数理解析研究所講究録
第 1730 巻 2011 年 28-44

On some variational problems relating to vortices[1]

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Abstract

The purpose of this note is to present recent results concerning the equilibrium vortices from the variational points of view. We also revisit the author's unpublished old note in appendix, which is one of the origin of the author's works mentioned here.

1 Introduction

This note is concerned with the recent studies around the following functional sometimes called the *free energy* functional of the mean field of vortices:

$$J_{\beta}(\psi)=\frac{1}{2}\int_{\Omega}|\nabla\psi|^{2}dx-\beta\log\int_{\Omega}e^{\psi}\,dx, \qquad \psi\in H_{0}^{1}(\Omega) \tag{1.1}$$

where $\Omega\subset\mathbf{R}^{2}$ is a bounded domain with smooth boundary $\partial\Omega$ and β is a positive parameter.

The functional $J_{\beta}(\psi)$ was first introduced in [3] and [11] independently to study the mean field limit of vortices in Ω based on the theory of Onsager [19] for 2-dimensional turbulence. According to their argument, the minimizers of $J_{\beta}(\psi)$ represent (the stream function of) the mean field of infinitely many vortices of *one kind*. It is easy to see that the classical Trudinger-Moser inequality ensures that

$$\inf_{\psi\in H_{0}^{1}(\Omega)}J_{\beta}(\psi)>-\infty \quad \text{if } \beta\leqq 8\pi \tag{1.2}$$

and consequently a minimizer exists if $\beta<8\pi$.

Recently we considered two kinds of similar functionals to $J_{\beta}(\psi)$:

$$J_{\beta,\mathcal{P}}(\psi)=\frac{1}{2}\int_{M}|\nabla_{g}\psi|^{2}\,dv_{g}-\beta\int_{[-1,1]}\log\left(\int_{M}e^{\alpha\psi}dv_{g}\right)\mathcal{P}(d\alpha),$$

$$F_{\lambda}(\psi)=\frac{1}{2}\int_{\Omega}|\nabla\psi|^{2}dx-\lambda\int_{\Omega}e^{\psi}\,dx,$$

[1]This work is supported by Grant-in-Aid for Scientific Research (No.22540231), Japan Society for the Promotion of Science.
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where $\mathcal{P}$ is a Borel probability measure on the interval $[-1,1](\ni \alpha)$ and λ is another positive parameter. For some technical reasons, we consider $J_{\beta,\mathcal{P}}(\psi)$ on

$$\mathcal{E} = \{\psi \in H^1(M) \mid \int_M \psi = 0\},$$

where (M,g) is a 2-dimensional Riemannian manifold without boundary.

In the special case that $\mathcal{P}$ concentrates on $\alpha = 1 \in [-1,1]$, that is, $\mathcal{P} = \delta_1$, the generalized functional $J_{\beta,\mathcal{P}}(\psi)$ reduces to $J_\beta(\psi)$ on $\mathcal{E}$. It is considered that $\mathcal{P}$ gives a relative circulation number density of vortices [21] and $\mathcal{P} = \delta_1$ means that all vortices have the same circulations. Concerning $J_{\beta,\mathcal{P}}(\psi)$, we get similar results to (1.2):

Theorem 1.1 ([16]).

$$\inf_{\psi\in\mathcal{E}} J_{\beta,\mathcal{P}}(\psi) > -\infty \quad \text{if } \beta \le \overline{\beta} := \frac{8\pi}{\max\{\int_{I_+}\alpha^2\mathcal{P}(d\alpha), \int_{I_-}\alpha^2\mathcal{P}(d\alpha)\}}, \tag{1.3}$$

where $I_+ = (0,1]$, $I_- = [-1,0)$. *Consequently,* $J_{\beta,\mathcal{P}}(\psi)$ *has a minimizer if* $\beta < \overline{\beta}$.

The proof is based on our previous result [18] corresponding to the case $\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}$. However, due to the presence of the general probability measure $\mathcal{P}$, we need to develop new argument considering the measure over the product space $I \times \Omega$, see section 3. Moreover it should be remarked that the optimality of the inequality (1.3) is not known for every probability measure $\mathcal{P}$ [16], though (1.3) is optimal when $\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}$ [18].

Concerning $F_\lambda(\psi)$, the similarity to $J_\beta(\psi)$ appears when we consider the critical points of them. Indeed, the Euler-Lagrange equations of $J_\beta(\psi)$ and $F_\lambda(\psi)$ are as follows:

$$-\Delta\psi = \beta\frac{e^\psi}{\int_\Omega e^\psi\,dx} \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega. \tag{1.4}$$

$$-\Delta\psi = \lambda e^\psi \quad \text{in } \Omega, \qquad \psi = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Therefore each critical point of (1.4) is linked to that of (1.5) under the relation $\beta/\int_\Omega e^\psi dx = \lambda$.

