)\} \neq \emptyset$,

(2a.3) $\{Z \in \mathbf{Suc}(X) \mid Z \notin L, f_{L_0 \to L}(Z_\ominus) \in \mathbf{G}^*_L, Z_\ominus \in \mathbf{Ant}(X)\} \neq \emptyset$.

If (2a.1) holds, then clearly, we have $X \in L$. If (2a.3) holds, then by $\mathbf{for}(Z_\ominus) \to \mathbf{for}(Z) \in \mathbf{K4}$, we have $X \in L$.

Therefore, we can assume that (2a.2) holds. Then there exists a sequent Z such that the following four conditions hold:

(2a.4) $Z \in \mathbf{Ant}(X)$,

(2a.5) $Z \notin L$,

(2a.6) $f_{L_\to L}(Z_\ominus) \in \mathbf{G}^*_L$,

(2a.7) $Z_\ominus \in \mathbf{Suc}(X)$.

By (2a.5) and the induction hypothesis of (2a), we have

$$\mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset. \tag{2a.8}$$

Also, by (2a.5), we have $Z \notin \mathbf{K4}$, and using (2a.6) and (2a.8), we have

$$f_{L_0 \to L}(Z) = f_{L_0 \to L}(Z_\ominus).$$

Using (2a.4) and the induction hypothesis of (2b), we have

$$\mathbf{for}(Z) \equiv_L \mathbf{for}(f_{L_0\to L}(Z)) = \mathbf{for}(f_{L_0\to L}(Z_\ominus)) \equiv_L \mathbf{for}(Z_\ominus).$$

Using (2a.4) and (2a.6), we obtain $X \in L$.

We show (2b). If $\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) \neq \emptyset$, then by (2a), we have $X \in L$, and thus,

$$\mathbf{for}(X) \equiv_L (\bot \supset \bot) = f_{L_0\to L}(X).$$

Therefore, we can assume that

$$\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset. \tag{2b.1}$$

We have

$$X = (\Box\mathbf{for}(\mathbf{Ant}(X, n-1)), \mathbf{ant}(X_\ominus) \to \mathbf{suc}(X_\ominus), \Box\mathbf{for}(\mathbf{Suc}(X, n-1))).$$

By the induction hypothesis of (2b), we have

$$\begin{aligned}
\mathbf{for}(X_\ominus) &\equiv_L \mathbf{for}(f_{L_0\to L}(X_\ominus)),\\[1em]
\bigwedge \mathbf{for}(\mathbf{Ant}(X, n-1)) &\equiv_L \bigwedge \mathbf{for}(f_{L_0\to L}(\mathbf{Ant}(X, n-1)))\\
&\equiv_L \bigwedge \mathbf{for}(f_{L_0\to L}(\mathbf{Ant}(X, n-1)) - L)\\
&\equiv_L \bigwedge \mathbf{for}(f_{L_0\to L}(\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L\}))\\
&\equiv_L \bigwedge \mathbf{for}(\mathcal{S}_{0,a} \cup \mathcal{S}'_{0,a}),\\[1em]
\bigvee \Box\mathbf{for}(\mathbf{Suc}(X, n-1)) &\equiv_L \bigvee \mathbf{for}(f_{L_0\to L}(\mathbf{Suc}(X, n-1)))\\
&\equiv_L \bigvee \mathbf{for}(\mathcal{S}_{0,s} \cup \mathcal{S}'_{0,s}),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{S}'_{0,a} &= f_{L_0\to L}(\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\}),\\
\mathcal{S}'_{0,s} &= f_{L_0\to L}(\{Z \in \mathbf{Suc}(X, n-1) \mid f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\}).
\end{aligned}$$

Therefore, we have

$$\mathbf{for}(X) \equiv_L \mathbf{for}(\Box\mathbf{for}(\mathcal{S}_{0,a} \cup \mathcal{S}'_{0,a}), \mathbf{ant}(f_{L_0\to L}(X_\ominus)) \to \mathbf{suc}(f_{L_0\to L}(X_\ominus)), \Box\mathbf{for}(\mathcal{S}_{0,s} \cup \mathcal{S}'_{0,s})). \tag{2b.2}$$

If $f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L$, then by (2b.1), (1a) and (1b), we have

$$\begin{aligned}
&\mathbf{for}(f_{L_0\to L}(X))\\
= \;&\mathbf{for}(\Box\mathbf{for}(\mathcal{S}_{0,a}), \mathbf{ant}(f_{L_0\to L}(X_\ominus)) \to \mathbf{suc}(f_{L_0\to L}(X_\ominus)), \Box\mathbf{for}(\mathcal{S}_{0,s}))\\
\equiv_L \;&\mathbf{for}(\Box\mathbf{for}(\mathcal{S}_{0,a} \cup \mathcal{S}'_{0,a}), \mathbf{ant}(f_{L_0\to L}(X_\ominus)) \to \mathbf{suc}(f_{L_0\to L}(X_\ominus)), \Box\mathbf{for}(\mathcal{S}_{0,s} \cup \mathcal{S}'_{0,s}))\\
\equiv_L \;&\mathbf{for}(X).
\end{aligned}$$

