数理解析研究所講究録
第 1730 巻 2011 年 107-116

Decay property for some hyperbolic-type equations with dissipation

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Abstract

In this article we report on the decay properties for several hyperbolic-type equations with dissipation. We first review the general theory on the decay structure for symmetric hyperbolic systems with relaxation, which was established in [7, 5]. Then we study the dissipative Timoshenko system as an example which is not covered by the general theory in [7, 5]. We observe that the decay structure of the dissipative Timoshenko system is of the regularity-loss type. Finally, we discuss the dissipative plate equation and verify a similar regularity-loss property in the decay structure.

1 Introduction

In this article we discuss the decay property for hyperbolic-type equations with dissipation. First we consider a class of symmetric hyperbolic systems with relaxation

$$A^0u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{1.1}$$

where u is an m vector function, and A^0, A^j and L are $m \times m$ real symmetric (constant) matrices such that $A^0 > 0$ and $L \geq 0$. For such systems, the dissipative structute is characterized by the *stability condition* formulated in [5]. It is known that the stability condition for (1.1) is equivalent to the property

$$\text{Type (I)}: \qquad \operatorname{Re}\lambda(i\xi) \leq -c|\xi|^2/(1+|\xi|^2), \tag{1.2}$$

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where $\lambda(i\xi)$ denotes the eigenvalues of the system obtained by taking the Fourier transform of (1.1), and c is a positive constant. In this situation, we have the decay estimate of the standard type.

On the other hand, there is an example which is not in the above class. Consider the dissipative Timoshenko system

$$\begin{aligned} &w_{tt} - (w_x - \psi)_x = 0, \\ &\psi_{tt} - a^2\psi_{xx} - (w_x - \psi) + \psi_t = 0, \end{aligned} \tag{1.3}$$

which is written in the form of (1.1) with a non-symmetric L and verifies the stability condition. However, the dissipative structure of this system with $a \neq 1$ is not charcterized by (1.2) but the property

$$\text{Type (II)}: \qquad \mathrm{Re}\,\lambda(i\xi) \leq -c|\xi|^2/(1+|\xi|^2)^2. \tag{1.4}$$

This dissipative structure is very weak in high frequency region. Consequently, we have the *regularity-loss* not only in the optimal decay estimate but also in the dissipation part of the energy estimate.

A similar dissipative structure of the regularity-loss type is also found for the dissipative plate equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = 0. \tag{1.5}$$

The dissipative structure of this equation is characterized by

$$\mathrm{Re}\,\lambda(i\xi) \leq -c|\xi|^4/(1+|\xi|^2)^3, \tag{1.6}$$

which is just the same as (1.4) in high frequency region. We will explain the difficulty caused by the regularity-loss property in solving the corresponding nonlinear problem.

2 Symmetric hyperbolic systems

We consider a class of symmetric hyperbolic systems with relaxation

$$A^0 u_t + \sum_{j=1}^{n} A^j u_{x_j} + Lu = 0, \tag{2.1}$$

where $u = u(x,t)$ is an m-vector function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and $t \geq 0$, and A^0, A^j and L are $m \times m$ real symmetric (constant) matrices such that A^0 is positive definite and L is nonnegative definite.

First we review the decay property for (2.1). Take the Fourier transform of (2.1) to get

$$A^0\hat{u}_t + i|\xi|A(\omega)\hat{u} + L\hat{u} = 0, \tag{2.2}$$

where $A(\omega) = \sum_{j=1}^{n} A^j\omega_j$, $\omega = \xi/|\xi| \in S^{n-1}$. The following structural condition was introduced in [7, 5] to derive the decay estimate of solutions to the system (2.1).

Condition (K): There exists $K(\omega)$ with the following properties:

(i) $K(\omega)A^0$ is real skew-symmetric for each $\omega \in S^{n-1}$.

(ii) $(K(\omega)A(\omega))_1 + L$ is real symmetric and positive definite for each $\omega \in S^{n-1}$, where X_1 denotes the symmetric part of X.

Theorem 2.1 ([7]). *Under the condition* (K) *we have*

$$|\hat{u}(\xi,t)| \le Ce^{-c\rho(\xi)t}|\hat{u}_0(\xi)| \tag{2.3}$$

for $\xi \in \mathbb{R}^n$ and $t \ge 0$, where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$.

The pointwise estimate (2.3) yields the following decay estimate of solutions to (2.1).

Corollary 2.2 ([7]). *Under the condition* (K) *we have*

$$\|\partial_x^k u(t)\|_{L^2} \le C(1+t)^{-n/4-k/2}\|u_0\|_{L^1} + Ce^{-ct}\|\partial_x^k u_0\|_{L^2}, \tag{2.4}$$

where $k \ge 0$; this decay estimate is without loss of regularity.