The behavior of the sequence of solutions of (1.4) and (1.5) are now well studied by several authors. Especially based on the argument in [2] (see also [17]), we are able to get a subsequence satisfying $\int_\Omega e^{\psi_n}\,dx \longrightarrow \infty$ if $\{(\psi_n,\beta_n)\}$ is a sequence of solutions of (1.4) satisfying that $\{\psi_n\}$ is unbounded in $L^\infty(\Omega)$ and $\{\beta_n\}$ is bounded. Therefore, the behaviors of unbounded sequence of solutions of (1.4) reduce to those of (1.5) satisfying $\lambda_n = \beta_n / \int_\Omega e^{\psi_n}\, dx \longrightarrow 0$. For such a sequence of solutions we know the following behaviors:

Fact 1.2 ([15]). Let $\{(\psi_n, \lambda_n)\}$ be a sequence of solutions for (1.5) satisfying $\|\psi_n\|_{L^\infty(\Omega)} \longrightarrow \infty$ and $\lambda_n \longrightarrow 0$. Then, taking subsequence if necessary, $\beta_n = \lambda_n \int_\Omega e^{\psi_n}\, dx \longrightarrow 8\pi m$ for some positive integer m and $\{\psi_n\}$ blows-up at m-points, that is, there is a blow-up set $\mathscr{S} = \{\kappa_1, \dots, \kappa_m\} \subset \Omega$ of distinct m-points such that $\|\psi_n\|_{L^\infty(\omega)} = O(1)$ for every $\omega \subset\subset \overline{\Omega} \setminus \mathscr{S}$ and $\{\psi_n(x)\}$ have a limit for $x \in \overline{\Omega} \setminus \mathscr{S}$ while $\psi_n|_{\mathscr{S}} \longrightarrow +\infty$. Moreover the limiting function ψ_∞ has the form

$$\psi_\infty(x) = 8\pi \sum_{j=1}^{m} G(x, \kappa_j), \tag{1.6}$$

where $G(x,y)$ is the Green function of $-\Delta$ under the Dirichlet condition. Furthermore, $(\kappa_1, \cdots, \kappa_m) \in \Omega^m$ is a critical point of

$$H^m(x_1, \dots, x_m) = \frac{1}{2} \sum_{i=1}^{m} R(x_i) + \frac{1}{2} \sum_{\substack{1 \le i,j \le m \\ i \ne j}} G(x_i, x_j),$$

where $R(x) = K(x,x)$ is the Robin function of Ω and $K(x,y) = G(x,y) - \frac{1}{2\pi} \log |x-y|^{-1}$.

The next result shows a deeper link between H^m and $\{\psi_n\}$:

Theorem 1.3 ([9]). *Assume the situation of Fact 1.2 and suppose that $\mathscr{S}$ is a non-degenerate critical point of H^m. Then ψ_n is a non-degenerate critical point of F_{λ_n} for n large enough.*

The above theorem has been already established by Gladiali and Grossi [7] for the case $m = 1$. There, the conclusion is obtained using a contradiction argument and some "Pohozaev type" identities. If $m > 1$ the problem is much more complicated and we need new ideas to derive the claim.

It should be remarked that H^m appeared in Fact 1.2 and Theorem 1.3 is nothing but the Hamiltonian of vortices of one kind, see Section 2. Therefore we are able to observe some recursive structures of vortices and its mean fields, that is, *the mean fields generated by vortices of one kind reduce to vortices of one kind.*

In the following sections, we observe some basic facts of vortices and add some comments on the above theorems. We also see a classical problem relating to them in Appendix.

2 Short note on vortices

The function H^m is rather popular in fluid mechanics. This is the Hamiltonian of *vortices* in two-dimensional incompressible non-viscous fluid.

Formally speaking, N-vortices is a set $\{(x_j(t), \Gamma_j)\}_{j=1,\cdots,N}$ $(\subset \Omega\times(\boldsymbol{R}\backslash\{0\}))$ that forms a vorticity field $\omega(x,t) = \sum_{j=1}^N \Gamma_j \delta_{x_j(t)}$ satisfying the Euler vorticity equation

$$\text{(2.1)} \qquad \frac{\partial \omega}{\partial t} + (\boldsymbol{v}\cdot\nabla)\,\omega = 0,$$

where $\boldsymbol{v} = \nabla^\perp \int_\Omega G(x,y)\omega(y,t)dy$ is the velocity field of the fluid determined by the vorticity field $\omega(x,t)$. Here $\nabla^\perp = \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)$ and we assume that Ω is simply connected for simplicity. δ_p is the Dirac measure supported at a point $p\,(\in\Omega)$ and Γ_j is the intensity (circulation) of the vortex at $x_j(t)$. From the Kelvin circulation law, the intensity Γ_j is considered to be conserved. From other several physical considerations, the form $\sum_{j=1}^N \Gamma_j\delta_{x_j(t)}$ is considered to be preserved during the time evolution.

It is true that the model "vortices" made many success to understand the motion of real fluid, but it should be noticed that the velocity field $\boldsymbol{v} = \sum_{j=1}^N \Gamma_j \nabla^\perp G(x, x_j(t))$ determined by the vorticity field $\sum_{j=1}^N \Gamma_j\delta_{x_j(t)}$ makes the kinetic energy $\frac{1}{2}\int_\Omega |\boldsymbol{v}|^2 dx$ infinite. Moreover it is difficult to understand, even in the sense of distributions, how it satisfies the vorticity equation (2.1). Nevertheless the motion of vortices have been "known" from 19th century. Indeed, they are considered to move according to the following equations:

$$\text{(2.2)} \qquad \Gamma_i \frac{dx_i}{dt} = \nabla_i^\perp H^{N,\boldsymbol{\Gamma}}(x_1,\cdots,x_N)\left(= \left(\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,2}}, -\frac{\partial H^{N,\boldsymbol{\Gamma}}}{\partial x_{i,1}}\right)\right),$$

where

$$H^{N,\boldsymbol{\Gamma}}(x_1,\cdots,x_N) = \frac{1}{2}\sum_{j=1}^N \Gamma_j^2 K(x_j,x_j) + \frac{1}{2}\sum_{1\le j,k\le N,\, j\ne k} \Gamma_j\Gamma_k G(x_j,x_k)$$

and $x_i = (x_{i,1}, x_{i,2})$. It is easy to see that the value of $H^{N,\boldsymbol{\Gamma}}$ is preserved under the time evolution of vortices. Therefore $H^{N,\boldsymbol{\Gamma}}$ is called the *Hamiltonian* of vortices.