If $f_{L_0\to L}(X_\ominus) \in \mathbf{G}^*_L$ and $X \in \mathbf{K4}$, then we can easily observe

$$\mathbf{for}(X) \equiv_L (\bot \supset \bot) = f_{L_0\to L}(X).$$

Therefore, we can assume that

$$f_{L_0\to L}(X_\ominus) \in \mathbf{G}_L^* \text{ and } X \notin \mathbf{K4}.$$

Then by (2b.1), we have

$$f_{L_0\to L}(X) = f_{L_0\to L}(X_\ominus).$$

Also, using (2b.2) and Lemma 2.3, we have

$$\mathcal{S}_{0,a} \cup \mathcal{S}'_{0,a} \cup \mathcal{S}_{0,s} \cup \mathcal{S}'_{0,s} \subseteq \mathbf{G}_L \text{ implies } \mathbf{for}(X) \equiv_L f_{L_0\to L}(X_\ominus) = f_{L_0\to L}(X).$$

On the other hand, by (2b.1), we have $\mathbf{Suc}(X, L) = \emptyset$, and thus,

$$\begin{aligned}\mathcal{S}_{0,a} \cup \mathcal{S}'_{0,a} \cup \mathcal{S}_{0,s} \cup \mathcal{S}'_{0,s} &= f_{L_0\to L}((\mathbf{Ant}(X, n-1) - L) \cup \mathbf{Suc}(X, n-1)) \\ &= f_{L_0\to L}((\mathbf{Ant}(X, n-1) \cup \mathbf{Suc}(X, n-1)) - L).\end{aligned}$$

Therefore, we have only to show

$$f_{L_0\to L}((\mathbf{Ant}(X, n-1) \cup \mathbf{Suc}(X, n-1)) - L) \subseteq \mathbf{G}_L. \tag{2b.3}$$

Suppose that $Z \in (\mathbf{Ant}(X, n-1) \cup \mathbf{Suc}(X, n-1)) - L$. Then

$$n = 1 \text{ implies } f_{L_0\to L}(Z) = Z \in \mathbf{G}_L(0). \tag{2b.4}$$

By $Z \notin L$ and (2a), we have $Z \notin \mathbf{K4}$ and $\mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset$, and thus,

$$n > 1 \text{ and } f_{L_0\to L}(Z_\ominus) \in \mathbf{G}_L^* \text{ imply } f_{L_0\to L}(Z) = f_{L_0\to L}(Z_\ominus) \in \mathbf{G}_L^* \subseteq \mathbf{G}_L. \tag{2b.5}$$

Also, by $Z \notin L$, and the induction hypothesis of (5),

$$n > 1 \text{ and } f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}_L^* \text{ imply } f_{L_0\to L}(Z) \in \mathbf{G}_L(n). \tag{2b.6}$$

By (2b.4), (2b.5), and (2b.6), we obtain (2b.3).

We show (2c). Suppose that $X \neq X'$ and $f_{L_0\to L}(X) = f_{L_0\to L}(X')$. Then by (2b), we have

$$\mathbf{for}(X) \equiv_L f_{L_0\to L}(X) = f_{L_0\to L}(X') \equiv_L \mathbf{for}(X').$$

If either $X \in L$ or $X' \in L$, then we obtain the lemma. We assume that $X \notin L \supseteq L_0$ and $X' \notin L \supseteq L_0$. Then we have $X, X' \in \mathbf{G}_{L_0}(n)$. Using $X \neq X'$, we have $\mathbf{for}(X') \supset \mathbf{for}(X) \equiv_{L_0} \mathbf{for}(X)$, and thus,

$$\mathbf{for}(X') \supset \mathbf{for}(X) \equiv_L \mathbf{for}(X).$$

Using $\mathbf{for}(X) \equiv_L \mathbf{for}(X')$, we have $X \in L$.

We show (2d). By Lemma 2.2(4) and Lemma 2.2(5), we can assume that $n > 1$. Suppose that $X \in \mathbf{G}_{L_0}^L(n)$. Then we have

$$f_{L_0\to L}(X) = X_L.$$

Therefore, we have only to show the following three conditions:

(2d.1) $f_{L_0\to L}(X_\ominus) \in \mathbf{G}_L(n-1) - \mathbf{G}^*_L(n-1)$,

(2d.2) $\mathcal{S}_{0,a} \cup \mathcal{S}_{0,s} = \mathbf{G}_L(n-1)$,

(2d.3) $\mathcal{S}_{0,a} \cap \mathcal{S}_{0,s} = \emptyset$.

Also, by $X \in \mathbf{G}^L_{L_0}(n)$, we have

$$X \in \mathbf{G}^+_{L_0}(n), \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset, \text{ and } f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L, \tag{2d.4}$$

and using Lemma 2.4,

$$X_\ominus \in \mathbf{G}_{L_0}(n-1), \mathbf{Suc}(X_\ominus, L) \cup \mathbf{as}(X_\ominus, L) = \emptyset, \text{ and } f_{L_0\to L}((X_\ominus)_\ominus) \notin \mathbf{G}^*_L. \tag{2d.5}$$

We show (2d.1). By $\mathbf{Suc}(X, L) = \emptyset$, we have $X_\ominus \notin L$. Using (2d.5) and the induction hypothesis of (5), we have

$$f_{L_0\to L}(X_\ominus) \in \mathbf{G}_L(n-1).$$

Using $f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L$, we obtain (2d.1).