Under the condition (K), it is shown that the system (2.2) has a Lyapunov function of the form

$$E[\hat{u}] = \langle A^0\hat{u},\, \hat{u}\rangle - \frac{\alpha|\xi|}{1+|\xi|^2}\langle iK(\omega)A^0\hat{u},\, \hat{u}\rangle, \tag{2.5}$$

where α is a small positive constant, and $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbb{C}^m$. In fact, we have the following inequality:

$$\frac{\partial}{\partial t}E[\hat{u}] + \frac{c|\xi|^2}{1+|\xi|^2}|\hat{u}|^2 + c|(I-P)\hat{u}|^2 \le 0, \tag{2.6}$$

where P is the orthogonal projection onto $\ker(L)$. It follows from (2.6) that

$$\frac{\partial}{\partial t}E[\hat{u}] + c\rho(\xi)E[\hat{u}] \le 0, \tag{2.7}$$

where $\rho(\xi) = |\xi|^2/(1+|\xi|^2)$. Solving this ordinary differential inequality, we obtain the desired pointwise estimate (2.3).

Also, (2.6) easily gives an energy estimate for (2.1). In fact, multiplying (2.6) by $(1+|\xi|^2)^s$ and integrating with respect to t and ξ, we obtain

$$\|u(t)\|_{H^s}^2 + \int_0^t \|\partial_x u(\tau)\|_{H^{s-1}}^2 + \|(I-P)u(\tau)\|_{H^s}^2\, d\tau \le C\|u_0\|_{H^s}^2, \tag{2.8}$$

where $s \ge 0$. Note that there is no regularity-loss in the dissipation part of this energy estimate.

Next we review the general theory on the dissipative structure for the system (2.1). Let $\lambda = \lambda(i\xi)$ be the eigenvalues which are the solutions to the characteristic equation

$$\det(\lambda A^0 + i|\xi|A(\omega) + L) = 0. \tag{2.9}$$

The following structural condition was introduced in [5] to characterize the dissipative structure of the system (2.1).

Stability condition: Let $\mu \in \mathbb{R}$, $\varphi \in \mathbb{R}^m$ and $\omega \in S^{n-1}$. If $L\varphi = 0$ and $\mu A^0\varphi + A(\omega)\varphi = 0$, then $\varphi = 0$.

Theorem 2.3 ([5]). *The following four conditions are equivalent.*

(a) *Stability condition.*

(b) *Condition* (K).

(c) $\mathrm{Re}\,\lambda(i\xi) \le -c|\xi|^2/(1+|\xi|^2)$ *for any* $\xi \in \mathbb{R}^n$.

(d) $\mathrm{Re}\,\lambda(i\xi) < 0$ *for any* $\xi \in \mathbb{R}^n$ *with* $\xi \ne 0$.

We say that the system (2.1) is *strictly dissipative* if $\mathrm{Re}\,\lambda(i\xi) < 0$ for any $\xi \in \mathbb{R}^n$ with $\xi \ne 0$. The above theorem says that the system (2.1) is strictly dissipative if and only if it has the dissipative srtucture of Type (I) in (1.2); there is no other type of dissipativity such as Type (II) in (1.4).

3 Dissipative Timoshenko system

As an example of the strict dissipativity of Type (II), we consider the dissipative Timoshenko system:

$$\begin{cases} w_{tt} - (w_x - \psi)_x = 0, \\ \psi_{tt} - a^2\psi_{xx} - (w_x - \psi) + \gamma\psi_t = 0, \end{cases} \tag{3.1}$$

where a and γ are positive constants. Putting $u = w_t$, $v = w_x - \psi$, $y = \psi_t$ and $z = a\psi_x$, we transform (3.1) into the equivalent first order system:

$$\begin{cases} v_t - u_x + y = 0, \\ u_t - v_x = 0, \\ z_t - ay_x = 0, \\ y_t - az_x - v + \gamma y = 0. \end{cases} \tag{3.2}$$

The system is written in the vector form as

$$U_t + AU_x + LU = 0, \tag{3.3}$$

where

$$U = \begin{pmatrix} v \\ u \\ z \\ y \end{pmatrix}, \quad A = -\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & a & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & \gamma \end{pmatrix}.$$

Since A is symmetric, (3.3) is a symmetric hyperbolic system with L being not symmetric but nonnegative definite. Moreover, this system verifies the stability condition formulated in the previous section. In fact, we have:

Claim 3.1. *The dissipative Timoshenko system* (3.3) *with a non-symmetric L satisfies the stability condition: If $L\varphi = 0$ and $\mu\varphi + A\varphi = 0$ for $\mu \in \mathbb{R}$ and $\varphi \in \mathbb{R}^4$, then $\varphi = 0$.*

To study the dissipative structure of the above Timoshenko system (3.3), we compute the eigenvalues $\lambda(i\xi)$ which are the solutions to the characteristic equation $\det(\lambda + i\xi A + L) = 0$. A direct comptation shows that

- If $a = 1$, then $\operatorname{Re}\lambda(i\xi) \le -c\xi^2/(1+\xi^2)$ for any $\xi \in \mathbb{R}$.
- If $a \ne 1$, then $\operatorname{Re}\lambda(i\xi) \le -c\xi^2/(1+\xi^2)^2$ for any $\xi \in \mathbb{R}$.

This implies that the dissipative Timoshenko system (3.3) is strict dissipative in the sense of Type (I) for $a = 1$ and of Type (II) for $a \ne 1$. More precisely, when $a \ne 1$, one can verify that the eigenvalues $\lambda(i\xi)$ behave as

$$\begin{aligned} \lambda(i\xi) &= \pm\, i\xi \pm \frac{\sigma}{2}(i\xi)^{-1} + \sigma^2\gamma(i\xi)^{-2} + O(|\xi|^{-3}), \\ \lambda(i\xi) &= \pm\, ai\xi - \frac{\gamma}{2} + O(|\xi|^{-1}) \end{aligned}$$

for $|\xi| \to \infty$, where $\sigma = 1/(a^2-1)$. The above computations show that the general theory in Theorem 2.3 on the dissipative structure can not be applied to the system (2.1) with a non-symmetric L.

Although the dissipative Timoshenko system (3.3) is classified into Type (II) for $a \ne 1$, the corresponding system obtained by taking the Fourier transform admits a Lyapunov function of the form

$$E[\hat{U}] = |\hat{U}|^2 + \frac{\alpha_1}{1+\xi^2}\Big\{ -\operatorname{Re}(\hat{v}\bar{\hat{y}} + a\hat{u}\bar{\hat{z}}) + \frac{\alpha_2\xi}{1+\xi^2}\operatorname{Re}(i\hat{v}\bar{\hat{u}} + i\hat{y}\bar{\hat{z}})\Big\}, \tag{3.4}$$

where $a \neq 1$, and α_1 and α_2 are small positive constants. In fact, by straightforward computations, we can show that

$$\frac{\partial}{\partial t}E[\hat{U}] + cD[\hat{U}] \leq 0 \tag{3.5}$$

for $a \neq 1$, where

$$D[\hat{U}] = \frac{\xi^2}{(1+\xi^2)^2}(|\hat{u}|^2 + |\hat{z}|^2) + \frac{1}{1+\xi^2}|\hat{v}|^2 + |\hat{y}|^2.$$

We can rewrite (3.5) as

$$\frac{\partial}{\partial t}E[\hat{U}] + c\eta(\xi)E[\hat{U}] \leq 0,$$

where $\eta(\xi) = \xi^2/(1+\xi^2)^2$. This gives the following pointwise estimate of solutions to (3.3) in the Fourier space.

Theorem 3.2 ([1]). *When $a \neq 1$, we have*

$$|\hat{U}(\xi,t)| \leq Ce^{-c\eta(\xi)t}|\hat{U}_0(\xi)| \tag{3.6}$$

for $\xi \in \mathbb{R}$ and $t \geq 0$, where $\eta(\xi) = \xi^2/(1+\xi^2)^2$.

The above pointwise estimate yields the corresonding decay estimate of solutions to (3.3).

Corollary 3.3 ([1]). *When $a \neq 1$, we have*

$$\|\partial_x^k U(t)\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1} + C(1+t)^{-l/2}\|\partial_x^{k+l}U_0\|_{L^2}, \tag{3.7}$$

where $k,\ l \geq 0$.

This result shows that we have the decay rate $t^{-l/2}$ for $t \to \infty$ only by assuming the additional l-th order regularity on the initial data. This regularity-loss property in the optimal decay estimate would be the typical phenomena in the strict dissipativity of Type (II).