H^m referred in Fact 1.2 corresponds to the special case $N = m$ and $\boldsymbol{\Gamma} = (\Gamma_1,\cdots,\Gamma_m) = (1,\cdots,1)$, that is, m-vortices of *one kind.* Therefore $\mathscr{S}$ in Theorem 1.2 is a stationary point of vortices of *one kind.*

Further classical information on vortices readable for researchers of mathematics shall be obtained in [6].

It should be remarked that this special Hamiltonian H^m also relates to $J_\beta(\psi)$. Indeed suppose all the intensities of vortices is equivalent to some constant Γ. Then the Hamiltonian of m-vortices $H^{m,\Gamma}$ reduces to $\Gamma^2 H^m$. In this situation, the Gibbs measure associated to this Hamiltonian is given as follows:

$$\mu^m = \frac{e^{-\tilde{\beta}\Gamma^2 H^m(x_1,\cdots,x_m)}}{\int_{\Omega^m} e^{-\tilde{\beta}\Gamma^2 H^m(x_1,\cdots,x_m)}dx_1\cdots dx_m} dx_1\cdots dx_m,$$

where $\tilde{\beta}$ is the parameter called the inverse temperature. The canonical Gibbs measure is considered in statistical mechanics to give the possibility of the state for given *energy* H^m under the fixed (inverse) temperature. If $\tilde{\beta} > 0$ (as usual), the low-energy state is likely to occur. On the contrary, if $\tilde{\beta} < 0$, the high energy states have more possibility to occur, which is considered to give some reason why there are often observed large-scale long-lived structures in two-dimensional turbulence. One of the most famous example of such structures is the Jupiter's great red spot. The idea to relate such structures to negative temperature states of equilibrium vortices is first proposed by Onsager [19].

Using the canonical Gibbs measure, we are able to get the probability (density) of one vortex observed at $x \in \Omega$ from

$$\rho^m(x_1) = \int_{\Omega^{m-1}} \mu^m dx_2\cdots dx_m,$$

which is equivalent to every vortices from the symmetry of H^m. Now we assume that total vorticity is equivalent to 1, that is, $\Gamma = \frac{1}{m}$ and suppose $\tilde{\beta} = \beta_\infty \cdot m$ for some fixed $\beta_\infty \in (-8\pi, +\infty)$. Then we get ρ satisfying the following equation at the limit of ρ^m as $m \longrightarrow \infty$:

$$\rho(x) = \frac{e^{-\beta_\infty G\rho(x)}}{\int_\Omega e^{-\beta_\infty G\rho(x)}dx}, \tag{2.3}$$

where G is the Green operator given by $G\rho(x) = \int_\Omega G(x,y)\rho(y)dy$ ([3, Thorem 2.1]). This ρ is called the *mean field* of the equilibrium vortices of one kind. It should be remarked that when the solution of (2.3) is unique, ρ^m weakly converges to ρ, and not unique, to some *superposition* of ρ.

These argument was established mathmatically rigorously by Caglioti-Lions-Marchioro-Pulvirenti [3] and Kiessling [11] independently based on the argument developped by Messer-Sphon [14], see also [13]. The relation between the mean field (2.3) and $J_\beta(\psi)$ is obvious because (2.3) means

$\psi := -\beta_\infty G\rho$ and $\beta := -\beta_\infty$ satisfy (1.4), which is the Euler-Lagrange equation of $J_\beta(\psi)$.

As we mentioned above, $J_\beta(\psi)$ corresponds to $J_{\beta,\mathcal{P}}(\psi)$ with $\mathcal{P} = \delta_1$. The case $\mathcal{P} = \frac{1}{2}\delta_1 + \frac{1}{2}\delta_{-1}$, for example, corresponding to the case that the original system of vortices are neutral and two kinds, that is, there are two kinds of intensities whose absolute values are equivalent and numbers of vortices with each intensicities are same, see [10, 20] for huristic derivations. A general measure $\mathcal{P}$ gives the most general situation of vortices of (continuously) infinitely many kinds with general ratio [21].

3 Comment on the proof of Theorem 1.1

The proof is based on our previous result [18] corresponding to the case $\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}$. However, for general probability measure $\mathcal{P}$, we need to develop new argument. Our argument is based on the precise behavior of blow-up sequence of the solutions of the mean field equation. To this purpose, we consider the measure over the product space $I \times M$ for general $\mathcal{P}$. Here we only mention why we need such argument.

Suppose $\mathcal{P}(\alpha) = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$. To simplify the presentation, we explain the idea assuming $M = \Omega$. Let $\psi_{+,n}$ and $\psi_{-,n}$ be solutions of

$$
\begin{aligned}
-\Delta\psi_{+,n} &= \frac{\beta_n}{2}\frac{e^{\psi_n}}{\int_\Omega e^{\psi_n}dx}, \quad \psi_{+,n} = 0 \text{ on } \partial\Omega,\\
-\Delta\psi_{-,n} &= \frac{\beta_n}{2}\frac{e^{-\psi_n}}{\int_\Omega e^{-\psi_n}dx}, \quad \psi_{-,n} = 0 \text{ on } \partial\Omega.
\end{aligned}
$$

Then ψ_n is given as $\psi_n = \psi_{+,n} - \psi_{-,n}$ and $(\psi_{+,n}, \psi_{-,n})$ satisfies

$$
\begin{aligned}
-\Delta\psi_{+,n} &= \frac{\beta_n}{2}\frac{V_+(x)e^{\psi_{+,n}}}{\int_\Omega V_+(x)e^{\psi_{+,n}}dx}, \quad \text{where } V_+(x) = e^{-\psi_{-,n}},\\
-\Delta\psi_{-,n} &= \frac{\beta_n}{2}\frac{V_-(x)e^{\psi_{-,n}}}{\int_\Omega V_-(x)e^{\psi_{-,n}}dx}, \quad \text{where } V_-(x) = e^{-\psi_{+,n}}
\end{aligned}
$$

which are mean field equations of one kind with *valuable* coefficients. We have already studied rather general such cases in [17], which is applicable for some extent in these cases.

