数理解析研究所講究録
第 1729 巻 2011 年 152-162

Nonstandard arguments and recursive arguments

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Abstract

We give a new nonstandard method for conservation proofs over $\mathrm{B}\Sigma_2^0$ using a combination of recursion theory and nonstandard analysis.

1 Introduction

Techniques from nonstandard analysis play an important role in Reverse Mathematics. In [12, 13], Keisler gives nonstandard characterizations for the big five subsystems of second-order arithmetic. In [16, 20, 21, 22], several nonstandard techniques for analysis in second-order arithmetic are developed, and in [8, 17], Impens and Sanders show that several theorems of nonstandard analysis are equivalent to the Π_1-transfer principle. Also, combinatorics is an important topic in Reverse Mathematics (see, e.g., [18, 19]). Especially, Ramsey's theorem for pairs (RT_2^2) plays an important role in Reverse Mathematics as an intermediate axiom between RCA_0 and ACA_0. There are many theorems of combinatorics and model theory that are provable from RT_2^2 (see, e.g., [3, 5, 6]). Thus, determining the exact strength of RT_2^2 is very important. It is well-known that RT_2^2 implies $\mathrm{B}\Sigma_2^0$. On the other hand, Cholak, Jockusch and Slaman ([1]) show that $\mathsf{RCA}_0+\mathrm{RT}_2^2+\mathrm{I}\Sigma_2^0$ is a Π_1^1-conservative extension of $\mathsf{RCA}_0+\mathrm{I}\Sigma_2^0$, *i.e.*, the first-order part of RT_2^2 is not stronger than $\mathrm{I}\Sigma_2^0$. Then, the question arises: is $\mathsf{RCA}_0+\mathrm{RT}_2^2+\mathrm{B}\Sigma_2^0$ a Π_1^1-conservative extension of $\mathsf{RCA}_0+\mathrm{B}\Sigma_2^0$? A partial answer to this question is given by Slaman, Chong and Yang ([2]). They showed that $\mathsf{RCA}_0+\mathrm{COH}+\mathrm{B}\Sigma_2^0$, $\mathsf{RCA}_0+\mathrm{ADS}+\mathrm{B}\Sigma_2^0$ and $\mathsf{RCA}_0+\mathrm{CAC}+\mathrm{B}\Sigma_2^0$ are Π_1^1-conservative extensions of $\mathsf{RCA}_0+\mathrm{B}\Sigma_2^0$. Here, COH, ADS and CAC are all combinatorial principles weaker than RT_2^2.

In this paper, we will introduce a new approach for conservation proofs over $\mathrm{B}\Sigma_2^0$. We will show how to use recursion-theoretic arguments within nonstandard arithmetic and give new proofs of the conservation theorems for WKL and COH over $\mathsf{RCA}_0+\mathrm{B}\Sigma_2^0$ (see [4] and [2] for the original proofs, respectively). It is well-known that the nonstandard approach works well for combinatorics (see, e.g., [7]). For Ramsey's theorem, the nonstandard proof of ACA_0 implies $\mathrm{RT}(k)$ is known [14, Theorem 2.2.16]. This proof can be formalized in the system of non-standard second-order arithmetic corresponding to ACA_0 introduced in [23]. In this proof, the Π_1^0-transfer principle is the key element. In the nonstandard arithmetic, the Π_1^0-transfer principle is conservative over $\mathrm{B}\Sigma_2^0$, and this fact plays a key role for the conservation proofs in this paper.

Nonstandard arithmetic

Let $\mathcal{L}$ be the language of first-order arithmetic, and let $\mathcal{L}_2$ be the language of second-order arithmetic. For a finite set of unary predicates $\bar{A}$, an $\mathcal{L}\cup\bar{A}$-structure is a pair $M=(M;\bar{A}^M)$

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where $A^M \subseteq M$ for any $A \in \bar{A}$. Let $\mathcal{L}^*$ be the language of nonstandard arithmetic, *i.e.*, $\mathcal{L}^* = \mathcal{L} \cup \{V^{\mathrm{s}}, V^*, \surd\}$ where V^{s} and V^* are unary predicate symbols denoting the standard and nonstandard universe respectively, and $\surd$ is a function symbol denoting the embedding from the standard universe into the nonstandard universe. An $\mathcal{L}^* \cup \bar{A}$ structure is a triple $\mathfrak{M} = (M, M^*, \surd)$ such that $M = (\{x \mid \mathfrak{M} \models x \in V^{\mathrm{s}}\}; \bar{A}^M)$ and $M^* = (\{x \mid \mathfrak{M} \models x \in V^*\}; \bar{A}^{M^*})$ are $\mathcal{L} \cup \bar{A}$-structures and $\surd$ is a mapping from M to M^*. We usually use the identification $M \cong \surd(M) \subseteq M^*$, *i.e.*, identify $a \in M$ with $\surd(a) \in M^*$.

An $\mathcal{L} \cup \bar{A}$-structure M is said to be a model of $\mathrm{I}\Sigma^0_n$ (resp. $\mathrm{B}\Sigma^0_n$) if $(M, \bar{A}^M) \models \mathrm{I}\Sigma^0_n$ (resp. $\mathrm{B}\Sigma^0_n$) as a second order structure. In other words, $(M; \bar{A})$ satisfies the induction axioms (resp. bounding axioms) for $\Sigma^{\bar{A}}_n$ formulas.

Definition 1.1. For a finite set of unary predicates $\bar{A}$, we define axioms for $\mathcal{L}^* \cup \bar{A}$ as follows:

- BNS consists of the following:
  - $\surd$ is an embedding (with respect to $+, \times, \bar{A}$-structures) from V^{s} to V^*,
  - V^* is an end extension of $\surd(V^{\mathrm{s}})$,
  - $V^{\mathrm{s}} \models \mathrm{I}\Sigma^0_1$ and $V^* \models \mathrm{I}\Sigma^0_1$.
- $\Pi^0_n\mathrm{TP}$: $\forall \bar{x} \in V^{\mathrm{s}}(V^{\mathrm{s}} \models \varphi(\bar{x}, \bar{A}) \leftrightarrow V^* \models \varphi(\bar{x}, \bar{A}))$ for any $\varphi \in \Pi^{\bar{A}}_n$ formulas.