We show (2d.2). By $\mathbf{Suc}(X, L) = \emptyset$, we have $\mathcal{S}_{0,s} = \mathcal{S}_{0,s} - L$, and hence

$$\mathcal{S}_{0,a} \cup \mathcal{S}_{0,s} = f_{L_0\to L}(\{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\}).$$

Using the induction hypothesis of (5), we obtain (2d.2).

We show (2d.3). Suppose that $Z \in \mathcal{S}_{0,a} \cap \mathcal{S}_{0,s}$. Then there exist sequents

$$Z' \in \mathbf{Ant}(X, n-1) - L \text{ and } Z'' \in \mathbf{Suc}(X, n-1)$$

such that $Z = f_{L_0\to L}(Z') = f_{L_0\to L}(Z'')$. By $X \in \mathbf{G}^+_L(n)$, we have $Z' \neq Z''$, and using (2c), we obtain $Z' \in L$, which is in contradiction with $Z' \in \mathbf{Ant}(X_n - 1) - L$.

We show (2e). By Lemma 2.2(4), we can assume that $n > 1$.

Suppose that $X \in \mathbf{G}^L_{L_0}(n)$. Then we have the conditions:

$$\text{(2d.1), (2d.2), (2d.3), (2d.4), and (2d.5).}$$

shown in the proof of (2d). For brevity, we refer to Y as $f_{L_0\to L}(X)$. Then by (2d), we have

$$Y = f_{L_0\to L}(X) = X_L \in \mathbf{G}^+_L(n).$$

By the definition,

$$\begin{aligned} & f_{L\to L_0}(f_{L_0\to L}(X)) \\ = \; & f_{L\to L_0}(Y) \\ = \; & (\Box\mathbf{for}(\mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a}), \mathbf{ant}(f_{L\to L_0}(Y_\ominus)) \to \mathbf{suc}(f_{L\to L_0}(Y_\ominus)), \Box\mathbf{for}(\mathcal{S}_{1,s} \cup \mathcal{S}_{2,s})), \end{aligned}$$

where $\mathcal{S}_{1,a}$, $\mathcal{S}_{1,s}$, $\mathcal{S}_{2,a}$, $\mathcal{S}_{2,s}$, and $\mathcal{S}_{3,a}$ are as in Definition 2.1. On the other hand, we have

$$X = (\Box\mathbf{for}(\mathbf{Ant}(X, n-1)), \mathbf{ant}(X_\ominus) \to \mathbf{suc}(X_\ominus), \Box\mathbf{for}(\mathbf{Suc}(X, n-1))).$$

By (2d.4), we have $\mathbf{Suc}(X, L) = \emptyset$, and thus,

$$\mathbf{Ant}(X, n-1) = (\mathbf{Ant}(X, n-1) - L) \cup \mathcal{S}_{3,a},$$

$$\mathbf{Suc}(X, n-1) = \mathbf{Suc}(X, n-1) - L.$$

Therefore, we have only to show the following five conditions:

(2e.1) $X_\ominus = f_{L\to L_0}(Y_\ominus)$,

(2e.2) $\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\} = \mathcal{S}_{1,a}$,

(2e.3) $\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\} = \mathcal{S}_{2,a}$,

(2e.4) $\{Z \in \mathbf{Suc}(X, n-1) \mid f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\} = \mathcal{S}_{1,s}$,

(2e.5) $\{Z \in \mathbf{Suc}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\} = \mathcal{S}_{2,s}$.

We show (2e.1). By (2d.1), we have

$$Y_\ominus = (f_{L_0\to L}(X))_\ominus = (X_L)_\ominus = f_{L_0\to L}(X_\ominus).$$

Using (2d.5) and the induction hypothesis of (2e), we obtain

$$f_{L\to L_0}(Y_\ominus) = f_{L\to L_0}(f_{L_0\to L}(X_\ominus)) = X_\ominus.$$

We show (2e.2) and (2e.4). By $Y = X_L$, (2d.1), and (2d.2), we have

$$\mathbf{Ant}(Y, n-1) = \mathcal{S}_{0,a} \text{ and } \mathbf{Suc}(Y, n-1) = \mathcal{S}_{0,s}.$$

By (2a) and $\mathbf{Suc}(X, L) = \emptyset$, we have

$$Z \in \mathbf{Ant}(X, n-1) \text{ and } Z \notin L \text{ imply } \mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset$$

and

$$Z \in \mathbf{Suc}(X, n-1) \text{ implies } Z \notin L \text{ and } \mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset.$$

Using the induction hypothesis of (2e), we have

$$\begin{aligned}
\mathcal{S}_{1,a} &= f_{L\to L_0}(\mathbf{Ant}(Y, n-1)) \\
&= f_{L\to L_0}(\mathcal{S}_{0,a}) \\
&= f_{L\to L_0}(f_{L_0\to L}(\{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\})) \\
&= \{Z \in \mathbf{Ant}(X, n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{S}_{1,s} &= f_{L\to L_0}(\mathbf{Suc}(Y, n-1)) \\
&= f_{L\to L_0}(\mathcal{S}_{0,s}) \\
&= f_{L\to L_0}(f_{L_0\to L}(\{Z \in \mathbf{Suc}(X, n-1) \mid f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\})) \\
&= \{Z \in \mathbf{Suc}(X, n-1) \mid f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}^*_L\},
\end{aligned}$$

and hence, we obtain (2e.2) and (2e.4).

