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ON THE TIME-GLOBAL EXISTENCE FOR NON-NEWTONIAN TWO-PHASE FLOW WITH DIFFERENT DENSITIES

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1. INTRODUCTION

Two-phase fluid flow problem in a most crude and 'non-regularized' form may be stated as follows: Consider two disjoint domains $\Omega_+(t)$ and $\Omega_-(t)$ separated by a hypersurface $\Gamma(t)$ so that $\Omega_+(t)\cup\Omega_-(t)\cup\Gamma(t)=\Omega\subset\mathbb{R}^n$ for each $t\geq 0$. n is either 2 or 3 but can be ≥ 4 in general. Each domain is filled with different incompressible fluid whose velocity field obeys the Navier-Stokes or non-Newtonian flow equation. Namely, let v be the flow velocity and p be the pressure. Then on each $\Omega_\pm(t)$, we have

$$(1.1)\qquad \begin{cases} \rho_\pm(v_t+v\cdot\nabla v)=\operatorname{div}(\tau_\pm(e(v)))-\nabla p & \text{on } \Omega_\pm(t),\\ \operatorname{div} v=0 & \text{on } \Omega. \end{cases}$$

Here $\rho_\pm$ is the density of the fluid occupying the domain $\Omega_\pm(t)$, $e(v)=(\nabla v+\nabla v^t)/2$ is the symmetric part of ∇v and $\tau_\pm(e(v))$ is the stress tensor times viscosity coefficient. For the Navier-Stokes equation, $\tau_\pm(e(v))=2\alpha_\pm e(v)$ with possibly different viscosity constants $\alpha_\pm$, which reduces to $\operatorname{div}\tau_\pm(e(v))=\alpha_\pm\Delta v$. For some two-phase non-Newtonian fluid flow equation, we may consider as an example $\tau_\pm(e(v))=\alpha_\pm(1+|e(v)|^2)^q e(v)$ for some $q>0$. The separating hypersurface $\Gamma(t)$ moves with the fluid, which is often called the kinematic condition. There should be natural jump conditions for stress tensor and pressure, which I do not go in for the moment. While it is easy to imagine that this is a very natural problem to consider as a setting for two-phase fluid flow, it is an impossible problem to obtain some reasonable global in time existence results for the Cauchy problem for general data. One of the reasons for the difficulty is the occurrence of singularities of interface $\Gamma(t)$. The flow may not be regular enough to keep the interface 'hypersurface-like' as time evolves, even if the initial data may be regular. On the other hand it is a very important and natural engineering problem and one would like to have a good framework and algorithm to capture the time evolution numerically.

In recent years the phase field method has been successfully employed to model such two-phase fluid flow problem ([3, 5, 6, 10]). Much of these works concern the model formulations and numerical analysis and they pose very interesting analytical problems. In this note I focus on the model proposed by Shen and Yang [12] and discuss the relevant sharp-interface problem. I also indicate how one can analyze the global existence issue using the recent developments on the related phase field equations, particularly [8]. The model has attractive features such as good energy law and the resulting built-in stability. The reference [12] reports the numerical stability for the scheme even under a severe density difference between the two phases such as air bubbles in water. One important feature of the approach of the present note is that it incorporates the effect of surface tension on the fluid and the surface energy at the same time. There have been many attempts to incorporate the surface tension to the two-phase flow problems. To do so, one needs to define the mean curvature of $\Gamma(t)$ in some weak form. Since mean curvature is the second order quantity, $\Gamma(t)$ needs to be sufficiently regular (even in some weak sense) to define it. On the other hand the flow field is not regular enough to allow such regularity to $\Gamma(t)$, so there is a fine balance between the regularity property of the fluids and regularizing effect of the moving $\Gamma(t)$ itself. The different densities add more difficulties to the problem. To define some type of approximate mean curvature, we first need to define the surface energy of the moving interface. In Section 2 we quickly review the phase field approximation of the surface energy. In Section 3 we review the expression of mean curvature. In the subsequent sections, we discuss the topic of this note, the two-phase flow problems.

2. Surface energy

The phase field method starts out by introducing the phase function which we call ϕ. Namely let ϕ be a phase field variable of two-phase fluid with $\phi = 1$ indicating the pure $\Omega_+(t)$ phase and $\phi = -1$ indicating the pure $\Omega_-(t)$ phase at the point. For the values between ± 1, we regard $(\phi+1)/2$ as a mixture ratio of the two fluids. Let $W : [-1,1] \to \mathbb{R}$ be defined by $W(s) = (1-s^2)^2/2$ which has local minima at ± 1. Suppose that we have a thin layer where transition from one phase to the other occurs smoothly, and additionally assume that the thickness of the thin layer is of order ε, which I think to be infinitesimally small compared to the domain size. Now introduce the following energy functional

$$E_\varepsilon(\phi) = \int_\Omega \frac{\varepsilon|\nabla\phi|^2}{2} + \frac{W(\phi)}{\varepsilon}\,dx. \tag{2.1}$$