The ordinary differential inequality (3.5) also gives an energy estimate for $a \neq 1$. In fact, multiplying (3.5) by $(1+\xi^2)^s$ and integrating with respect to t and ξ, we have

$$\|U(t)\|_{H^s}^2 + \int_0^t \|\partial_x(u,z)(\tau)\|_{H^{s-2}}^2 + \|v(\tau)\|_{H^{s-1}}^2 + \|y(\tau)\|_{H^s}^2\, d\tau \leq C\|U_0\|_{H^s}^2, \tag{3.8}$$

where $s \geq 0$. In the dissipation part of this energy estimate for $a \neq 1$, we have the regularity-loss for the componets (v, u, z), and this regularity-loss property would be the typical phenomena in the strict dissipativity of type (II).

The above regularity-loss property causes the difficulty in showing the global existence of solutions to the nonlinear dissipative Timoshenko system. This difficulty can be overcome by applying the time weighted energy method together with the optimal decay estimates of lower-order derivatives of solutions. For the details, we refer to [2].

4 Dissipative plate equation

As a hyperbolic-type equation having the dissipative structure of the regularity-loss type, we consider the dissipative plate equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = 0. \tag{4.1}$$

We take the Fourier transform of (4.1) to get

$$(1 + |\xi|^2)\hat{u}_{tt} + \hat{u}_t + |\xi|^4 \hat{u} = 0. \tag{4.2}$$

The fundamental solutions $G(x,t)$ and $H(x,t)$ of (4.1) are given by the formulas

$$G(x,t) = \mathcal{F}^{-1}\Big[\frac{e^{\lambda_+(\xi)t} - e^{\lambda_-(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\Big](x),$$

$$H(x,t) = \mathcal{F}^{-1}\Big[\frac{(1+\lambda_+(\xi))e^{\lambda_-(\xi)t} - (1+\lambda_-(\xi))e^{\lambda_+(\xi)t}}{\lambda_+(\xi) - \lambda_-(\xi)}\Big](x),$$

where $\lambda_\pm(\xi)$ are the eigenvalues given explicitly by

$$\lambda_\pm(\xi) = \frac{-1 \pm \sqrt{1 - 4|\xi|^4(1+|\xi|^2)}}{2(1+|\xi|^2)}.$$

Then the solution to (4.1) is expressed by using the above fundamental solutions as

$$u(t) = G(t) * (u_0 + u_1) + H(t) * u_1, \tag{4.3}$$

where u_0 and u_1 are the initial data for u and u_t, respectively, and $*$ denotes the convolution with respect to $x \in \mathbb{R}^n$.

We see that

$$\operatorname{Re}\lambda_\pm(\xi) \leq -c|\xi|^4/(1+|\xi|^2)^3.$$

Therefore the the dissipative structure of (4.1) is similar to that of Type (II) in high frequency region. Also, a direct computation applied to (4.2) gives

$$\frac{\partial}{\partial t}\tilde{E} + \tilde{D} = 0, \tag{4.4}$$

where

$$\tilde{E} = (1+|\xi|^2)^2|\hat{u}_t|^2 + \{\frac{1}{2} + (1+|\xi|^2)|\xi|^4\}|\hat{u}|^2 + (1+|\xi|^2)\mathrm{Re}(\hat{u}_t\bar{\hat{u}}),$$

$$\tilde{D} = (1+|\xi|^2)|\hat{u}_t|^2 + |\xi|^4|\hat{u}|^2.$$

Here we note that $\tilde{E}$ is equivalent to $(1+|\xi|^2)^2E$, where $E = |\hat{u}_t|^2 + (1+|\xi|^2)|\hat{u}|^2$. Therefore, multiplying (4.4) by $(1+|\xi|^2)^{s-2}$ and integrating with respect to t and ξ, we have the energy estimate for (4.1) as

$$\|u_t(t)\|_{H^s}^2 + \|u(t)\|_{H^{s+1}}^2 + \int_0^t \|u_t(\tau)\|_{H^{s-1}}^2 + \|\partial_x^2 u(\tau)\|_{H^{s-2}}^2\, d\tau \le CE_0^2, \qquad (4.5)$$

where $s \ge 0$ and $E_0 = \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s}$. We have 1 regularity-loss in the dissipation part of the energy estimate (4.5).

On the other hand, we can deduce from (4.4) the following pointwise estimate of solutions to (4.2).

Theorem 4.1 ([6]). *For the solution of* (4.2) *we have*

$$|\hat{u}_t(\xi,t)|^2 + (1+|\xi|^2)|\hat{u}(\xi,t)|^2 \le Ce^{-c\eta(\xi)t}\{|\hat{u}_1(\xi)|^2 + (1+|\xi|^2)|\hat{u}_0(\xi)|^2\} \qquad (4.6)$$

for $\xi \in \mathbb{R}^n$ *and* $t \ge 0$, *where* $\eta(\xi) = |\xi|^4/(1+|\xi|^2)^3$.