Note that we can easily show that BNS implies $\Pi^0_0\mathrm{TP}$.

2 $\mathrm{B}\Sigma^0_2$ and $\Pi^0_1\mathrm{TP}$

In this section, we prove that $\mathrm{BNS} + \Pi^0_1\mathrm{TP}$ is a (first-order) conservative extension of $\mathrm{B}\Sigma^0_2$. To prove this, we use a version of Friedman's self-embedding theorem.

From now on, we identify an $\mathcal{L} \cup \bar{A}$ formula φ with an $\mathcal{L}^* \cup \bar{A}$ formula φ^{s}, where φ^{s} is a formula constructed by replacing $\forall x$ (resp. $\exists x$) in φ into $\forall x \in V^{\mathrm{s}}$ (resp. $\exists x \in V^{\mathrm{s}}$).

Theorem 2.1. *Let $n \geq 1$. Then,* $\mathrm{BNS} + \Pi^0_n\mathrm{TP} + (V^{\mathrm{s}}, V^* \models \mathrm{I}\Sigma^0_{n-1})$ *proves* $\mathrm{B}\Sigma^0_{n+1}$. *In other words, for any finite set of unary predicates $\bar{A}$, if $M = (M; \bar{A}^M)$ and $M^* = (M^*; \bar{A}^{M^*})$ are models of* $\mathrm{I}\Sigma^0_{n-1}$ *such that M^* is an elementary end extension of M with respect to $\Pi^{\bar{A}}_n$ formulas, then M is a model of* $\mathrm{B}\Sigma^0_{n+1}$.

Proof. This proof is essentially due to Theorem B of [15]. Let $M \doteq (M; \bar{A}^M)$ and $M^* = (M^*; \bar{A}^{M^*})$ are models of $\mathrm{B}\Sigma^0_n$ such that M^* is an elementary end extension of M with respect to $\Pi^{\bar{A}}_n$ formulas. Let $\theta(x, y) \equiv \forall z \theta_0(x, y, z)$ be a $\Pi^{\bar{A}}_n$ formula, and let $a \in M$ such that $M \models \forall x < a \exists y \theta(x, y)$. We will show that there exists $b \in M$ such that $M \models \forall x < a \exists y < b \theta(x, y)$. By $\Pi^0_n\mathrm{TP}$, for any $c \in M^* \setminus M$, we have $M^* \models \forall x < a \exists y < c \theta(x, y)$. Take $d \in M^* \setminus M$. Then, for any $c \in M^* \setminus M$, we have $M^* \models \forall x < a \exists y < c \forall z < d \theta_0(x, y, z)$. Then, there exists $b \in M$ such that $M^* \models \forall x < a \exists y < b \forall z < d \theta_0(x, y, z)$ by underspill for $\Sigma^{\bar{A}}_{n-1}$ formula, which is available from $M^* \models \mathrm{I}\Sigma^0_{n-1}$. (Note that $\forall x < a \exists y < b \forall z < d \theta_0(x, y, z)$ is equivalent to a $\Sigma^{\bar{A}}_{n-1}$ formula since $M^* \models \mathrm{I}\Sigma^0_{n-1}$.) Thus, we have $M \models \forall x < a \exists y < b \theta(x, y)$. This means that M satisfies $\mathrm{B}\Pi^0_n$, which is equivalent to $\mathrm{B}\Sigma^0_{n+1}$. □

The following lemma is a modification of a version of Friedman's self-embedding theorem. See also [11, page 166, Exercise 12.2]

Lemma 2.2. *Let M and N be countable recursively saturated models of $\mathrm{B}\Sigma^0_{n+1}$ such that $\mathrm{SSy}(M) = \mathrm{SSy}(N)$. Let $a \in M$ and $b, c \in N$ such that $M \models \exists x\psi(x,a)$ implies $N \models \exists x < b\psi(x,c)$ for any Π_n formulas $\psi(x,y)$. Then, there exists an embedding $f : M \to N$ such that $f(M) \subseteq_e N$, $f(M) < b$, $f(a) = c$ and f is an elementary embedding with respect to Π_n formulas.*

Proof. We will construct sequences $\{a_i\}_{i<\omega} = M$ and $\{c_i\}_{i<\omega} \subseteq_e N_{<b}$ such that $a_0 = a$, $c_0 = c$ and $M \models \exists x\psi(x,\bar{a_i})$ implies $N \models \exists x < b\psi(x,\bar{c_i})$ for any Π_n formulas by a back and forth argument, where $\bar{a_i} = (a_0, \ldots, a_i)$ and $\bar{c_i} = (c_0, \ldots, c_i)$. We fix enumerations $M = \{p_k\}_{k\in\omega}$ and $N = \{q_k\}_{k\in\omega}$ such that each element of $d \in N$ occurs infinitely often in $\{q_k\}_{k\in\omega}$.

Assume that we have already constructed $\{a_j\}_{j<i}$ and $\{c_j\}_{j<i}$ which satisfy the desired conditions. If $i = 2k+1$, put $a_i = p_k$. By recursive saturation, there exists $\alpha \in M$ such that for any $\theta(x) \in \Pi_n$, $\ulcorner\theta(x)\urcorner \in \mathrm{code}(\alpha) \leftrightarrow \exists z\theta(\langle\bar{a_i}, z\rangle)$. Since $\mathrm{SSy}(M) = \mathrm{SSy}(N)$, there exists $\beta \in N$ such that $\mathrm{SSy}(\alpha) = \mathrm{SSy}(\beta)$. Then, $q(y) = \{\ulcorner\theta(x)\urcorner \in \mathrm{code}(\beta) \to \exists z\theta(\langle\bar{c}_{i-1}, y, z\rangle) \wedge y < b \mid \theta(x) \in \Pi_n\}$ is a recursive type over N (we can easily check that $q(y)$ is finitely satisfiable). Take a solution c' of $q(y)$, and define $c_i = c'$. Then $\{a_j\}_{j\leq i}$ and $\{c_j\}_{j\leq i}$ satisfy the desired conditions.

