数理解析研究所講究録
第1730巻 2011年 18-27

Uniqueness of Stationary Solutions to a One-dimensional Bipolar Hydrodynamic Model of Semiconductors

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Abstract

We consider a one-dimensional bipolar hydrodynamic model of semiconductors. We are concerned with the uniqueness of stationary solutions in particular. The most difficult point is to obtain the bounded estimate of the solutions.

1 Introduction

We study the stationary problem of bipolar hydrodynamic model of semiconductors on the open interval $I=(0,1)$.

$$\begin{cases} j_x = 0, \quad \left(\dfrac{j^2}{n}+n\right)_x = n\phi_x - j, \\ k_x = 0, \quad \left(\dfrac{k^2}{h}+h\right)_x = -h\phi_x - k, \\ \phi_{xx} = n - h - D(x), \end{cases} \tag{1.1}$$

where h and n are the density of the electron and the hole respectively. j and k are the current density of the electron and the hole respectively. ϕ denotes the electrostatic potential. The doping profile $D \in C(\bar{I})$ is a known function, which represents the density of impurities in semiconductors.

Then we consider the boundary problem (1.1) and the boundary conditions

$$\begin{aligned} &n|_{x=0} = n|_{x=1} = n_d\ (>0), \quad h|_{x=0} = h|_{x=1} = h_d\ (>0), \\ &\phi|_{x=0} = 0,\ \phi|_{x=1} = \phi_r\ (>0). \end{aligned} \tag{1.2}$$

From the physical point of view, condition (1.2) represents Ohmic contacts (see [M], [S1] and [S2]).

n-MOS FET.

The problem (1.1)–(1.2) represents the motion of the electron and the hole in MOS FET (metal-oxide-semiconductor field-effect transistor). We introduce n-MOS FET in particular. N-MOS FET consists of the body, the source and the drain (see Figure 1). The body consists of p region, where much hole and little electron exist. The source and the drain consist of n^+ region, where much more electron and little more hole exist. With sufficient gate voltage, the hole of the body is driven away from the gate, forming an n-channel at the interface between the p region and the oxide (SiO_2). This conducting channel extends between the source and the drain, and current is conducted through it when a voltage is applied between the source and the drain. In Figure 1, n_d, h_d, and ϕ_r represents the density of the electron in the source and the drain, the density of the hole in the source and the drain, and the electrostatic potential at the drain respectively. Under this boundary condition, we consider the motion of the electron and the hole on n-channel.

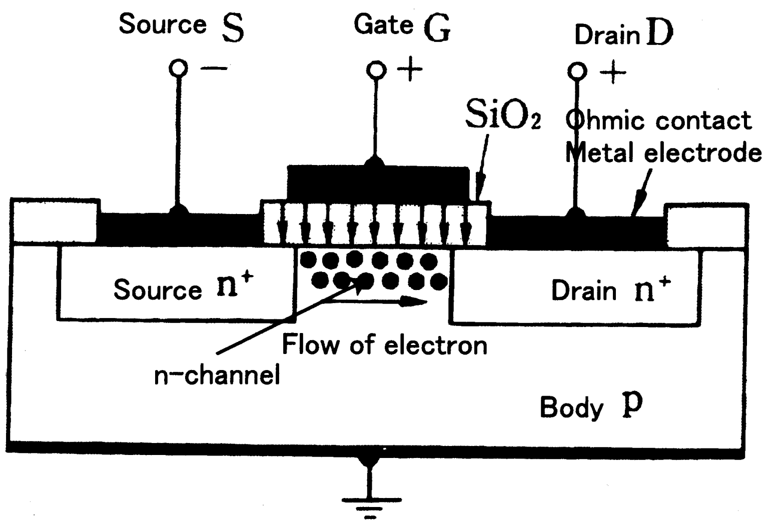

Figure 1: n-MOS FET

Related results. The hydrodynamic model of semiconductors was introduced by Bløtekjær [B]. In engineering, this model is used for practical applications, such as the simulation of the flash memory and semiconductors with high bias.