If we follow the argument of [18] for general $\mathcal{P}$, however, we need the solution $\psi_{\alpha,n}$ of the following problem for each α:

$$
-\Delta\psi_{\alpha,n} = \beta_n\alpha\frac{e^{\alpha\psi_n}}{\int_\Omega e^{\alpha\psi_n}dx}, \quad \psi_{\alpha,n} = 0 \text{ on } \partial\Omega,
$$

The solution $\psi_{\alpha,n}$ ($\alpha \in [-1,1]$) form a one-parameter family of functions and it is hard to extract a subsequence of $\psi_{\alpha,n}$ with good behavior for *every* α.

To overcome this difficulty, we found that it is sufficient to introduce the measure

$$\mu_n = \beta_n \frac{e^{\alpha v_n}}{\int_\Omega e^{\alpha v_n}} \mathcal{P}(d\alpha)dx$$

over the product space $I \times \Omega$ and to consider a weaker limit over it.

The details are rather complicated and we omit them. Here we only mention that we are not able to avoid the possibility of boundary blow-up of ψ_n, which prevent us to handle several proceeding arguments of the proof. Therefore we only get the result on problem over a manifold (M,g) without boundary. We note that this is also a problem for more simpler cases $\mathcal{P} = t\delta_1 + (1-t)\delta_{-1}$ considered in [18].

4 Sketch of the proof of Theorem 1.3

Similarly to [7], we prove Theorem 1.3 arguing by contradiction. For this purpose we assume the existence of a sequence $\{v_n\}$ of non-degenerate critical point of F_{λ_n} as $n \longrightarrow \infty$. Using the standard argument v_n is a non-trivial solution of the linearized problem of (1.5):

$$-\Delta v = \lambda_n e^{\psi_n} v \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial\Omega. \tag{4.1}$$

Without loss of generality we may assume that $\|\psi_n\|_{L^\infty(\Omega)} \equiv 1$.

Taking sufficiently small $\bar{R} > 0$, we may assume that for each κ_j there exists a sequence $\{x_{j,n}\}$ satisfying

$$x_{j,n} \longrightarrow \kappa_j, \quad u_n(x_{j,n}) = \max_{B_{\bar{R}}(x_{j,n})} u_n(x) \longrightarrow \infty.$$

Then we re-scale u_n and v_n around $x_{j,n}$ as follows:

$$\widetilde{\psi}_{j,n}(\widetilde{x}) = \psi_n(\delta_{j,n}\widetilde{x} + x_{j,n}) - \psi_n(x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0),$$

$$\widetilde{v}_{j,n}(\widetilde{x}) = v_n(\delta_{j,n}\widetilde{x} + x_{j,n}) \quad \text{in } B_{\frac{\bar{R}}{\delta_{j,n}}}(0),$$

where the scaling parameter $\delta_{j,n}$ is chosen to satisfy $\lambda_n e^{\psi_n(x_{j,n})}\delta_{j,n}^2 = 1$. From the standard argument based on the estimate concerning the blow-up behavior of ψ_n [12] and the classification result of the solutions of (1.5) and (4.1) in the whole space [4, 5], there exist $\boldsymbol{a}_j \in \mathbf{R}^2$, $b_j \in \mathbf{R}$ for each j and subsequences of u_n and v_n satisfying

$$\widetilde{\psi}_{j,n} \longrightarrow \log \frac{1}{\left(1 + \frac{|\widetilde{x}|^2}{8}\right)^2}, \quad \widetilde{v}_{j,n} \longrightarrow \frac{\boldsymbol{a}_j \cdot \widetilde{x}}{8 + |x|^2} + b_j \frac{8 - |\widetilde{x}|^2}{8 + |\widetilde{x}|^2},$$

locally uniformly. We shall show $\boldsymbol{a}_j = \mathbf{0}$ and $b_j = 0$.

The proof is divided into 3 steps:

Step 1: We show the following asymptotic behavior for (a subsequence of) v_n:

$$\frac{v_n}{\lambda_n^{\frac{1}{2}}} \longrightarrow 2\pi \sum_{j=1}^{m} C_j \boldsymbol{a}_j \cdot \nabla_y G(x, \kappa_j) \tag{4.2}$$

locally uniformly in $\overline{\Omega} \setminus \cup_{j=1}^{m} B_{2\bar{R}}(\kappa_j)$, where $C_j > 0$ is some constant.

Step 2: Using the fact that $\mathscr{S}$ is a non-degenerate critical point of H^m, we show $\boldsymbol{a}_j = \mathbf{0}$ for every j.

Step 3: We show $b_j = 0$ for every j and consequently we show the uniform convergence $v_n \longrightarrow 0$ in Ω, which contradicts $\|v_n\|_{L^\infty(\Omega)} \equiv 1$.

Concerning Step 1, we have simplified the argument of [7] by using the argument of the same authors in [8].