If $i = 2k+2$ and $q_k > \max\{\bar{c}_{i-1}\}$, put $c_i = c_0$ and $a_i = a_0$. If $i = 2k+2$ and $q_k \leq \max\{\bar{c}_{i-1}\}$, put $c_i = q_k$. By recursive saturation, there exists $\beta \in N$ such that for any $\theta(x) \in \Sigma_n$, $\ulcorner\theta(x)\urcorner \in \mathrm{code}(\beta) \leftrightarrow \forall z < b\theta(\langle\bar{c_i}, z\rangle)$. Since $\mathrm{SSy}(N) = \mathrm{SSy}(M)$, there exists $\alpha \in M$ such that $\mathrm{SSy}(\beta) = \mathrm{SSy}(\alpha)$. Then, $p(x) = \{\ulcorner\theta(x)\urcorner \in \mathrm{code}(\alpha) \to \forall z\theta(\langle\bar{a}_{i-1}, x, z\rangle) \mid \theta(x) \in \Sigma_n\}$ is a recursive type over M. To show that $p(x)$ is finitely satisfiable, let $\theta_0(x), \ldots, \theta_{l-1}(x) \in \Sigma$ such that $N \models \bigwedge_{k<l} \forall z < b\theta_k(\langle\bar{c_i}, z\rangle)$. Then, $N \models \forall y < b\exists x \leq \max\{\bar{c}_{i-1}\} \bigwedge_{k<l} \forall z \leq y\theta_k(\langle\bar{c}_{i-1}, x, z\rangle)$. Since $\{a_j\}_{j<i}$ and $\{c_j\}_{j<i}$ satisfy the desired conditions, we have $M \models \forall y\exists x \leq \max\{\bar{a}_{i-1}\} \bigwedge_{k<l} \forall z \leq y\theta_k(\langle\bar{a}_{i-1}, x, z\rangle)$ (note that there is a Σ_n formula which is equivalent to $\exists x \leq \max\{\bar{u}_{i-1}\} \bigwedge_{k<l} \forall z \leq y\theta_k(\langle\bar{u}_{i-1}, x, z\rangle)$ over $\mathrm{B}\Sigma^0_n$). Then, by $M \models \mathrm{B}\Sigma^0_{n+1}$, we have $M \models \exists x \leq \max\{\bar{a}_{i-1}\}\forall y\exists y' > y \bigwedge_{k<l} \forall z \leq y'\theta_k(\langle\bar{a}_{i-1}, x, z\rangle)$. Thus, $M \models \exists x \leq \max\{\bar{a}_{i-1}\} \bigwedge_{k<l} \forall z\theta_k(\langle\bar{a}_{i-1}, x, z\rangle)$, which means that $p(x)$ is finitely satisfiable. Take a solution a' of $p(x)$, and define $a_i = a'$. Then $\{a_j\}_{j\leq i}$ and $\{c_j\}_{j\leq i}$ satisfy the desired conditions.

Define a function $f : M \to N$ as $f(a_i) = c_i$. Then, we can easily check that f is the desired embedding. □

Note that in the previous proof, we only used $M \models \mathrm{B}\Sigma^0_{n+1}$ and $N \models \mathrm{B}\Sigma^0_n$.

Theorem 2.3. *Let M be a countable recursively saturated model of $\mathrm{B}\Sigma_{n+1}$. Then, there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and f is an elementary embedding with respect to Π_n formulas.*

Proof. Let M be a countable recursively saturated model of $\mathrm{B}\Sigma_{n+1}$, and let N be a copy of M, *i.e.*, $M \cong N$. Define a recursive type $p(x)$ over M as $p(x) = \{\exists y\theta(y) \to \exists y < x\theta(y) \mid \theta \in \Pi_n\}$. Then, there exists $b \in N$ such that $N \models p(b)$. Define $a = 0 \in M$ and $c = 0 \in N$, then, M, N, a, b, c enjoy the requirements of the previous lemma. □

Theorem 2.4. *Let $\bar{A}$ be a finite set of unary predicates, and let $M = (M; \bar{A}^M)$ be a countable recursively saturated model of $\mathrm{I}\Sigma^0_1$. Then, $M \models \mathrm{B}\Sigma^0_{n+1}$ if and only if there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and f is an elementary embedding with respect to $\Pi^{\bar{A}}_n$ formulas.*

Proof. The proof of the forward direction is an easy generalization of the previous lemma and theorem. We will prove the reverse direction by induction on n. Assume that there exists a self-embedding $f : M \to M$ such that $f(M) \subsetneq_e M$ and f is an elementary embedding with respect to

$\Pi_n^{\bar{A}}$ formulas. By induction hypothesis, we have $M \models \mathrm{B}\Sigma_n^0$. Then, the triple (M, M, f) is a model of $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{I}\Sigma_{n-1}^0)$. Thus, we have $M \models \mathrm{B}\Sigma_{n+1}^0$ by Theorem 2.1. $\square$

Corollary 2.5. $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{I}\Sigma_{n-1}^0)$ *and* $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0)$ *are conservative extensions of* $\mathrm{B}\Sigma_{n+1}^0$ *(with respect to* $\mathcal{L} \cup \bar{A}$*-sentences). In other words, for any* $\mathcal{L} \cup \bar{A}$*-sentence* φ*, the following are equivalent.*

1. $\mathrm{B}\Sigma_{n+1}^0 \vdash \varphi$.
2. $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{I}\Sigma_{n-1}^0) \vdash (V^s \models \varphi)$.
3. $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0) \vdash (V^s \models \varphi)$.