We first introduce mathematical results for the unipolar hydrodynamics model (i.e., h =const. and $k = 0$ in (1.1).). The pioneer work in this field is Degond and Markowich [DM]. They investigated the existence and the uniqueness of stationary solutions. Subsequently, Nishibata and Suzuki [NS] proved the existence and the asymptotic stability of stationary solutions.

Next, we survey the bipolar case (1.1). Hattori and Zhu [HZ] discussed the stability of stationary solutions for the Cauchy problem. On the other hand, Li and Zhou [LZ] studied the existence and some limits of stationary solutions to a one-dimensional Dirichlet problem. Moreover, Gasser, Hsiao and Li [GHL] considered the asymptotic stability of classical solutions for the Cauchy problem. However, the doping profile is not considered in [LZ] and [GHL].

Now, in practical applications such as the simulation of n-MOS FET, the hydrodynamic model (1.1) is treated under the following conditions (see [M] and [S1]):

(C1) (1.1) is supplemented by the Dirichlet boundary conditions, such as (1.2).

(C2) The doping profile $D(x)$ has large derivative, that is, $D(x)$ is not flat.

In addition to the above papers, there are other mathematical papers for the bipolar case. Unfortunately, few results satisfy (C1) and (C2). In this paper, we shall consider a solution for the boundary condition (1.2) and an arbitrary doping profile.

From (1.1), we have

$$\begin{cases} j = \dfrac{\phi_r}{\displaystyle\int_0^1 \frac{1}{n} dx}, & \left(\dfrac{j^2}{2n^2} + \log n\right)_{xx} = n - h - D(x) + j\dfrac{n_x}{n^2}, \\ k = -\dfrac{\phi_r}{\displaystyle\int_0^1 \frac{1}{h} dx}, & \left(\dfrac{k^2}{2h^2} + \log h\right)_{xx} = h - n + D(x) + k\dfrac{h_x}{h^2}. \end{cases} \tag{1.3}$$

Then our main theorem is as follows.

Theorem 1 (Main Theorem[T]). *If ϕ_r is small enough, the boundary value problem* (1.2) *and* (1.3) *has a unique classical solution.*

In the present paper, we devote to proving the uniqueness. The proof of existence can be found in [T].

2 Uniqueness of classical solutions

In this section, we consider the uniqueness of classical solutions. To do this, we first introduce the following notation:

$$D_M := \max_{x \in \bar{I}} |D(x)|, \; a := \max\{3, D_M\},$$
$$C_m := \min\{n_d, h_d\}, \; C_M := \max\{n_d, h_d\}.$$

We first assume the following proposition.

Proposition 2. *If ϕ_r is small enough, classical solutions to the boundary value problem* (1.2) *and* (1.3) *satisfy*

$$C_m e^{a(x^2-1)} \leq n(x) \leq C_M e^{a(1-x^2)}, \quad C_m e^{a(x^2-1)} \leq h(x) \leq C_M e^{a(1-x^2)}. \quad (2.1)$$

The proof of Proposition 2 is postponed in the next section. Using this proposition, we prove the uniqueness of classical solutions.

Theorem 3. *If ϕ_r is small enough, the classical solution to the boundary value problem* (1.2)–(1.3) *is unique.*

Proof. In view of (1.3) and Proposition 2, choosing ϕ_r small enough, j and k are also small. Therefore, for simplicity, we consider the case where $j = k = 0$.

Now we assume (n_1, h_1) and (n_2, h_2) are classical solutions to the boundary problem (1.2)–(1.3).

Then, from (1.3), we have

$$(\log n_2 - \log n_1)_{xx} = (n_2 - n_1) - (h_2 - h_1), \quad (2.2)$$
$$(\log h_2 - \log h_1)_{xx} = (h_2 - h_1) - (n_2 - n_1). \quad (2.3)$$

Set $m := \log n_2 - \log n_1$ and $g := \log h_2 - \log h_1$.