Step 2 is based on the observation that

$$-\Delta \psi_{x_i} = \lambda e^{\psi} \psi_{x_i}, \tag{4.3}$$

holds for every solution ψ of (1.5), that is, $\psi_{x_i} = \frac{\partial \psi}{\partial x_i}$ is always a solution of (4.1) except for the boundary condition. Then using the Green identity, we get

$$\int_{\partial B_R(\kappa_j)} \left(\frac{\partial}{\partial \nu} (\psi_n)_{x_i} v_n - (\psi_n)_{x_i} \frac{\partial}{\partial \nu} v_n \right) d\sigma = 0. \tag{4.4}$$

for every κ_j and sufficiently small $R\,(> 2\bar{R}) > 0$. From the know asymptotic behaviors of ψ_n and v_n, we are able to see that the limit of the above identity is a linearly combination of the integration

$$I_{ij} := \int_{\partial B_R(z_1)} \left\{ \frac{\partial}{\partial \nu_x} G_{x_i}(x, z_2) G_{y_j}(x, z_3) - G_{x_i}(x, z_2) \frac{\partial}{\partial \nu_x} G_{y_j}(x, z_3) \right\} d\sigma_x.$$

We are able to calculate this and get some *localized* versions of the known Rellich-Pohozaev type identity for the Green function:

$$I_{ij} = I_{ij}(z_1, z_2, z_3) = \begin{cases} 0 & (z_1 \neq z_2,\ z_1 \neq z_3), \\ \frac{1}{2} R_{x_i x_j}(z_1) & (z_1 = z_2 = z_3), \\ G_{x_i y_j}(z_1, z_3) & (z_1 = z_2,\ z_1 \neq z_3), \\ G_{x_i x_j}(z_1, z_2) & (z_1 \neq z_2,\ z_1 = z_3). \end{cases} \tag{4.5}$$

Collecting the limit of (4.4) for all $j = 1, \cdots, m$, we get

$$0 = 16\pi^2 \,\mathrm{Hess} H^m|_{(x_1,\cdots,x_m)=(\kappa_1,\cdots,\kappa_m)} \, {}^t(C_1 \boldsymbol{a}_1, \cdots, C_m \boldsymbol{a}_m).$$

This gives $\boldsymbol{a}_j = \boldsymbol{0}$ from the assumption that $\mathrm{Hess}H^m$ is invertible at $\mathscr{S}$.

Concerning Step 3, we have also simplified the corresponding argument in [7].

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Appendix "On λ-dependence of solutions of $-\Delta u = \lambda u_+^\beta$"

Bibliographical notes The purpose of this appendix is to carry my old note entitled "On λ-dependence of solutions of $-\Delta u = \lambda u_+^\beta$"(「$-\Delta u = \lambda u_+^\beta$ の解の λ 依存性について」), which is written in Japanese in 1992 for publication in the proceeding of a conference. The article, however, have not been published yet, which is inconvenient to me for easy referring of it. I think the work is one of the origin of my recent works mentioned in this article and seems worth publishing even now. Therefore I would like to present it below with some comments in Concluding remarks.

A.1 On the problem

We consider the following nonlinear eigenvalue problem on a bounded domain $\Omega \subset \mathbf{R}^2$ with smooth boundary.

$$
\text{(P)} \qquad \begin{cases} -\Delta u_\lambda = \lambda f(u_\lambda) & \text{in } \Omega \text{ for } \lambda > 0 \\ u_\lambda = -b_\lambda & \text{(unknown constant) on } \partial\Omega \\ \displaystyle\int_\Omega \lambda f(u_\lambda) dx = 1 \end{cases}
$$

The typical f we consider is $f(t) = (t_+)^\beta$, where $t_+ = \max(0, t)$ and $\beta(> 1)$ is a constant (we only mention these cases here). In this note we are concerned with the behavior of some variational solutions of this problem as $\lambda \longrightarrow \infty$.

These kinds of problems are studied in relation to the steady states of an incompressible ideal fluid in Ω with uniform density. Indeed, let $\lambda f(u_\lambda) \underset{\to}{=} \omega$ (vorticity), $(\frac{\partial u_\lambda}{\partial y}, -\frac{\partial u_\lambda}{\partial x}) \underset{\to}{=} v$ (velocity). Then they satisfy the Euler (vorticity) equation

$$
\begin{cases} \dfrac{D\omega}{Dt} = (\dfrac{\partial}{\partial t} + v \cdot \nabla)\omega = 0 & \text{in } \Omega, \\ v \cdot n = 0 & \text{on } \partial\Omega, \end{cases}
$$

where n is the unit outer normal vector of $\partial\Omega$, see, e.g., Turkington [T] for details. In [T], the existence and the behavior of solutions as $\lambda \longrightarrow \infty$ are studied when f is the Heaviside function. We follow the argument of [T] for the proof in this note. (It is mentioned in [T] that the similar results for all $\beta > 0$ would be obtained by similar arguments, but we are able to prove only when $\beta > 1$ yet.)