For people who are not familiar with this functional, it is instructive to consider the minimizing problem of E_ε with $\Omega = \mathbb{R}$ and with fixed boundary values $\phi(-\infty) = -1$ and $\phi(\infty) = +1$. The minimizer satisfies the Euler-Lagrange equation $-\varepsilon\phi'' + W'(\phi)/\varepsilon = 0$ and one can check that $\phi(x) = \tanh(x/\varepsilon)$ is a solution to this equation, and in fact is the unique minimizer of E_ε with $\phi(0) = 0$ with the stated boundary values at both sides of infinity. In fact, multiply ϕ' to the Euler-Lagrange equation, and integrate in x from $-\infty$ to x. One then obtains $-\frac{\varepsilon^2(\phi')^2}{2} + W(\phi) = 0$ holding on $\mathbb{R}$. Thus we have $\varepsilon\phi' = \sqrt{2W(\phi)} = (1-\phi^2)$ by the definition of W. Note that $g(x) = \tanh(x)$

satisfies $g' = 1 - g^2$, thus the above claim that $\phi(x) = \tanh(x/\varepsilon)$ follows. Since $\varepsilon\phi' = \sqrt{2W(\phi)}$, one can compute $E_\varepsilon(\phi)$:

$$E_\varepsilon(\phi) = \int_{\mathbb{R}} \varepsilon(\phi')^2\,dx = \int_{\mathbb{R}} \phi'\sqrt{2W(\phi)}\,dx = \int_{-1}^{1} \sqrt{2W(s)}\,ds(=:\sigma)$$

where the last equality follows by the change of variable $s = \phi(x)$. So for the simple one-dimensional problem, we note immediately that changing from -1 to 1 costs at least σ which is a constant depending only on W, not ϕ.

Consider then the multi-dimensional situation. Suppose that domain $\Omega \subset \mathbb{R}^n$ are divided into two domains Ω_+ and Ω_- separated by some hypersurface Γ which we assume to be sufficiently smooth for the moment, for example, C^2, and also suppose Γ is inside of Ω to avoid technicalities coming from boundary issue. Let $d : \Omega$ be the signed distance function to Γ, namely, $d(x) = \operatorname{dist}(x, \Gamma)$ if $x \in \Omega_+$ and $d(x) = -\operatorname{dist}(x, \Gamma)$ if $x \in \Omega_-$. It is well-known that d is a C^2 function in some neighorhood of Γ. On Γ the vector field ∇d defines the unit normal to Γ pointing towards Ω_+ and Δd coincides with the mean curvature of Γ. Now define $\phi(x) = \tanh(d(x)/\varepsilon)$ in the neighborhood of Γ and suitably taper off ϕ to constant ± 1 away from Γ so that for very small $\varepsilon > 0$, $\phi = 1$ inside Ω_+ away from Γ, and $= -1$ inside Ω_- away from Γ. The energy (2.1) for ϕ may be computed rather explicitly. By ignoring exponentially small numbers and using $|\nabla d| = 1$ and $\tanh(\cdot)' = \sqrt{2W(\tanh(\cdot))}$,

$$E_\varepsilon(\phi) = \int_\Omega \frac{1}{\varepsilon}(\tanh(\cdot)')^2\,dx = \int_\Omega (\tanh(\cdot)')\sqrt{2W(\tanh(\cdot))}\,\frac{|\nabla d|}{\varepsilon}\,dx.$$

By the Co-area formula (see for example [13]), we have

$$= \int_{-\infty}^{\infty} ds \int_{\{d(x)/\varepsilon = s\}} (\tanh(\cdot)')\sqrt{2W(\tanh(\cdot))}\,d\mathcal{H}^{n-1}.$$

Here $\mathcal{H}^{n-1}$ is the $n-1$-dimensional Hausdorff measure. Since the integrand inside is constant,

$$= \int_{-\infty}^{\infty} ds\mathcal{H}^{n-1}(\{d(x)/\varepsilon = s\})(\tanh(\cdot)')\sqrt{2W(\tanh(\cdot))}.$$

Since $\mathcal{H}^{n-1}(\{d(x)/\varepsilon = s\})$ is nearly equal to $\mathcal{H}^{n-1}(\Gamma)$, we obtain

$$\approx \sigma\mathcal{H}^{n-1}(\Gamma) \tag{2.2}$$

when $\varepsilon \approx 0$. The argument above just says that if ϕ is $\tanh(d(x)/\varepsilon)$, then $\sigma^{-1}E_\varepsilon(\phi)$ approximate the surface measure of Γ. This looks like a very special and specific choice of ϕ. But we now know that such approximation holds for a surprisingly very generic situation whenever we deal with variational problems involving E_ε. We do not go further into the up-to-date results on this but I hope that the reader do not feel uncomfortable thinking

$\sigma^{-1}E_\varepsilon \approx \mathcal{H}^{n-1}(\Gamma)$. Similar heuristic argument also indicates that, for any $\psi \in C_c(\Omega)$,

$$\sigma^{-1}\int_\Omega \psi\left(\frac{\varepsilon|\nabla\phi|^2}{2}+\frac{W(\phi)}{\varepsilon}\right)dx \approx \int_\Gamma \psi\, d\mathcal{H}^{n-1}$$

as $\varepsilon \approx 0$. Somewhat a crude rule of thumb is that

$$\frac{(\tanh(\cdot)')^2}{\varepsilon}dx \approx \sigma\mathcal{H}^{n-1}\lfloor_\Gamma$$

in the following computations.