We show (2e.3) and (2e.5). By (2d), we have only to show

$$Z \in \mathbf{Ant}(X, n-1) \text{ if and only if } f_{L_0\to L}(Z_\ominus) \in \mathbf{Ant}(Y_\ominus) \tag{2e.6}$$

for any $Z \in \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L\}$. Let Z be a sequent in $\mathbf{G}_{L_0}(n-1) - L$ satisfying $f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L$. Then by (1a) and (1b), we have

$$Z \in \mathbf{Ant}(X, n-1) \text{ implies } f_{L_0\to L}(Z) \in \mathbf{Ant}(Y)$$

and

$$Z \in \mathbf{Suc}(X, n-1) \text{ implies } f_{L_0 \to L}(Z) \in \mathbf{Suc}(Y).$$

Using (2d), we have

$$Z \in \mathbf{Ant}(X, n-1) \text{ if and only if } f_{L_0 \to L}(Z) \in \mathbf{Ant}(Y). \tag{2e.7}$$

Also, by $Z \notin L$ and (2a), we have $\mathbf{Suc}(Z, L) \cup \mathbf{as}(Z, L) = \emptyset$, and using $Z \notin L \supseteq \mathbf{K4}$,

$$f_{L_0 \to L}(Z) = f_{L_0 \to L}(Z_\ominus). \tag{2e.8}$$

Moreover, by $Z \in \mathbf{G}_{L_0}(n-1) - L$, we have $Z_\ominus \in \mathbf{G}_{L_0}(n-2) - L$, and using the induction hypothesis of (2f), we have

$$f_{L_0 \to L}(Z_\ominus) \in \bigcup_{i=0}^{n-2} \mathbf{G}_L(i),$$

and therefore,

$$f_{L_0 \to L}(Z_\ominus) \in \mathbf{Ant}(Y) \text{ if and only if } f_{L_0 \to L}(Z_\ominus) \in \mathbf{Ant}(Y_\ominus). \tag{2e.9}$$

By (2e.7), (2e.8), and (2e.9), we obtain (2e.6).

We show (2f). Let X be a sequent in $\mathbf{G}_{L_0}(n) - L$. By $X \notin L$ and (2b), we have

$$f_{L_0 \to L}(X) \notin L. \tag{2f.1}$$

Also, by $X \notin L$ and (2a), we have

$$\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset.$$

If $f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L$, then by (2d), we have $f_{L_0 \to L}(X) \in \mathbf{G}^+_L(n)$, and using (2f.1), we obtain the lemma. If $f_{L_0 \to L}(X_\ominus) \in \mathbf{G}^*_L$, then we have $f_{L_0 \to L}(X) = f_{L_0 \to L}(X_\ominus)$, and using $X_\ominus \in \mathbf{G}_{L_0}(n-1)$, (2f.1), and the induction hypothesis of (2f), we obtain

$$f_{L_0 \to L}(X) = f_{L_0 \to L}(X_\ominus) \in \bigcup_{i=0}^{n-2} \mathbf{G}_L(i).$$

We show (3). By Lemma 2.2(5) and Lemma 2.2(6), we can assume that $n > 1$. Let Y be a sequent in $\mathbf{G}^+_L(n)$. Then

$$f_{L \to L_0}(Y) = (\Box\mathbf{for}(\mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a}), \mathbf{ant}(f_{L \to L_0}(Y_\ominus)) \to \mathbf{suc}(f_{L \to L_0}(Y_\ominus)), \Box\mathbf{for}(\mathcal{S}_{1,s} \cup \mathcal{S}_{2,s})), \tag{3.1}$$

where $\mathcal{S}_{1,a}$, $\mathcal{S}_{1,s}$, $\mathcal{S}_{2,a}$, $\mathcal{S}_{2,s}$, and $\mathcal{S}_{3,a}$ are as in Definition 2.1.

We show (3a). By the induction hypothesis of (3a),

$$\mathbf{for}(Y_\ominus) \equiv_L \mathbf{for}(f_{L \to L_0}(Y_\ominus)),$$

$$\begin{aligned} \bigwedge\mathbf{for}(\mathbf{Ant}(Y, n-1)) &\equiv_L \bigwedge\mathbf{for}(f_{L \to L_0}(\mathbf{Ant}(Y, n-1))) \\ &= \bigwedge\mathbf{for}(\mathcal{S}_{1,a}), \end{aligned}$$

$$\begin{aligned} \bigvee\mathbf{for}(\mathbf{Suc}(Y, n-1)) &\equiv_L \bigvee\mathbf{for}(f_{L \to L_0}(\mathbf{Suc}(Y, n-1))) \\ &= \bigvee\mathbf{for}(\mathcal{S}_{1,s}). \end{aligned}$$

Also, we note that each member of $\mathcal{S}_{3,a}$ is provable in L. Therefore,

$$\begin{array}{ll} & \mathbf{for}(f_{L\to L_0}(Y)) \\ \equiv_L & \mathbf{for}(\Box\mathbf{for}(\mathcal{S}_{1,a}\cup\mathcal{S}_{2,a}\cup\mathcal{S}_{3,a}), \mathbf{ant}(Y_\ominus)\to\mathbf{suc}(Y_\ominus), \Box\mathbf{for}(\mathcal{S}_{1,s}\cup\mathcal{S}_{2,s})) \\ \equiv_L & \mathbf{for}(\Box\mathbf{for}(\mathbf{Ant}(Y,n-1)\cup\mathcal{S}_{2,a}), \mathbf{ant}(Y_\ominus)\to\mathbf{suc}(Y_\ominus), \Box\mathbf{for}(\mathbf{Suc}(Y,n-1)\cup\mathcal{S}_{2,s})). \end{array}$$