A.2 The variational problem

Fix $p > \beta$ and we consider the function space

$$K_p \underset{\leftarrow}{=} \{\omega \in L^{p^\sharp}(\Omega)\,;\ w \geq 0 \text{ a.e. in } \Omega,\ \int_\Omega \omega\, dx = 1\},\ \text{where } p^\sharp \underset{\leftarrow}{=} 1 + \frac{1}{p}.$$

We consider the following functional $E_\lambda : K_p \longrightarrow \mathbf{R} \cup \{-\infty\}$ over this space:

$$E(\omega_\lambda) \underset{\leftarrow}{=} \frac{1}{2}\int_\Omega G\omega \cdot \omega dx - \frac{\lambda}{\beta^\sharp}\int_\Omega (\frac{\omega}{\lambda})^{\beta^\sharp} dx$$

Here $G : L^{p^\sharp}(\Omega) \longrightarrow W^{2,p^\sharp}(\Omega) \cap W_0^{1,p^\sharp}(\Omega)$ is the Green operator of $-\Delta$ with the 0-Dirichlet condition. Under these situations, we know the following fact:

Theorem (H.Berestycki & H.Brezis [BB]). *$\exists \omega_\lambda \in K_p$ s.t.*

$$E_\lambda(\omega_\lambda) = \max_{\omega \in K_p} E_\lambda(\omega)$$

and $\exists 1 b_\lambda \in \mathbf{R}$ for each ω_λ s.t.

$$\omega_\lambda = \lambda(G\omega_\lambda - b_\lambda)_+^\beta,$$

i.e., (u_λ, b_λ) satisfies (P) for $u_\lambda \underset{\leftarrow}{=} G\omega_\lambda - b_\lambda$.

We note that it is not clear whether ω_λ is unique or not. Similar problems in higher dimensional spaces are also treated in [BB] in more general settings. We also mention that the argument in the existence part of the results in [T] is similar to that in the above result.

A.3 Main Theorem

Theorem . *$\exists R > 1$ that is a constant independent of λ s.t.*

$$\operatorname{diam}\operatorname{supp}\omega_\lambda = \operatorname{diam}\operatorname{supp}(u_\lambda)_+ \leq 2R\epsilon \quad \textit{for sufficiently large } \lambda,$$

where ϵ is determined by the relation $\pi\lambda\epsilon^2 = 1$.

In consequence we get the following fact:

$$\operatorname{diam}\operatorname{supp}\omega_\lambda = \operatorname{diam}\operatorname{supp}(u_\lambda)_+ \longrightarrow 0 \quad \text{as } \lambda \longrightarrow 0.$$

As is mentioned above, we don't know the uniqueness of ω_λ for each λ in general. Nevertheless the estimate above is uniform for all solutions of the above variational problem. As a consequence, we get the following fact (using some general facts for the Green function [BF, Lemma 5.1]:

Corollary . *For every $X_\lambda \in \operatorname{supp} \omega_\lambda$, we have*

$$\|G\omega_\lambda - g(\cdot, X_\lambda)\|_{W^{1,p}(\Omega)} = \|u_\lambda + b_\lambda - g(\cdot, X_\lambda)\|_{W^{1,p}(\Omega)} = o(1)$$

for each $1 \leq p < 2$ as $\varepsilon \longrightarrow 0$, where $g(\cdot, \cdot)$ denotes the Green function of $-\Delta$ under the 0-Dirichlet condition.

There exist several results of blow-up phenomenon similar to (P). For example, we have result of M. S. Berger & L. E. Fraenkel ([BF]) or A. Ambrosetti & J. Yang ([AY]) but their variational characterization of solutions are different from us and consequently the arguments for the proofs are different. It should be noticed that our argument is only applicable in $\mathbf{R}^2$ from technical reasons. We also note that the result in [BF] is also obtained in $\mathbf{R}^2$ by using another speciality of $\mathbf{R}^2$. Now we would like to start the proof of the theorem.

Proof. We prove that $\exists R > 1$ s.t. $\forall y \in \operatorname{supp} \omega_\lambda$

$$(*) \qquad \int_{\Omega \setminus B_{R\epsilon}(y)} \omega_\lambda(x) dx < \frac{1}{2} \quad \text{for sufficiently large } \lambda.$$

Here $B_r(x)$ denotes the open ball centred at x with radius r. This is sufficient for the conclusion because it is equivalent to

$$\int_{\Omega \cap B_{R\epsilon}(y)} \omega_\lambda(x) dx > \frac{1}{2}.$$

Then, suppose $\operatorname{diam} \operatorname{supp} \omega_\lambda > 2R\epsilon$, and we have

$$\exists y_1, \exists y_2 \in \operatorname{supp} \omega_\lambda \quad \text{s.t.} \quad B_{R\epsilon}(y_1) \cap B_{R\epsilon}(y_2) = \emptyset.$$

This leads

$$1 = \int_\Omega \omega_\lambda dx \geq \int_{\Omega \cap B_{R\epsilon}(y_1)} \omega_\lambda(x) dx + \int_{\Omega \cap B_{R\epsilon}(y_2)} \omega_\lambda(x) dx > 1,$$

which is a contradiction. Therefore we prove $(*)$. Since it holds that

$$\operatorname{supp} \omega_\lambda = \operatorname{supp} (G\omega_\lambda - b_\lambda)_+^\beta = \operatorname{supp} (G\omega_\lambda - b_\lambda)_+,$$

we have

$$y \in \operatorname{supp} \omega_\lambda \Longrightarrow b_\lambda \leq G\omega_\lambda(y).$$

Here we assume $\operatorname{diam} \Omega \leq 1$. (We can realize this by scale change and can use the same argument.) Then we get

$$G\omega_\lambda(y) \leq N\omega_\lambda(y) \underset{\leftarrow}{=} \frac{1}{2\pi} \int_\Omega \log \frac{1}{|x-y|} \omega_\lambda(x) dx \quad \text{(Newton Potential)}.$$