Proof. We have proved $1 \to 2$ in Theorem 2.1, and $2 \to 3$ is trivial. We will show $\neg 1 \to \neg 3$. Let φ be an $\mathcal{L} \cup \bar{A}$-sentence such that $\mathrm{B}\Sigma_{n+1}^0 \nvdash \varphi$. Then, there exists a countable model $M_0 \models \mathrm{B}\Sigma_{n+1}^0 + \neg\varphi$. We can easily construct an elementary extension $M \supseteq M_0$ such that M is recursively saturated. By the previous lemma, there exists a $\Pi_n^{\bar{A}}$ elementary embedding $f : M \to M$. Then, the triple (M, M, f) is a model of $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0) + (V^s \models \neg\varphi)$. Thus, $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0) \nvdash (V^s \models \varphi)$. $\square$

Note that the previous corollary implies that $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0)$ (as a system of nonstandard second-order arithmetic) is a Π_1^1 conservative extension of $\mathrm{B}\Sigma_{n+1}^0$ as a second-order theory. In fact, $\mathrm{BNS} + \Pi_1^0\mathrm{TP} + (V^s, V^* \models \mathsf{WKL}_0 + \mathrm{B}\Sigma_2^0)$ is a (full second-order) conservative extension of $\mathsf{WKL}_0 + \mathrm{B}\Sigma_2^0$. In general, it is not known whether $\mathrm{BNS} + \Pi_n^0\mathrm{TP} + (V^s, V^* \models \mathrm{B}\Sigma_{n+1}^0)$ is a full second-order conservative extension of $\mathrm{B}\Sigma_{n+1}^0$ or not. Tin Lok Wong kindly informed the author that by Theorem B of [15], we have $\mathrm{BNS} + \Pi_n^0\mathrm{TP}$ is a full second-order conservative extension of $\mathrm{B}\Sigma_n^0$.

3 First jump control and $\Pi_1^0\mathrm{TP}$

In this section we will show that several conservation results over $\mathrm{B}\Sigma_2^0$ can be proved by combining some well-known first jump control arguments from the recursion theory, such as a version of the finite injury priority argument, with the transfer principle. In a model $\mathfrak{M} = (M, M^*, \mathrm{id}_M)$ of $\mathrm{BNS} + \Pi_1^0\mathrm{TP}$, we can use methods of nonstandard analysis by considering M as the standard universe and M^* as the nonstandard universe which satisfies the restricted transfer principle.

The following notion of resplendency plays a key role to use our constructions in Subsections 3.1 and 3.2 repeatedly.

Definition 3.1 (Resplendency). Let $\mathcal{L}_0$ be a first-order language, and let M be an $\mathcal{L}_0$-structure. Then, M is said to be *resplendent* if for every $\bar{a} \in M$, for every new unary predicate symbol A and for every $\mathcal{L}_0 \cup \{A\}$-formula $\psi(\bar{x}, A)$ such that $\mathrm{Th}(M; \mathcal{L}_0 \cup M) \cup \{\psi(\bar{a}, A)\}$ is consistent, M can be expanded into $\mathcal{L}_0 \cup \{A\}$-structure $(M; A^M)$ such that $(M; A^M) \models \psi(\bar{a}, A)$.

M is said to be *chronically resplendent* if for every $\bar{a} \in M$, for every new unary predicate symbol A and for every $\mathcal{L}_0 \cup \{A\}$-formula $\psi(\bar{x}, A)$ such that $\mathrm{Th}(M; \mathcal{L}_0 \cup M) \cup \{\psi(\bar{a}, A)\}$ is consistent, M can be expanded into $\mathcal{L}_0 \cup \{A\}$-structure $(M; A^M)$ such that $(M; A^M) \models \psi(\bar{a}, A)$ and $(M; A^M)$ is resplendent.

Theorem 3.1 (Chronical resplendency and recursive saturation [11, 14]). *Let* $\mathfrak{A}$ *be a first-order structure with a finite language. Then, the following are equivalent.*

1. $\mathfrak{A}$ *is recursively saturated.*

2. $\mathfrak{A}$ *is resplendent.*

3. $\mathfrak{A}$ *is chronically resplendent.*

Proof. See [11, Theorem 15.7, Corollary 15.13] and [14, Propositions 1.9.2, 1.9.3, 1.9.4]. $\square$

We next define the fix notation $\Phi^{\tau}_{e,s}$ to simulate recursive arguments using oracles in nonstandard arithmetic. Let $\bar{A}$ be a finite set of predicates. We fix a universal Π^0_1 formula $\Phi(e,x,\bar{X},Y) \equiv \forall n\Theta(n,x,\bar{X}[n],Y[n])$, *i.e.*, for any Π^0_1 formula $\varphi(x,\bar{X},Y)$, there exists $e<\omega$ such that $\mathrm{I}\Sigma^0_1 \vdash \Phi(e,x,\bar{X},Y) \leftrightarrow \varphi(x,\bar{X},Y)$.

Within $M=(M,\bar{A}^M) \models \mathrm{I}\Sigma^0_1$, given $s, e=(e',a)\in M$ and $\tau \in 2^{<M}$ such that $\mathrm{lh}(\tau)\geq s$, we write $\Phi^{\bar{A},\tau}_{e,s}\uparrow$ for $\forall n\leq s\Theta(e',a,\bar{A}^M[n],\tau\restriction n)$, and we write $\Phi^{\bar{A},\tau}_{e,s}\downarrow$ for $\neg(\Phi^{\bar{A},\tau}_{e,s}\uparrow)$. We often omit $\bar{A}$ and write $\Phi^{\tau}_{e,s}\uparrow$ if the oracle $\bar{A}$ is fixed. Then, for any Π^0_1 formula $\varphi(x,\bar{X},Y)$ for any $a\in M$ and for any $G^M\subseteq M$, there exists $e'<\omega$ such that $M^G=(M;G^M)\models\varphi(a,\bar{A}^M,G^M)\Leftrightarrow M^G\models \forall s\Phi^{G^M[s]}_{(e',a),s}\uparrow$.

The next lemma shows that controlling the first jump implies controlling Π_1 transfer principle.