Integrating $((2.2)+(2.3))(m+g)$ from 0 to 1, we have $\|(m+g)_x\|=0$, where $\|\cdot\|$ is the L^2-norm. This yields

$$m+g=0. \tag{2.4}$$

Then, from (2.4), we have

$$\begin{aligned} m-g =&2m,\\ (n_2-n_1)-(h_2-h_1) =&m\int_0^1 e^{\log n_1+\theta m}d\theta - g\int_0^1 e^{\log h_1+\theta g}d\theta\\ =&\left(\int_0^1 e^{\log n_1+\theta m}d\theta + \int_0^1 e^{\log h_1+\theta g}d\theta\right)m. \end{aligned}$$

Thus we obtain

$$\int_0^1 (n_2-n_1-h_2+h_1)(m-g)dx \geq 0. \tag{2.5}$$

Computing $\int_0^1 (2.2)\times m dx + \int_0^1 (2.3)\times g dx$, from (2.5), we have

$$\|m_x\|^2+\|g_x\|^2 = -\int_0^1 (n_2-n_1-h_2+h_1)(m-g)dx \leq 0.$$

Therefore, we find $n_2=n_1$ and $h_2=h_1$. We can complete the proof. □

3 Bounded estimate of classical solutions

The aim in this section is to prove Proposition 2.

Separating three parts, we prove Proposition 2. First, we show that solutions are positive.

Lemma 4. *If $n(x)$ and $h(x)$ are classical solutions to the boundary value problem* (1.1)–(1.2), *$n(x)$ and $h(x)$ are positive.*

Proof. We prove only $n(x)$ is positive. Since solutions are positive at the boundary and continuous, it suffices to prove $n(x)\neq 0$ for any $x\in I$.

First, we notice that n is positive near $x=0$. If $j=0$, from $(1.1)_1$, we have $(\log n)_x=\phi_x$ near $x=0$. Integrating the equality from 0 to x, we have

$$\log n(x)=\phi(x)+\log n_d.$$

Since the right-hand side is bounded on I, $n(x) \neq 0$ for any $x \in I$.

Next, we consider the case where $j \neq 0$. Integrating $(1.1)_1$ from 0 to x, we have

$$\frac{j^2}{n(x)} = -n(x) + \frac{j^2}{(n_d)^2} + n_d + \int_0^x (n\phi_y - j)dy$$

near $x = 0$. Then we similarly find that $n(x) \neq 0$ for any $x \in I$. □

Second, we prove that any solution is bounded from above.

Lemma 5. *If ϕ_r is small enough, classical solutions to the boundary value problem* (1.1)–(1.2) *satisfy*

$$n(x) \leq C_M e^{a(1-x^2)}, \quad h(x) \leq C_M e^{a(1-x^2)}. \tag{3.1}$$

Proof. We prove only $(3.1)_1$. We can handle the other inequality similarly.

We set $\tilde{n} = e^{ax^2} n$ and $\tilde{h} = e^{ax^2} h$. Then, from $(1.3)_1$, we have

$$\left(\frac{1}{n} - \frac{j^2}{n^3}\right) e^{-ax^2} \tilde{n}_{xx} + (\cdots)\, \tilde{n}_x - 2a \left(1 - \frac{j^2}{n^2}\right) + 8a^2x^2 \frac{j^2}{n^2} + 2ax\frac{j}{n} \\ + D(x) = e^{-ax^2}(\tilde{n} - \tilde{h}). \tag{3.2}$$