Indeed we are able to represent $G\omega_\lambda(y) = \int_\Omega g(x,y)\omega_\lambda(x)dx$ and set $g(x,y) = \frac{1}{2\pi}\log\frac{1}{|x-y|} - h(x,y)$. Then the above inequality follows the facts that $h(x,y) \geq 0$ if $\operatorname{diam}\Omega \leq 1$ and $\omega_\lambda \geq 0$. Next we deform $E_\lambda(\omega_\lambda)$ as follows:

$$\begin{aligned} E_\lambda(\omega_\lambda) &= \frac{1}{2}\int_\Omega G\omega_\lambda \cdot \omega_\lambda dx - \frac{\lambda}{\beta^\sharp}\int_\Omega (\frac{\omega_\lambda}{\lambda})^{\beta^\sharp} dx, \\ &= \frac{1}{2}b_\lambda + \frac{1}{2}\int_\Omega (G\omega_\lambda - b_\lambda)\omega_\lambda dx - \frac{\lambda}{\beta^\sharp}\int_\Omega (\frac{\omega_\lambda}{\lambda})^{\beta^\sharp} dx. \end{aligned}$$

Here we recall $\omega_\lambda = \lambda(G\omega_\lambda - b_\lambda)_+^\beta$, which guarantees

$$= \frac{1}{2}b_\lambda - (\frac{1}{\beta^\sharp} - \frac{1}{2})\lambda\int_\Omega (\frac{\omega_\lambda}{\lambda})^{\beta^\sharp} dx.$$

From the argument above, we get

$$E_\lambda(\omega_\lambda) \leq \frac{1}{2}N\omega_\lambda(y) - (\frac{1}{\beta^\sharp} - \frac{1}{2})\lambda\int_\Omega (\frac{\omega_\lambda}{\lambda})^{\beta^\sharp} dx$$

for $\forall y \in \operatorname{supp}\omega_\lambda$. By the way, since ω_λ attains maximum value of E_λ over K_p we get the following fact:

Fact. $\exists C$ independent of λ s.t.

$$E_\lambda(\omega_\lambda) \geq \frac{1}{4\pi}\log\frac{1}{\epsilon} - C.$$

(This is obtained from $E_\lambda(\lambda I_{B_\epsilon(x_0)}) \leq E_\lambda(\omega_\lambda)$ since $\lambda I_{B_\epsilon(x_0)} \in K_p$ for some fixed $x_0 \in \Omega$ (here $I_{B_\epsilon(x_0)}$ is the characteristic function of $B_\epsilon(x_0)$). See almost same result in [T].) Using these, we get

$$0 \leq \frac{1}{4\pi}\int_\Omega \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx - (\frac{1}{\beta^\sharp} - \frac{1}{2})\lambda\int_\Omega (\frac{\omega_\lambda}{\lambda})^{\beta^\sharp} dx + C.$$

for $\forall y \in \operatorname{supp}\omega_\lambda$. Moreover, for every $R > 1$, we have

$$\begin{aligned} \int_\Omega \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx = &\int_{\Omega\cap B_{R\epsilon}(y)} \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx \\ &+ \int_{\Omega\setminus B_{R\epsilon}(y)} \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx. \end{aligned}$$

Since $\log\frac{\epsilon}{|x-y|} \leq 0$ on $\Omega\setminus B_\epsilon(y)$, it follows that

$$\begin{aligned} \int_{\Omega\cap B_{R\epsilon}(y)} \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx &\leq \int_{\Omega\cap B_\epsilon(y)} \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx \\ &\leq \epsilon^{\frac{2}{\beta+1}}\|\log|\cdot|\|_{L^{\beta+1}(B_1(0))}\|\omega_\lambda\|_{L^{\beta^\sharp}(\Omega)} (\underset{\rightarrow}{=} C'\epsilon^{\frac{2}{\beta+1}}\|\omega_\lambda\|_{L^{\beta^\sharp}(\Omega)}). \end{aligned}$$

Since $\log\frac{\epsilon}{r}$ is monotone decreasing with respect to r and $\omega_\lambda \geq 0$, we get

$$\int_{\Omega\backslash B_{R\epsilon}(y)} \log\frac{\epsilon}{|x-y|}\omega_\lambda(x)dx \leq \log\frac{1}{R}\int_{\Omega\backslash B_{R\epsilon}(y)} \omega_\lambda(x)dx.$$

Summarizing these, we get

$$\frac{1}{4\pi}\log R\int_{\Omega\backslash B_{R\epsilon}(y)} \omega_\lambda(x)dx \leq \frac{1}{4\pi}C'\epsilon^{\frac{2}{\beta+1}}\|\omega_\lambda\|_{L^{\beta\sharp}(\Omega)} - \left(\frac{1}{\beta\sharp}-\frac{1}{2}\right)\lambda\int_\Omega \left(\frac{\omega_\lambda}{\lambda}\right)^{\beta\sharp}dx + C.$$

Here we set $p' > 1$ satisfying $\frac{1}{p}+\frac{1}{p'} = 1$ for each $p > 1$. Using Young's inequality, we get

$$\begin{aligned}\log R\int_{\Omega\backslash B_{R\epsilon}(y)} \omega_\lambda(x)dx \leq{}& \frac{(\frac{C'}{\eta})^{(\beta\sharp)'}}{(\beta\sharp)'} + \frac{(\eta\epsilon^{\frac{2}{\beta+1}}\|\omega_\lambda\|_{L^{\beta\sharp}(\Omega)})^{\beta\sharp}}{\beta\sharp} \\ & - 4\pi\left(\frac{1}{\beta\sharp}-\frac{1}{2}\right)(\pi\epsilon^2)^{\beta\sharp-1}\|\omega_\lambda\|^{\beta\sharp}_{L^{\beta\sharp}(\Omega)} + 4\pi C.\end{aligned}$$

Since we assumed $\beta > 1$ we have $\frac{1}{\beta\sharp} - \frac{1}{2} > 0$. Moreover we note $\frac{2}{\beta+1}\cdot\beta\sharp = 2(\beta\sharp - 1)$. Therefore taking sufficient η, we have an upper bound of the right-hand side of the above inequality not depending on R, λ, and ω_λ. Therefore we may take a sufficiently large R. □

A.4 Other remarks

A.4.1 On the blow-up point

It should be mentioned that for solutions obtained by the variational problem in [T], we are able to get some characterization of the blow-up point easily, though it is not mentioned in [BF] studying another variational solutions.