From the definition of $\mathcal{S}_{2,a}$ and $\mathcal{S}_{2,s}$, we have

$$Z\in\mathcal{S}_{2,a} \text{ implies } f_{L_0\to L}(Z_\ominus)\in\mathbf{Ant}(Y_\ominus)$$

and

$$Z\in\mathcal{S}_{2,s} \text{ implies } f_{L_0\to L}(Z_\ominus)\in\mathbf{Suc}(Y_\ominus).$$

Considering

$$Y=(\Box\mathbf{for}(\mathbf{Ant}(Y,n-1)), \mathbf{ant}(Y_\ominus)\to\mathbf{suc}(Y_\ominus), \Box\mathbf{for}(\mathbf{Suc}(Y,n-1))),$$

we have only to show

$$Z\in\mathcal{S}_{2,a}\cup\mathcal{S}_{2,s} \text{ implies } \mathbf{for}(Z)\equiv_L\mathbf{for}(f_{L_0\to L}(Z_\ominus)).$$

Suppose that $Z\in\mathcal{S}_{2,a}\cup\mathcal{S}_{2,s}$. Then we have $Z\notin L$ and $f_{L_0\to L}(Z_\ominus)\in\mathbf{G}^*_L$. By $Z\notin L$ and (2a), we have $\mathbf{Suc}(Z,L)\cup\mathbf{as}(Z,L)=\emptyset$, and using $f_{L_0\to L}(Z_\ominus)\in\mathbf{G}^*_L$, we have

$$f_{L_0\to L}(Z)=f_{L_0\to L}(Z_\ominus).$$

Using the induction hypothesis of (3a), we have

$$\mathbf{for}(Z)\equiv_L\mathbf{for}(f_{L_0\to L}(Z))=\mathbf{for}(f_{L_0\to L}(Z_\ominus)).$$

We show (3b). Suppose that $Y\neq Y'$ and $f_{L\to L_0}(Y)=f_{L\to L_0}(Y')$. Then by (3a), we have

$$\mathbf{for}(Y)\equiv_L f_{L\to L_0}(Y)=f_{L\to L_0}(Y')\equiv_L\mathbf{for}(Y').$$

If either $Y\in L$ or $Y'\in L$, then we obtain the lemma. We assume that $Y\notin L$ and $Y'\notin L$. Then we have $Y,Y'\in\mathbf{G}_L(n)$. Using $Y\neq Y'$, we have

$$\mathbf{for}(Y')\supset\mathbf{for}(Y)\equiv_L\mathbf{for}(Y).$$

Using $\mathbf{for}(Y)\equiv_L\mathbf{for}(Y')$, we have $Y\in L$.

We show (3c). By (3.1), we have only to show the following six conditions:

(3c.1) $f_{L\to L_0}(Y_\ominus)\in\mathbf{G}_{L_0}(n-1)-\mathbf{G}^*_{L_0}(n-1)$,

(3c.2) $\mathcal{S}_{1,a}\cup\mathcal{S}_{2,a}\cup\mathcal{S}_{3,a}\cup\mathcal{S}_{1,s}\cup\mathcal{S}_{2,s}=\mathbf{G}_{L_0}(n-1)$,

(3c.3) $(\mathcal{S}_{1,a}\cup\mathcal{S}_{2,a}\cup\mathcal{S}_{3,a})\cap(\mathcal{S}_{1,s}\cup\mathcal{S}_{2,s})=\emptyset$,

(3c.4) $\mathcal{S}_{1,s}\cap L=\emptyset$,

(3c.5) $\mathbf{as}(f_{L\to L_0}(Y),L)=\emptyset$,

(3c.6) $f_{L_0\to L}((f_{L\to L_0}(Y))_\ominus)\notin\mathbf{G}^*_L$.

We show (3c.1). We note that $Y_\ominus \in \mathbf{G}_L(n-1) - \mathbf{G}_L^*(n-1)$. By $Y_\ominus \in \mathbf{G}_L(n-1)$ and the induction hypothesis of (7), we have

$$f_{L\to L_0}(Y_\ominus) \in \mathbf{G}_{L_0}(n-1) \tag{3c.7}$$

and

$$f_{L\to L_0}(Y_\ominus) \notin L \text{ and } f_{L_0\to L}((f_{L\to L_0}(Y_\ominus))_\ominus) \notin \mathbf{G}_L^*. \tag{3c.8}$$

By $Y_\ominus \notin \mathbf{G}_L^*(n-1)$ and the induction hypothesis of (9), we have

$$f_{L\to L_0}(Y_\ominus) \notin \{Z \in \mathbf{G}_{L_0}^*(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}_L^*\}.$$

Using (3c.8), we have

$$f_{L\to L_0}(Y_\ominus) \notin \mathbf{G}_{L_0}^*(n-1).$$

Using (3c.7), we obtain (3c.1).