Lemma 3.2. *Let $\bar{A}$ be a finite set of unary predicates, and let $M=(M,\bar{A}^M)$ and $M^*=(M^*,\bar{A}^{M^*})(\supseteq M)$ be $\mathcal{L}\cup\bar{A}$ structures such that $\mathfrak{M}=(M,M^*,\mathrm{id}_M)\models \mathrm{BNS}+\Pi^0_1\mathrm{TP}$. Let G be a new unary predicate, and let $G^{M^*}\subseteq M^*$, $G^M\subseteq M$ such that $G^M=M\cap G^{M^*}$. Define expansion of M and M^* as $M^G=(M;G^M)$ and $M^{*G}=(M^*;G^{M^*})$. Then, the following are equivalent.*

1. *For any $e\in M$, either $(\exists s\in M\ M^{*G}\models\Phi^{\bar{A},G^{M^*}[s]}_{e,s}\downarrow)$ or $(M^{*G}\models\forall s\Phi^{\bar{A},G^{M^*}[s]}_{e,s}\uparrow)$ holds.*

2. *$\mathfrak{M}^G=(M^G,M^{*G},\mathrm{id}_M)\models\mathrm{BNS}+\Pi^0_1\mathrm{TP}$ as an $\mathcal{L}^*\cup\bar{A}\cup\{G\}$-structure.*

Proof. In this proof, we omit $\bar{A}$ for Φ. The implication $2\to 1$ is trivial. Note that for any $e\in M$, the assertion $(\exists s\in M\ M^{*G}\models\Phi^{G^{M^*}[s]}_{e,s}\downarrow)$ is equivalent to $(M^G\models\exists s\Phi^{G^M[s]}_{e,s}\downarrow)$ since $G^{M^*}[s]=G^M[s]$ for any $s\in M$.

To show $1\to 2$, we only need to show that for any Π^0_1 formula $\forall n\varphi(n,x,\bar{X},Y)$ and $a\in M$, $M^G\models\forall n\varphi(n,a,\bar{A}^M,G^M)$ implies $M^{*G}\models\forall n\varphi(n,a,\bar{A}^{M^*},G^{M^*})$. Let $\forall n\varphi(n,x,\bar{X},Y)$ be a Π^0_1 formula, and let $a\in M$. Then, there exists $e'<\omega$ such that $\mathrm{I}\Sigma^0_1\vdash\forall n\varphi(n,x,\bar{X},Y)\leftrightarrow\forall s(\Phi^{\bar{X},Y[s]}_{(e',x),s}\uparrow)$. Let $e=(e',a)\in M$. Then $\exists s\in M\ M^{*G}\models\Phi^{G^{M^*}[s]}_{e,s}\downarrow$ means that $M^G\models\exists n\neg\varphi(n,a,\bar{A}^M,G^M)$, and $M^{*G}\models\forall s\Phi^{G^{M^*}[s]}_{e,s}\uparrow$ means that $M^{*G}\models\forall n\varphi(n,a,\bar{A}^{M^*},G^{M^*})$. This completes the proof. $\square$

Finally, we prepare a basic property for Δ^0_1 definable sets.

Lemma 3.3. *Let $\bar{A}$ be a finite set of unary predicates. Let $M=(M;\bar{A}^M)$ be a model of $\mathrm{B}\Sigma^0_n$, and let $B^M\in\Delta^0_1(M,\bar{A}^M)$. Then, $(M;\bar{A}^M\cup\{B^M\})$ is a model of $\mathrm{B}\Sigma^0_n$. Moreover, if $M=(M;\bar{A}^M)$ is recursively saturated, then $(M;\bar{A}^M\cup\{B^M\})$ is recursively saturated.*

Proof. We can easily show that for any $\Sigma^{\bar{A}\cup\{B\}}_1$ formula φ, there exists a $\Sigma^{\bar{A}}_1$ formula ψ such that $(M;\bar{A}^M\cup\{B^M\})\models\varphi\leftrightarrow\psi$. $\square$

3.1 Conservation proof for WKL

In this part, we will prove that $\mathsf{WKL}_0 + \mathrm{B}\Sigma^0_2$ is a Π^1_1 conservative extension of $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$. We will combine the proof of the low basis theorem for binary trees with the previous nonstandard arguments.

Lemma 3.4. *Let $\bar{A}$ be a finite set of unary predicates. Let $M = (M; \bar{A}^M)$ be a countable recursively saturated model of $\mathrm{B}\Sigma^0_2$ and let $T \in \bar{A}^M$ be an infinite binary tree in M. Then, there exists $G \subseteq M$ such that $(M; \bar{A} \cup \{G\})$ is recursively saturated and*

($\dagger$) $(M; \bar{A}^M \cup \{G\}) \models \mathrm{B}\Sigma^0_2 + (G$ *is a path of* $T)$.

Proof. By Theorem 3.1, if we find $G^M \subseteq M$ which satisfies ($\dagger$), then we can redefine G such that $(M; \bar{A}^M \cup \{G\})$ is recursively saturated and G satisfies ($\dagger$) again. Thus, we only need to construct $G^M \subseteq M$ which satisfies ($\dagger$).

By Theorem 2.4, take a $\Pi^{\bar{A}}_1$-elementary end extension $M^* = (M^*; \bar{A}^{M^*}) \models \mathrm{I}\Sigma^0_1$ of M. Then, $\mathfrak{M} = (M, M^*, \mathrm{id}_M) \models \mathrm{BNS} + \Pi^0_1\mathrm{TP}$. We write T^* for a set $\{a \in M^* \mid M^* \models a \in A_T\}$ where $A_T \in \bar{A}$ such that $T = {A_T}^M$. We will imitate the first jump control construction to take a path of T^* which is low within $\mathcal{M}^* = (M^*, \Delta^0_1(M^*; \bar{A}^{M^*})) \models \mathsf{RCA}_0$. In $\mathcal{M}^*$, we can construct a sequence $\langle \eta(e,s) \in 2 \mid e < s, s \in M^* \rangle$ which satisfies the following:

For any s,

– if there exists $e < s$ such that

$$\eta(e,s) = 0 \wedge \neg(\exists \tau \in T^* \; |\tau| = s \wedge \forall i \leq e(\eta(i,s) = 0 \rightarrow \Phi^{\tau}_{i,s}\uparrow)), \tag{1}$$

then, $e_0 = \min\{e < s \mid e \text{ satisfies (1)}\}$ and

$$\eta(i, s+1) = \begin{cases} \eta(i,s) & i < e_0 \\ 1 & i = e_0 \\ 0 & e_0 < i \leq s, \end{cases}$$