Proposition 1. *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a sequence satisfying $\lambda_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and fix a solution ω_{λ_n} and a point $X_{\lambda_n} \in \operatorname{supp}\omega_{\lambda_n}$ for each λ_n. Here we may assume that $X_{\lambda_n} \longrightarrow X_\infty \in \overline{\Omega}$ as $n \longrightarrow \infty$, taking a subsequence if necessary (which we also denoted $\{X_{\lambda_n}\}$). Then it holds that*

$$h(X_\infty, X_\infty) = \min_{x\in\Omega} h(x,x).$$

Consequently we know that $X_\infty \in \Omega$ from the known fact on $h(x,x)$.

We note that this proposition insists on that every accumulating point of $\{X_{\lambda_n}\}$ is an interior point of Ω. Similar behavior is also studied in [BF] for another variational problem but conclude only $X_\infty \in \overline{\Omega}$. We get Proposition 1 quite easily because we characterize the solution by the variational problem of [BB]. We omit the proof of it but only mention that the translation invariant part of E_λ comes from $h(x,y)$ and this fact gives the result. It should be notices that the point where $\min_{x\in\Omega} h(x,x)$ has a special meaning in the fluid mechanics. In fact, the point is a stationary point of a vortex (that is, a vorticity field supported at one point) in an incompressible ideal fluid in Ω. Therefore our result implies that we constructed an approximating sequence of stationary solutions of the Euler equation to a stationary vortex keeping its total vorticity.

A.4.2 On the behavior of b_λ

We are able to calculate the behavior of b_λ though it is rather rough. We would like to omit the details of the proof but mention that the result follows by reducing the spherically symmetric case via the Schwarz symmetrization.

Proposition 2. *There exists a subsequence of λ_n satisfying*

$$b_{\lambda_n} = \frac{1}{2\pi}\log\frac{1}{\epsilon_n} - h(X_\infty, X_\infty) - \frac{1}{2\pi^2}\log T(\pi M(T)^{\beta-1})^{\frac{1}{2}} + o(1) \quad \textit{as } n \longrightarrow \infty$$

(here ϵ_n satisfies $\pi\lambda_n\epsilon_n^2 = 1$), i.e.,

$$b_{\lambda_n} \longrightarrow \infty \quad \textit{as } n \longrightarrow \infty.$$

Here T and $M(T)$ is determined as follows: let $\varphi(t)$ be the solution of the initial value problem of the following ordinary differential equation:

$$\begin{cases} -\frac{d^2}{dt^2}\varphi - \frac{1}{t}\frac{d}{dt}\varphi = \varphi_+^\beta, \\ \varphi(0) = 1, \\ \varphi'(0) = 0. \end{cases}$$

It is easy to see that the solution φ uniquely exists in $C^2[0,\infty)$ and strictly decreases to $-\infty$. Now T is the unique zero point of φ and $M(T) = -2\pi T\varphi'(T)$.

We note that inserting $\beta = 0$ formally into the conclusion (that is, we consider t_+^0 as the Heaviside function), we get the same conclusion in [T].

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Finally we note that this note reviews the author's master thesis.

Concluding remarks The functional E_λ closely relates to the mean field of vortices. Indeed the relation (2.3) is originally obtained as the Euler-Lagrange equation of the functional

$$f_{\beta_\infty}(\omega) = \frac{1}{2}\int_\Omega G\omega \cdot \omega dx + \frac{1}{\beta_\infty}\int_\Omega \omega \log \omega dx$$

over the class of densities of (positive) vortices with finite entropies:

$$P_{L\log L}(\Omega) = \{\omega \in L^1(\Omega)\,;\, \int_\Omega \omega \log \omega < \infty,\ \omega \geq 0 \text{ a.e. in } \Omega,\ \int_\Omega \omega dx = 1\}.$$

The functional f_{β_∞} is called the mean free energy functional of equilibrium vortices as $N \longrightarrow \infty$ [3] and this seems to give an exponential f counterpart of the problem (P) in negative temperature cases $\beta_\infty < 0$. Indeed we are able to know that f_{β_∞} is bounded from the above and there exists a maximizer of f_{β_∞} over $P_{L\log L}(\Omega)$ when $\beta_\infty \in (-8\pi, 0)$. Moreover similar blow-up behaviors, e.g., the characterization of the blow-up point, of this variational solution are observed when $\beta_\infty \longrightarrow -8\pi$ [3, Theorem 7.1].

The functional J_β we started this article is known as the (Toland type) dual functional to $f_{-\beta}$ and further studies concerning J_β (and $f_{-\beta}$) seems to be rather developed than E_λ because of several good structures in J_β (and $f_{-\beta}$). It seems, however, interesting now to revisit the old problem (P) and study several counterparts of the results mentioned in the body of this article. We would like to refer [CPY] for readers interested in the recent developments around (P).

References

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