We show (3c.2). By the induction hypothesis of (7), we have

$$\begin{aligned} \mathcal{S}_{1,a} \cup \mathcal{S}_{1,s} &= \{f_{L\to L_0}(Y') \mid Y' \in \mathbf{G}_L(n-1)\} \\ &= \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \notin \mathbf{G}_L^*\}. \end{aligned} \tag{3c.9}$$

Also, we have

$$\mathcal{S}_{2,a} \cup \mathcal{S}_{2,s} = \{Z \in \mathbf{G}_{L_0}(n-1) \mid Z \notin L, f_{L_0\to L}(Z_\ominus) \in \mathbf{G}_L^*\}. \tag{3c.10}$$

Therefore, we can easily observe (3c.2).

We show (3c.3). Clearly,

$$\mathcal{S}_{2,s} \cap (\mathcal{S}_{2,a} \cup \mathcal{S}_{3,a}) = \emptyset.$$

By (3c.9) and (3c.10), we have

$$\mathcal{S}_{2,s} \cap \mathcal{S}_{1,a} = \emptyset$$

and

$$\mathcal{S}_{1,s} \cap \mathcal{S}_{2,a} = \emptyset.$$

By (3c.9), we have

$$\mathcal{S}_{1,s} \cap \mathcal{S}_{3,a} = \emptyset.$$

Therefore, we have only to show

$$\mathcal{S}_{1,s} \cap \mathcal{S}_{1,a} = \emptyset. \tag{3c.11}$$

Suppose that $Z \in \mathcal{S}_{1,a} \cup \mathcal{S}_{1,s}$. Then there exist sequents $Z' \in \mathbf{Ant}(Y, n-1)$ and $Z'' \in \mathbf{Suc}(Y, n-1)$ such that $f_{L\to L_0}(Z') = f_{L\to L_0}(Z'') = Z$. By $Z' \in \mathbf{Ant}(Y, n-1)$ and $Z'' \in \mathbf{Suc}(Y, n-1)$, we have $Z' \neq Z''$, and using (3b), we have $Z \in L$, which is in contradiction in (3c.9). Hence, we obtain (3c.11).

We have (3c.4) from (3c.9).

We show (3c.5). Suppose that $Z \in \mathbf{as}(f_{L\to L_0}(Y), L)$. By (3c.8) and (2a), we have

$$\mathbf{as}(f_{L\to L_0}(Y_\ominus), L) = \emptyset.$$

Using (3.1) and (3c.2), we have the following five conditions:
(3c.12) $Z \in \mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a} \cup \mathcal{S}_{1,s} \cup \mathcal{S}_{2,s}$,
(3c.13) $Z \notin L$,
(3c.14) $f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L$,
(3c.15) $Z \in \mathcal{S}_{1,a} \cup \mathcal{S}_{2,a} \cup \mathcal{S}_{3,a}$ implies $Z_\ominus \in \mathbf{Suc}(f_{L\to L_0}(Y_\ominus))$,
(3c.16) $Z \in \mathcal{S}_{1,s} \cup \mathcal{S}_{2,s}$ implies $Z_\ominus \in \mathbf{Ant}(f_{L\to L_0}(Y_\ominus))$.
If $Z \in \mathcal{S}_{1,a} \cup \mathcal{S}_{1,s}$, then by

$$\mathcal{S}_{1,a} \cup \mathcal{S}_{1,s} = f_{L\to L_0}(\mathbf{Ant}(Y, n-1) \cup \mathbf{Suc}(Y, n-1)) \subseteq f_{L\to L_0}(\mathbf{G}_L(n-1))$$

and the induction hypothesis of (3c), we have $Z \in \mathbf{G}^L_{L_0}(n-1)$, and thus, $f_{L_0\to L}(Z_\ominus) \in \mathbf{G}^*_L$, which is in contradiction with (3c.14). If $Z \in \mathcal{S}_{2,a}$, then by the definition of $\mathcal{S}_{2,a}$, we have $f_{L_0\to L}(Z_\ominus) \in \mathbf{Ant}(Y_\ominus)$, and using the induction hypothesis of (3d) and Lemma 2.6(1),

$$Z_\ominus = f_{L\to L_0}(f_{L_0\to L}(Z_\ominus)) \in f_{L\to L_0}(\mathbf{Ant}(Y_\ominus)) \subseteq \mathbf{Ant}(f_{L\to L_0}(Y_\ominus)),$$

which is in contradiction with (3c.15) and (3c.1). If $Z \in \mathcal{S}_{2,s}$, then by the definition of $\mathcal{S}_{2,s}$, we have $f_{L_0\to L}(Z_\ominus) \in \mathbf{Suc}(Y_\ominus)$, and using the induction hypothesis of (3d) and Lemma 2.6(2),

$$Z_\ominus = f_{L\to L_0}(f_{L_0\to L}(Z_\ominus)) \in f_{L\to L_0}(\mathbf{Suc}(Y_\ominus)) \subseteq \mathbf{Suc}(f_{L\to L_0}(Y_\ominus)),$$

which is in contradiction with (3c.16) and (3c.1). If $Z \in \mathcal{S}_{3,a}$, then by the definition of $\mathcal{S}_{3,a}$, we have $Z \in L$, which is in contradiction with (3c.13).