– otherwise,

$$\eta(i, s+1) = \begin{cases} \eta(i,s) & i < s \\ 0 & i = s. \end{cases}$$

Let $\eta^e_s := \langle \eta(i,s) \mid i \leq e \rangle \in 2^{e+1}$, and let $I_e := \{\eta \in 2^{e+1} \mid \exists s \in M^* \eta = \eta^e_s\}$. Define $\bar{\eta}^e := \max I_e$ as the lexicographic order on I_e, and $s_e := \min\{s \in M^* \mid \eta^e_s = \bar{\eta}^e\}$. Then, by $\Pi^0_1\mathrm{TP}$, $e \in M$ implies $s_e \in M$ since $\bar{\eta}^e \in M$ and $(\exists s \; \eta^e_s = \bar{\eta}^e)$ can be expressed by a $\Sigma^{\bar{A}}_1$ formula within M^*. We can easily check the following:

- $i \leq j$ implies $s_i \leq s_j$ and $\bar{\eta}^i \subseteq \bar{\eta}^j$.
- $s_e \leq t$ implies $\bar{\eta}^e = \eta^e_t$.
- $T^e = \{\tau \in T^* \mid \forall i \leq e(\eta(i, s_e) = 0 \rightarrow \Phi^{\tau}_{i,|\tau|}\uparrow)\}$ is infinite as a subset of M^*.
- $i \leq j$ implies $T_i \subseteq T_j$.
- If $\eta(e, s_e) = 1$, $\tau \in T_e$ and $|\tau| > s_e$, then $\Phi^{\tau \upharpoonright s_e}_{e,s_e}\downarrow$.

Let $\alpha \in M^* \setminus M$. By Harrington's forcing argument for $\mathcal{M}^*$, there exists $G^{M^*} \subseteq M^*$ such that $(M^*; \bar{A}^{M^*} \cup \{G^{M^*}\}) \models \mathrm{I}\Sigma^0_1$ and G^{M^*} is a path of T^α. Define $G^M := G^{M^*} \cap M$, and define $\mathcal{L} \cup \bar{A} \cup \{G\}$-structures M^G and M^{*G} as $M^G = (M; \bar{A}^M \cup \{G^M\})$ and $M^{*G} = (M^*; \bar{A}^{M^*} \cup \{G^{M^*}\})$. Then, for any $n \in M$, we have $G^M[n] = G^{M^*}[n]$ which is in $T^\alpha \cap M \subseteq T$. Thus, G^M is a path of T.

Finally, we show that $\mathfrak{M}^G = (M^G, M^{*G}, \mathrm{id}_M) \models \Pi^0_1\mathrm{TP}$, which implies $(M; \bar{A}^M \cup \{G^M\}) \models \mathrm{B}\Sigma^0_2$ by Theorem 2.1. Note that for any $e \in M$ and for any $n \in M^*$, we have $G^{M^*}[n] \in T_e$ since $\alpha > s_e \in M$ and $T_\alpha \subseteq T_e$. Then, for any $e \in M$, we have $\Phi^{G^{M^*}[s_e]}_{e,s_e} \downarrow$ if $\eta(e, s_e) = 1$, and we have $\Phi^{G^{M^*}[s]}_{e,s} \uparrow$ for any $s \in M^*$ if $\eta(e, s_e) = 0$. Thus, by Lemma 3.2, we have $\mathfrak{M}^G = (M^G, M^{*G}, \mathrm{id}_M) \models \Pi^0_1\mathrm{TP}$. This completes the proof. $\square$

Theorem 3.5. $\mathsf{WKL_0} + \mathrm{B}\Sigma^0_2$ *is a* Π^1_1 *conservative extension of* $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2$.

Proof. Let $\varphi(X)$ be an arithmetical formula such that $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2 \nvdash \forall X\varphi(X)$. Then there exists a countable recursively saturated model (M, S) and $A_0 \in S$ such that $(M, S) \models \mathsf{RCA_0} + \mathrm{B}\Sigma^0_2 + \neg\varphi(A_0)$. Starting from a first-order countable recursively saturated model $(M; A_0)$, we use Lemma 3.3 and Lemma 3.4 ω-times and construct a sequence $\{A_i \subseteq M\}_{i<\omega}$ such that for each $N < \omega$, $(M; \{A_i\}_{i<N})$ is recursively saturated and satisfies $\mathrm{B}\Sigma^0_2$ and $(M, \{A_i\}_{i<\omega}) \models \mathsf{WKL_0}$. Then, we have $(M, \{A_i\}_{i<\omega}) \models \mathsf{WKL_0} + \mathrm{B}\Sigma^0_2 + \neg\varphi(A_0)$, which means that $\mathsf{WKL_0} + \mathrm{B}\Sigma^0_2 \nvdash \forall X\varphi(X)$. $\square$

3.2 Conservation proof for COH

In this part, we will prove that $\mathsf{RCA_0} + \mathrm{COH} + \mathrm{B}\Sigma^0_2$ is a Π^1_1 conservative extension of $\mathsf{RCA_0} + \mathrm{B}\Sigma^0_2$. For this, we will imitate the first jump control construction for a low$_2$ cohesive set in [1] with the nonstandard arguments. (Jockusch and Stephan first constructed a low$_2$ cohesive set in [9]. See also [10].)

We first define the notion of cohesiveness. Let $R \subseteq M$ and $M = (M; R) \models \mathrm{I}\Sigma^0_1$. For $i \in M$, define $R_i = \{x \in M \mid (x, i) \in R\}$. For $X, Y \subseteq M$, we write $X \subseteq_{\mathrm{al}} Y$ if $M \models \exists x\ \forall y \geq x(y \in X \to y \in Y)$. Then, $G \subseteq M$ is said to be R-*cohesive* if $M \models \forall i(G \subseteq_{\mathrm{al}} R_i \vee G \subseteq_{\mathrm{al}} {R_i}^c)$. The axiom COH of second-order arithmetic asserts that $\forall X \exists Y(Y$ is X-cohesive).

Lemma 3.6. *Let $\bar{A}$ be a finite set of unary predicates. Let $M = (M; \bar{A}^M)$ be a countable recursively saturated model of* $\mathrm{B}\Sigma^0_2$ *and let $R \in \bar{A}^M$. Then, there exists $G \subseteq M$ such that $(M; \bar{A} \cup \{G\})$ is recursively saturated and*

(†) $(M; \bar{A}^M \cup \{G\}) \models \mathrm{B}\Sigma^0_2 + (G$ *is R-cohesive*$)$.