Then we consider the case where $\tilde{n}$ attains the maximum at a certain point $x_1 \in I$. In this case, we notice that $\tilde{n}_x(x_1) = 0, \tilde{n}_{xx}(x_1) \leq 0$ and

$$\frac{j}{n(x_1)} = \frac{1}{n(x_1)} \frac{\phi_r}{\displaystyle\int_0^1 \frac{1}{n} dx} \leq \frac{1}{\tilde{n}(x_1) e^{-a(x_1)^2}} \frac{\phi_r}{\displaystyle\int_0^1 \frac{1}{\tilde{n}(x_1)} dx} \leq \phi_r e^a. \tag{3.3}$$

Choosing ϕ_r small enough such that $\phi_r \leq e^{-2a}$, we have $1 - j^2/(n(x_1))^2 < 0$, which means that (3.2) is subsonic at $x = x_1$. Moreover, at $x = x_1$, we have

$$-2a\left(1 - \frac{j^2}{n^2}\right) + 8a^2x^2\frac{j^2}{n^2} + 2ax\frac{j}{n} + D(x) \\ \leq -2a + 2ae^{-2a} + 8a^2e^{-2a} + 2ae^{-a} + D_M < 0.$$

Here we recall the definition of a.

Therefore, it follows from (3.2) that

$$\tilde{n}(x_1) < \tilde{h}(x_1). \tag{3.4}$$

If $\tilde{h}$ attains the maximum at a certain point $x = x_2 \in I$, we similarly find

$$\tilde{h}(x_2) < \tilde{n}(x_2). \tag{3.5}$$

Now we investigate the maximum of $\tilde{n}$ and $\tilde{h}$. Then the following four cases may occur.

Case 1 $\tilde{n}$ and $\tilde{h}$ attain the maximum at the boundary $x = 0, 1$.

Case 2 $\tilde{n}$ attains the maximum at the boundary $x = 0, 1$. $\tilde{h}$ attains the maximum at an interior point $x = x_2$.

Case 3 $\tilde{n}$ attains the maximum at an interior point $x = x_1$. $\tilde{h}$ attains the maximum at the boundary $x = 0, 1$.

Case 4 $\tilde{n}$ and $\tilde{h}$ attain the maximum at an interior point $x = x_1$ and $x = x_2$ respectively.

Case 1 and **Case 3**. In these cases, we notice that $\tilde{n}|_{x=0} = n_d$, $\tilde{n}|_{x=1} = n_d e^a$. Since $\tilde{n}(x) \leq n_d e^a$, we have $n(x) \leq n_d e^{a(1-x^2)} \leq C_M e^{a(1-x^2)}$.

Case 2. In this case, we notice that $\tilde{h}|_{x=0} = h_d$ and $\tilde{h}|_{x=1} = h_d e^a$. It follows from (3.4) that $\tilde{n}(x) \leq \tilde{n}(x_1) < \tilde{h}(x_1) \leq h_d e^a$. Therefore, we have $n(x) \leq h_d e^{a(1-x^2)} \leq C_M e^{a(1-x^2)}$.

Case 4. This case cannot occur. In fact, from (3.4) and (3.5), we have $\tilde{n}(x_1) < \tilde{h}(x_1) \leq \tilde{h}(x_2) < \tilde{n}(x_2)$. However, this contradicts the fact that $\tilde{n}$ attains the maximum at $x = x_1$.

We thus conclude $(3.1)_1$. □

From Lemma 5, we have the following.

Corollary 6. *If ϕ_r is small enough,*

$$j \leq C_M \phi_r e^a, \quad |\phi_x| \leq C, \tag{3.6}$$

where C depends only on n_d, h_d and D_M.

To complete the proof of Proposition 2, the remainder is to obtain the lower bound of n and h. This estimate is not similar to the above lemma, because (3.3) does not hold in this case. We first prove the following.

Lemma 7. *If ϕ_r is small enough, there exists a positive constant C such that classical solutions to the boundary value problem* (1.1) *and* (1.2) *satisfy*

$$C < n(x), \quad C < h(x), \tag{3.7}$$

where C depends only on n_d, h_d and D_M.