We show (3c.6). By (3.1), (3c.1), and (3c.2), we have

$$(f_{L\to L_0}(Y))_\ominus = f_{L\to L_0}(Y_\ominus) \in \mathbf{G}_{L_0}(n-1), \tag{3c.17}$$

and using the induction hypothesis of (3d), we have

$$f_{L_0\to L}((f_{L\to L_0}(Y))_\ominus) = f_{L_0\to L}(f_{L\to L_0}(Y_\ominus)) = Y_\ominus \notin \mathbf{G}^*_L.$$

We show (3d). By the induction hypothesis of (3d), we have

$$f_{L_0\to L}(f_{L\to L_0}(Y_\ominus)) = Y_\ominus,$$

Also, by (3c), we have

$$f_{L\to L_0}(Y) \in \mathbf{G}^L_{L_0}(n).$$

Therefore, by (3.1) and (3c.17), we have

$$\begin{aligned}
& f_{L_0\to L}(f_{L\to L_0}(Y)) \\
= \;& (f_{L\to L_0}(Y))_L \\
= \;& (\Box\mathbf{for}(f_{L_0\to L}(\mathcal{S}_{1,a})), \mathbf{ant}(Y_\ominus) \to \mathbf{suc}(Y_\ominus), \Box\mathbf{for}(f_{L_0\to L}(\mathcal{S}_{1,s}))).
\end{aligned}$$

Also, by the induction hypothesis of (3d), we have

$$\begin{aligned} f_{L_0\to L}(\mathcal{S}_{1,a}) &= \{f_{L_0\to L}(f_{L\to L_0}(Y')) \mid Y' \in \mathbf{Ant}(Y, n-1)\} \\ &= \{Y' \mid Y' \in \mathbf{Ant}(Y, n-1)\} \\ &= \mathbf{Ant}(Y, n-1), \end{aligned}$$

and similarly,

$$f_{L_0\to L}(\mathcal{S}_{1,s}) = \mathbf{Suc}(Y, n-1).$$

Hence, we obtain

$$\begin{aligned} & f_{L_0\to L}(f_{L\to L_0}(Y)) \\ = \ & (\Box\mathbf{for}(f_{L_0\to L}(\mathcal{S}_{1,a})), \mathbf{ant}(Y_\ominus) \to \mathbf{suc}(Y_\ominus), \Box\mathbf{for}(f_{L_0\to L}(\mathcal{S}_{1,s}))) \\ = \ & (\Box\mathbf{for}(\mathbf{Ant}(Y, n-1)), \mathbf{ant}(Y_\ominus) \to \mathbf{suc}(Y_\ominus), \Box\mathbf{for}(\mathbf{Suc}(Y, n-1))) \\ = \ & Y. \end{aligned}$$

We show (4). By (2d), we have

$$f_{L_0\to L}(\mathbf{G}^L_{L_0}(n)) \subseteq \mathbf{G}^+_L(n).$$

Also, by (3c) and (3d), we have

$$\mathbf{G}^+_L(n) = f_{L_0\to L}(f_{L\to L_0}(\mathbf{G}^+_L(n)) \subseteq f_{L_0\to L}(\mathbf{G}^L_{L_0}(n)).$$

We show (5). By (2a), (2b), and (4), we have

$$\begin{aligned} \mathbf{G}_L(n) &= \mathbf{G}^+_L(n) - L \\ &= f_{L_0\to L}(\mathbf{G}^L_{L_0}(n)) - L \\ &= f_{L_0\to L}(\mathbf{G}^L_{L_0}(n) - L) \\ &= \{f_{L_0\to L}(X) \mid X \in \mathbf{G}^+_{L_0}(n), \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L, X \notin L\} \\ &= \{f_{L_0\to L}(X) \mid X \in \mathbf{G}_{L_0}(n), X \notin L, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L\}. \end{aligned}$$

We show (6). By (3c), we have

$$f_{L\to L_0}(\mathbf{G}^+_L(n)) \subseteq \mathbf{G}^L_{L_0}(n).$$

Also, by (2d) and (2e), we have

$$\mathbf{G}^L_{L_0}(n) = f_{L\to L_0}(f_{L_0\to L}(\mathbf{G}^L_{L_0}(n))) \subseteq f_{L\to L_0}(\mathbf{G}^+_L(n)).$$

We show (7). By (2a), (3a), and (6), we have

$$\begin{aligned} f_{L\to L_0}(\mathbf{G}_L(n)) &= f_{L\to L_0}(\mathbf{G}^+_L(n) - L) \\ &= f_{L\to L_0}(\mathbf{G}^+_L(n)) - L \\ &= \mathbf{G}^L_{L_0}(n) - L \\ &= \{X \in \mathbf{G}^+_{L_0}(n) \mid \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L\} - L \\ &= \{X \in \mathbf{G}^+_{L_0}(n) \mid \mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L, X \notin L\} \\ &= \{X \in \mathbf{G}_{L_0}(n) \mid X \notin L, f_{L_0\to L}(X_\ominus) \notin \mathbf{G}^*_L\}. \end{aligned}$$

We show (8). By (7), we have

$$\{X \in \mathbf{G}^*_{L_0}(n) \mid X \notin L, f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L\} = \mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L. \tag{8.1}$$

Therefore, we have only to show

$$f_{L_0 \to L}(\mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L) \subseteq \mathbf{G}^*_L(n).$$