Proof. By Theorem 3.1, if we find $G^M \subseteq M$ which enjoys (†), then we can redefine G such that $(M; \bar{A}^M \cup \{G\})$ is recursively saturated and G enjoys (†) again. Thus, we only need to construct $G^M \subseteq M$ which enjoys (†).

By Theorem 2.4, take a $\Pi^{\bar{A}}_1$-elementary end extension $(M^*; \bar{A}^{M^*}) \models \mathrm{I}\Sigma^0_1$ of M. Then, $\mathfrak{M} = (M, M^*, \mathrm{id}_M) \models \mathrm{BNS} + \Pi^0_1\mathrm{TP}$. We write R^* for a set $\{a \in M^* \mid M^* \models a \in A_R\}$ where $A_R \in \bar{A}$ such that $R = {A_R}^M$. Note that $R_i = M \cap R^*_i$ for any $i \in M$. Take $\alpha \in M^* \setminus M$, and define a sequence $\sigma \in 2^\alpha$ as $\sigma(i) = 1 \leftrightarrow \alpha \in R^*_i$. For $\rho \in 2^{\leq\alpha}$, define R^*_ρ as

$$R^*_\rho = \left(\bigcap_{\rho(i)=1, i<|\rho|} R^*_i\right) \cap \left(\bigcap_{\rho(i)=0, i<|\rho|} {R^*_i}^c\right).$$

Then, for any $n \in M$, $R_{\sigma\restriction n} = R^*_{\sigma\restriction n} \cap M$ is unbounded in M. This can be proved by $\alpha \in R^*_{\sigma\restriction n}$ and Π^0_1TP. We will do the first jump control construction using a nonstandard oracle σ to take an R-cohesive set within $\mathcal{M}^* = (M^*, \Delta^0_1(M^*; \bar{A}^{M^*})) \models \mathsf{RCA}_0$. The idea of the following construction is essentially due to Theorem 4.3 of [1].

For $\tau \in 2^{<M^*}$, define $\mathrm{card}(\tau) := \mathrm{card}(\{i < |\tau| \mid \tau(i) = 1\})$. For $\tau, \tau' \in 2^{<M^*}$ and $X \subseteq M^*$, we write $\tau' \in (\tau, X)$ if $\tau' \subseteq \tau$ or $\tau' \supseteq \tau \wedge \forall i < |\tau'|(\tau'(i) = 0 \vee i < |\tau| \vee i \in X)$. In $\mathcal{M}^*$, we construct sequences $\langle \eta(e,s) \in 3 \mid e < s, s \in M^* \rangle$ and $\langle \tau(e,s) \in 2^{<s} \mid e < s, s \in M^* \rangle$ as follows:

(††) Let $\tau(-1,0) = \langle\rangle$. For each s, we do one of the following.

(I) If there exists $e < \min\{s, |\sigma|\}$ such that

$$\eta(e,s) = 1 \wedge \forall e' < e\ \eta(e',s) \neq 0 \wedge \exists \tau \in (\tau(e,s), R^*_{\sigma\restriction e+1})(|\tau| \leq s \wedge \Phi^\tau_{e,|\tau|}\downarrow), \quad (2)$$

then, let $e_0 = \min\{e < s \mid e \text{ satisfies } (2)\}$, $\tau_0 = \min\{\tau \in (\tau(e,s), R^*_{\sigma\restriction e+1}) \mid \Phi^\tau_{e,s}\downarrow\}$ and define

$$\eta(i,s+1) = \begin{cases} \eta(i,s) & i < e_0 \\ 2 & i = e_0 \\ 0 & e_0 < i \leq s, \end{cases} \qquad \tau(i,s+1) = \begin{cases} \tau(i,s) & i < e_0 \\ \tau_0 & e_0 \leq i \leq s. \end{cases}$$

(II) If (I) is false case and there exists $e < \min\{s, |\sigma|\}$ such that

$$\eta(e,s) = 0 \wedge \forall e' < e\ \eta(e',s) \neq 0 \wedge \exists \tau \in (\tau(e,s), R^*_{\sigma\restriction e+1})(|\tau| \leq s \wedge \mathrm{card}(\tau) \geq e), \quad (3)$$

then, let $e_0 = \min\{e < s \mid e \text{ satisfies } (3)\}$, $\tau_0 = \min\{\tau \in (\tau(e,s), R^*_{\sigma\restriction e+1}) \mid \mathrm{card}(\tau) \geq e\}$ and define

$$\eta(i,s+1) = \begin{cases} \eta(i,s) & i < e_0 \\ 1 & i = e_0 \\ 0 & e_0 < i \leq s, \end{cases} \qquad \tau(i,s+1) = \begin{cases} \tau(i,s) & i < e_0 \\ \tau_0 & e_0 \leq i \leq s. \end{cases}$$

(III) Otherwise, we define

$$\eta(i,s+1) = \begin{cases} \eta(i,s) & i < s \\ 0 & i = s, \end{cases} \qquad \tau(i,s+1) = \begin{cases} \tau(i,s) & i < s \\ \tau(s-1,s) & e_0 \leq i \leq s. \end{cases}$$

Let $\eta^e_s := \langle \eta(i,s) \mid i \leq e \rangle \in 3^{e+1}$, and let $I_e := \{\eta \in 3^{e+1} \mid \exists s \in M^* \eta = \eta^e_s\}$. Define $\bar{\eta}^e := \max I_e$ as the lexicographic order on I_e, $s_e := \min\{s \in M^* \mid \eta^e_s = \bar{\eta}^e\}$, and $\bar{\tau}^e := \tau(e, s_e)$.