Proof. We prove only $(3.7)_1$. In this proof, we use the same letter C to denote constants depending only on n_d, h_d and D_M.

To do this, we estimate Riemann invariants. From (1.1), we have

$$\left(\frac{j}{n} + \log n\right)_x = \frac{\phi_x - \dfrac{j}{n}}{1 + \dfrac{j}{n}}, \quad \left(\frac{j}{n} - \log n\right)_x = \frac{-\phi_x + \dfrac{j}{n}}{1 - \dfrac{j}{n}}. \tag{3.8}$$

Set $S_1 := \{x \in \bar{I}; j/n \le 2\}$ and $S_2 := \bar{I} \setminus S_1$.

If $x \in S_1$, integrating $(3.8)_1$ from 0 to x, we have

$$\frac{j}{n(x)} + \log n(x) - \frac{j}{n_d} - \log n_d = \int_0^x \frac{\phi_y - \dfrac{j}{n}}{1 + \dfrac{j}{n}} dy.$$

Since $0 < j/n < 2$ in this case, from Corollary 6, there exists C such that $\log n(x) \ge -C$, which yields $(3.7)_1$.

Next, we consider the case where $x \in S_2$. Set $x_l := \sup\{y; j/n(y) \le 2, y < x\}$. Choosing ϕ_r small enough, from Corollary 6, we find j/n is small enough at $x = 0$. Since $x = 0 \in S_1$ and n is continuous, we find $x_l > 0$ and $j/n(x_l) = 2$. Then, integrating (3.8) from x_l to x, we have

$$\frac{j}{n(x)} + \log n(x) - \frac{j}{n(x_l)} - \log n(x_l) = \int_{x_l}^x \frac{\phi_y - \dfrac{j}{n}}{1 + \dfrac{j}{n}} dy,$$

$$\frac{j}{n(x)} - \log n(x) - \frac{j}{n(x_l)} + \log n(x_l) = \int_{x_l}^x \frac{-\phi_y + \dfrac{j}{n}}{1 - \dfrac{j}{n}} dy.$$

Since $x_l \in S_1$, from Corollary 6, we have

$$\frac{j}{n(x)} + \log n(x) \ge -C, \ \frac{j}{n(x)} - \log n(x) \le C,$$

which yield $(3.7)_1$. □

Finally we prove Proposition 2.

Proof of Proposition 2. We deduce the bounded estimate from below as follows. We set $\tilde{n} = e^{-ax^2}n$ and $\tilde{h} = e^{-ax^2}h$. Then, if ϕ_r is small enough, from Lemma 7, j/n is also small enough. Then, investigating the minimum of $\tilde{n}$ and $\tilde{h}$ in a similar manner to Lemma 5, we can complete the proof. □

4 Open problem

Finally, we introduce an open problem. We consider the bipolar hydrodynamic model with the SRH (Shockley-Read-Hall) term $R(n,h)$,

$$\begin{cases} j_x = -R(n,h), & \left(\dfrac{j^2}{n} + n\right)_x = n\phi_x - j, \\ k_x = -R(n,h), & \left(\dfrac{k^2}{h} + h\right)_x = -h\phi_x - k, \\ \phi_{xx} = n - h - D(x), \end{cases} \tag{4.1}$$

where $R(n,h) = Q(n,h)(nh-1)$ have the form

$$R(n,h) = Q(n,h)(nh-1)$$

and Q is a bounded and locally Lipschitz continuous function on $R_+ \times R_+$. The SRH term represents the recombination-generation of the electron and the hole (see [M, Section 2.2] and [S1]) and is peculiar to the bipolar case. Moreover, the boundary data satisfy the following thermal equilibrium condition

$$n_d h_d = 1.$$

As long as the author knows, the existence of solutions to the boundary problem (4.1) and (1.2) has not been obtained yet. It is one of the difficult points that j and k are not constants.

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