Suppose that $X \in \mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L$ and $f_{L_0 \to L}(X) \notin \mathbf{G}^*_L(n)$; in other words, we suppose that $X \in \mathbf{G}^*_{L_0}(n)$, $\mathbf{Suc}(X, L) \cup \mathbf{as}(X, L) = \emptyset$, $f_{L_0 \to L}(X_\ominus) \notin \mathbf{G}^*_L$, $X \notin L$, and $f_{L_0 \to L}(X) \notin \mathbf{G}^*_L(n)$. Then by $f_{L_0 \to L}(X) \notin \mathbf{G}^*_L(n)$, there exists a sequent $Y \in \mathbf{G}_L(n)$ such that

$$\mathbf{Ant}(f_{L_0 \to L}(X)) \subsetneq \mathbf{Ant}(Y). \tag{8.2}$$

By (7), we have $f_{L \to L_0}(Y) \in \mathbf{G}_{L_0}(n)$, and using $X \in \mathbf{G}^*_{L_0}(n)$, we have only to show the following two conditions:

(8.3) $\mathbf{Ant}(X) \subseteq \mathbf{Ant}(f_{L \to L_0}(Y))$,
(8.4) $\mathbf{Ant}(X) \neq \mathbf{Ant}(f_{L \to L_0}(Y))$.

We show (8.3). Let Z be a sequent in $\mathbf{Ant}(X)$. By Lemma 2.6(1), we have

$$Z \in L \text{ implies } Z \in \mathcal{S}^*_{3,a} \subseteq \mathbf{Ant}(f_{L \to L_0}(Y)),$$

where $\mathcal{S}^*_{3,a}$ is as in Lemma 2.6. Therefore, we can assume that $Z \notin L$. Then we have

$$Z \in \{Z' \in \mathbf{Ant}(X) \mid Z' \notin L, f_{L_0 \to L}(Z'_\ominus) \in \mathbf{G}^*_L\} \cup \{Z' \in \mathbf{Ant}(X) \mid Z' \notin L, f_{L_0 \to L}(Z'_\ominus) \notin \mathbf{G}^*_L\}.$$

Using $X \in \mathbf{G}^L_{L_0}(n)$, Lemma 2.5(1), (1a), and (8.2), we have

$$f_{L_0 \to L}(Z) \in \mathbf{Ant}(f_{L_0 \to L}(X)) \subsetneq \mathbf{Ant}(Y),$$

and using Lemma 2.6(1),

$$f_{L \to L_0}(f_{L_0 \to L}(Z)) \in f_{L \to L_0}(\mathbf{Ant}(Y)) \subseteq \mathbf{Ant}(f_{L \to L_0}(Y)).$$

Using (2b) and (3a), we have

$$\mathbf{for}(f_{L \to L_0}(f_{L_0 \to L}(Z))) \equiv_L \mathbf{for}(Z).$$

Therefore, if $Z \in \mathbf{Suc}(f_{L \to L_0}(Y))$, then $f_{L \to L_0}(Y) \in L$, which is in contradiction with (7) and $Y \in \mathbf{G}_L(n)$. Hence, we have

$$Z \notin \mathbf{Suc}(f_{L \to L_0}(Y)). \tag{8.5}$$

Also, by (7) and $Y \in \mathbf{G}_L(n)$, we have $f_{L \to L_0}(Y) \in \mathbf{G}_{L_0}(n)$, and thus,

$$Z \in \mathbf{Ant}(X) \subseteq \mathbf{Ant}(f_{L \to L_0}(Y)) \cup \mathbf{Suc}(f_{L \to L_0}(Y)).$$

Using (8.5), we obtain

$$Z \in \mathbf{Ant}(f_{L \to L_0}(Y)).$$

We show (8.4). By (5) and (8.2), there exists a sequent

$$Z \in \mathbf{Ant}(Y) \cap \mathbf{Suc}(f_{L_0 \to L}(X)).$$

Using Lemma 2.6, we have

$$f_{L \to L_0}(Z) \in f_{L \to L_0}(\mathbf{Ant}(Y) \cap \mathbf{Suc}(f_{L_0 \to L}(X))) \subseteq \mathbf{Ant}(f_{L \to L_0}(Y)) \cap \mathbf{Suc}(f_{L \to L_0}(f_{L_0 \to L}(X))),$$

and using $X \in \mathbf{G}^L_{L_0}(n)$ and (2e), we obtain (8.4).

We show (9). By (8.1), we have only to show

$$\mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L \subseteq f_{L \to L_0}(\mathbf{G}^*_L(n)).$$

By (8), we have

$$f_{L \to L_0}(f_{L_0 \to L}(\mathbf{G}^L_{L_0}(n) \cap \mathbf{G}^*_{L_0} - L)) \subseteq f_{L \to L_0}(\mathbf{G}^*_L(n)),$$

and using (2e), we obtain (9).

References

[Gen35] G. Gentzen, *Untersuchungen über das logisch Schliessen*, Mathematische Zeitschrift, 39, 1934–35, pp. 176–210, 405–431.

[Sas10a] K. Sasaki, *Formulas in modal logic* **S4**, The Review of Symbolic Logic, 3, to appear.

[Sas10b] K. Sasaki, *Constructions of normal forms in modal logic K4*, Technical Report of the Nanzan Academic Society Information Sciences and Engineering, NANZAN-TR-2010-01, Seto, Japan: Nanzan University, 2010.