We will show that $e \in M$ implies $s_e \in M$. Fix ${}^\star e \in M$. Define ${}^\star\sigma = \sigma \restriction {}^\star e + 1 \in M$, and do the construction (††) by replacing σ with ${}^\star\sigma$. Let ${}^\star\eta(i,s)$, ${}^\star\tau(i,s)$, ${}^\star s_i, \ldots$ be the results of this construction. By $\mathrm{I}\Sigma^0_0$ in $\mathcal{M}^*$, we can easily show that $\forall i \leq {}^\star e(\eta(i,s) = {}^\star\eta(i,s) \wedge \tau(i,s) = {}^\star\tau(i,s))$ for any $s \in M^*$. Thus, for $i \leq {}^\star e$, we have ${}^\star s_i = \min\{s \in M^* \mid {}^\star\eta^i_s = {}^\star\bar{\eta}^i = \bar{\eta}^i\} = s_i$. Then, by Π^0_1TP, $s_i = {}^\star s_i \in M$ for $i \leq {}^\star e$ since "$\exists s\ {}^\star\eta^i_s = {}^\star\bar{\eta}^i$" can be expressed by a $\Sigma^{\bar{A}}_1$ formula within M^*.

We can easily check the following:

- $|\bar{\tau}^e| \leq s_e$
- $i \leq j$ implies $s_i \leq s_j$, $\bar{\eta}^i \subseteq \bar{\eta}^j$ and $\bar{\tau}^j \subseteq \bar{\tau}^j$.

- $s_e \leq t$ implies $\bar{\eta}^e = \eta_t^e$ and $\bar{\tau}^e = \tau(e,t)$.
- If $\eta(e,s_e) \geq 1$, then $\mathrm{card}(\bar{\tau}^e) \geq e$.
- If $\eta(e,s_e) = 2$ and $i \geq e$, then $\Phi_{e,s_e}^{\bar{\tau}^i \restriction s_e} \downarrow$.
- If $\eta(e,s_e) = 1$, then $\forall \tau' \in (\bar{\tau}^e, R^*_{\sigma \restriction e+1})\ \Phi_{e,|\tau'|}^{\tau'} \uparrow$.

Let $\beta = \min\{e \mid \eta(e,s_e) = 0\} \cup \{\alpha\}$. We will show that $\beta \in M^* \setminus M$ by way of contradiction. Assume $\beta \in M$. Then, we have $|\bar{\tau}^\beta| \leq s_\beta \in M$, $\mathrm{card}(\bar{\tau}^\beta) \geq \mathrm{card}(\bar{\tau}^{\beta-1}) \geq \beta - 1$, and $\forall \tau' \in (\bar{\tau}^\beta, R^*_{\sigma\restriction\beta+1})\,\mathrm{card}(\tau') < \beta$. Therefore, for any $n \in R^*_{\sigma\restriction\beta+1}$, we have $n \leq s_\beta$. This contradicts the fact that $M \cap R^*_{\sigma\restriction\beta+1}$ is unbounded in M.

Finally, we will define $\mathcal{L} \cup \bar{A} \cup \{G\}$-structures $M^G = (M; \bar{A}^M \cup \{G^M\})$ and $M^{*G} = (M^*; \bar{A}^{M^*} \cup \{G^{M^*}\})$, and show that G^M is R-cohesive and $\mathfrak{M}^G = (M^G, M^{*G}, \mathrm{id}_M) \models \Pi^0_1\mathrm{TP}$. Let $G^{M^*} = \{n \in M^* \mid n < |\bar{\tau}^\beta| \wedge \bar{\tau}^\beta(n) = 1\}$, and let $G^M = G^{M^*} \cap M$. Then, G^M is unbounded in M since $G^M[s_e] \supseteq \bar{\tau}^e$ and $\mathrm{card}(\bar{\tau}^e) \geq e$ for any $e \in M$. For any $e \in M$ and for any $t \in M^*$ such that $t \geq s_e$, we have $G^{M^*}[t] \in (\bar{\tau}^e, R^*_{\sigma\restriction e+1})$ since $\bar{\tau}^\beta \in (\bar{\tau}^e, R^*_{\sigma\restriction e+1})$. This implies $M^G \models G^M \subseteq_{\mathrm{al}} R_i \vee G^M \subseteq_{\mathrm{al}} R_i{}^c$ for any $e \in M$. This means that G^M is R-cohesive in M^G, and we also have $M^{*G} \models \forall s \Phi_{e,s}^{G^{M^*}[s]} \uparrow$ for any $e \in M$ such that $\eta(e,s_e) = 1$. On the other hand, if $e \in M$ and $\eta(e,s_e) = 2$, then $M^{*G} \models \forall s (\Phi_{e,s_e}^{G^{M^*}[s_e]} \downarrow)$. Thus, we have $\mathfrak{M}^G \models \Pi^0_1\mathrm{TP}$ by Theorem 3.2, which implies $(M; \bar{A}^M \cup \{G^M\}) \models \mathrm{B}\Sigma^0_2$ by Theorem 2.1. This completes the proof. □

Theorem 3.7. $\mathsf{RCA}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_2$ *is a* Π^1_1 *conservative extension of* $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2$.

Proof. Let $\varphi(X)$ be an arithmetical formula such that $\mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 \not\vdash \forall X \varphi(X)$. Then there exists a countable recursively saturated model (M,S) and $A_0 \in S$ such that $(M,S) \models \mathsf{RCA}_0 + \mathrm{B}\Sigma^0_2 + \neg\varphi(A_0)$. Starting from a first-order countable recursively saturated model $(M; A_0)$, we use Lemma 3.3 and Lemma 3.6 ω-times and construct a sequence $\{A_i \subseteq M\}_{i<\omega}$ such that for each $N < \omega$, $(M; \{A_i\}_{i<N})$ is recursively saturated and satisfies $\mathrm{B}\Sigma^0_2$ and $(M, \{A_i\}_{i<\omega}) \models \mathsf{RCA}_0 + \mathrm{COH}$. Then, we have $(M, \{A_i\}_{i<\omega}) \models \mathsf{RCA}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_2 + \neg\varphi(A_0)$, which means that $\mathsf{RCA}_0 + \mathrm{COH} + \mathrm{B}\Sigma^0_2 \not\vdash \forall X \varphi(X)$. □

Acknowledgments

The author would like to thank Dr. Sam Sanders and Dr. Tin Lok Wong for useful comments. This work was supported by Grant-in-Aid for Young Scientists (B) 21740061.

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