Making use of (4.6), we can show the decay estimates for the solution operators in (4.3). In fact, we have:

$$\|\partial_x^k G(t) * \phi\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|\phi\|_{L^1} + C(1+t)^{-\frac{l+1}{2}}\|\partial_x^{(k+l)_+}\phi\|_{L^2}, \qquad (4.7)$$

$$\|\partial_x^k H(t) * \psi\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{1}{2}}\|\psi\|_{L^1} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}\psi\|_{L^2}, \qquad (4.8)$$

where $l+1 \ge 0$ and $(k+l)_+ = \max\{k+l, 0\}$ in (4.7), and $l \ge 0$ in (4.8). Also, we have

$$\|\partial_x^k \partial_t G(t) * \phi\|_{L^2} \le C(1+)^{-\frac{n}{8}-\frac{k}{4}-1}\|\phi\|_{L^1} + C(1+t)^{-\frac{l}{2}}\|\partial_x^{k+l}\phi\|_{L^2}, \qquad (4.9)$$

$$\|\partial_x^k \partial_t H(t) * \psi\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{3}{2}}\|\psi\|_{L^1} + C(1+t)^{-\frac{l-1}{2}}\|\partial_x^{k+l}\psi\|_{L^2}, \qquad (4.10)$$

where $l \ge 0$ in (4.9), and $l \ge 1$ in (4.10). Notice that we have the regularity-loss in these optimal decay estimates. Applying these decay estimates to the solution formula (4.3), we can show the decay estimate of solutions to (4.1).

Theorem 4.2 ([6]). *Put* $E_1 = \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1}$. *Then the solution to* (4.1) *satisfies the decay estimate*

$$\|\partial_x^k u(t)\|_{H^{s-\sigma_1(k,n)}} \le CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}} \qquad (4.11)$$

for $k \geq 0$ *with* $\sigma_1(k,n) \leq s$. *Moreover, for each* j *with* $0 \leq j \leq 2$, *we have*

$$\|\partial_x^k u_t(t)\|_{H^{s-1-\sigma_1(k,n)-j}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}-\frac{j}{2}} \tag{4.12}$$

for $k \geq 0$ *with* $\sigma_1(k,n)+j \leq s-1$. *Here* $\sigma_1(k,n) = k + [\frac{n+2k-1}{4}]$.

Note that we have $1+\sigma_1(k,n)-k = 1+[\frac{n+2k-1}{4}]$ regularity-loss in the optimal decay estimate (4.11).

The above regularity-loss property causes the difficulty in showing the global existence of solutions to the nonlinear equation

$$u_{tt} - \Delta u_{tt} + \sum_{j,k=1}^{n} b^{jk}(\partial_x^2 u)_{x_j x_k} + u_t = 0, \tag{4.13}$$

where the nonlinear functions b^{jk} satisfy the structural conditions formulated in [3, 4]. This difficulty can be overcome by applying the time weighted energy method together with the optimal decay estimates of lower-order derivatives of solutions. As a result, we obtain the global existence and optimal decay of solutions under smallness and enough regularity assumptions on the initial data; see [3, 4] for the details. To state the results, we introduce some special notations. Define

$$\sigma(k,n) = \max\{\sigma_0(k),\ \sigma_1(k,n)\},$$

where $\sigma_0(k) = k + [\frac{k+1}{2}]$, and $\sigma_1(k,n)$ is given in Theorem 4.2. Put

$$s(n) = \begin{cases} 7, & n = 1,2, \\ 6, & n = 3, \\ [\frac{n}{2}] + [\frac{n}{4}] + 4, & n \geq 4, \end{cases}$$

which indicates the regularity of the initial data. Then our global existence result for the nonlinear equation (4.13) is stated as follows.

Theorem 4.3 ([4, 3]). *Let* $n \geq 1$ *and* $s \geq s(n)$. *Put* $E_1 = \|u_0\|_{H^{s+1}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1}$. *If* E_1 *is suitably small, then the equation* (4.13) *has a unique global solution which satisfies the following optimal decay estimates:*

$$\|\partial_x^k u(t)\|_{H^{s-1-\sigma(k,n)}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{4.14}$$

$$\|\partial_x^k u_t(t)\|_{H^{s-4-\sigma(k,n)}} \leq CE_1(1+t)^{-\frac{n}{8}-\frac{k}{4}-1} \tag{4.15}$$

for $k \geq 0$, *where* $\sigma(k,n) \leq s-1$ *in* (4.14) *and* $\sigma(k,n) \leq s-4$ *in* (4.15).

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