3. Mean curvature

Continuing with this specific choice of ϕ, let us now consider the first variation of E_ε. It is

$$\delta E_\varepsilon = -\varepsilon\Delta\phi + \frac{W'(\phi)}{\varepsilon}.$$

Using $-\tanh(\cdot)'' + W'(\tanh(\cdot)) = 0$, for $\phi = \tanh(d/\varepsilon)$, we have $\delta E_\varepsilon = -(\tanh(\cdot)')\Delta d$. Thus we may expect that $\delta E_\varepsilon = 0$ implies $\Delta d = 0$, which simply means that Γ is a minimal hypersurface. From the previous section we also note that for $g \in C_c(\Omega;\mathbb{R}^n)$,

$$\int_\Omega\left(-\varepsilon\Delta\phi+\frac{W'(\phi)}{\varepsilon}\right)\nabla\phi\cdot g\,dx = \int_\Omega -\frac{(\tanh(\cdot)')^2\Delta d}{\varepsilon}\nabla d\cdot g\,dx \tag{3.1}$$

$$\approx \int_\Gamma \sigma H\nu\cdot g\,d\mathcal{H}^{n-1}$$

where $H\,(= -\Delta d)$ is the mean curvature of Γ, ν is the unit normal to Γ pointing inwards Ω_+. We may call $H\nu$ as the mean curvature vector, and above indicates

$$\left(-\varepsilon\Delta\phi+\frac{W'(\phi)}{\varepsilon}\right)\nabla\phi\,dx \approx \sigma H\nu\,d\mathcal{H}^{n-1}\lfloor_\Gamma. \tag{3.2}$$

This correspondence can be proved rigorously in some generalized sense. We also have

$$\varepsilon^{-1}\int_\Omega\left(-\varepsilon\Delta\phi+\frac{W'(\phi)}{\varepsilon}\right)^2 dx = \int_\Omega \frac{(\tanh(\cdot)')^2}{\varepsilon}(\Delta d)^2\,dx \tag{3.3}$$

$$\approx \int_\Gamma \sigma H^2\,d\mathcal{H}^{n-1}.$$

Though these approximations seem reasonable, it is with some great care that one can establish how these approximations make sense and under what conditions. In full generality, these relations are rigorously established only during the last 10 years. Again I do not go into the details on how they make sense.

4. Mean curvature flow with transport term

Given a vector field $v(x,t)$, consider the following PDE:

$$\phi_t + v\cdot\nabla\phi = \Delta\phi - \frac{W'(\phi)}{\varepsilon^2}. \tag{4.1}$$

Substitute $\phi = \tanh(d(x,t)/\varepsilon)$, where we regard $\Gamma = \Gamma(t)$ as a evolving hypersurface and $d = d(x,t)$ as the signed distance function to $\Gamma(t)$. We then obtain

$$d_t + v\cdot\nabla d = \Delta d$$

which says that the velocity vector $V_{\Gamma(t)}$ of $\Gamma(t)$ satisfies

$$V_{\Gamma(t)} = (v\cdot\nu)\nu + H\nu. \tag{4.2}$$

When $v = 0$, one can check that

$$\frac{d}{dt}\mathcal{H}^{n-1}(\Gamma) = -\int_\Gamma H\nu\cdot V_\Gamma\, d\mathcal{H}^{n-1} = -\int_\Gamma H^2\, d\mathcal{H}^{n-1}$$

so that the hypersurface area is a decreasing function of time. When $v \neq 0$ and Γ is assumed to be regular enough,

$$\frac{d}{dt}\mathcal{H}^{n-1}(\Gamma) = -\int_\Gamma (H^2 + H(v\cdot\nu))\, d\mathcal{H}^{n-1} \le -\frac{1}{2}\int_\Gamma (H^2 - |v|^2)\, d\mathcal{H}^{n-1}.$$

If we would like to have bounded hypersurface area as Γ evolves in time, then we require naturally that

$$v \in L^2_{loc}([0,\infty); L^2(\mathcal{H}^{n-1}\lfloor_\Gamma)). \tag{4.3}$$

In [8] we investigated the conditions under which the condition (4.3) can be guaranteed and at the same time the correspondence between (4.1) and (4.2) is correct. Roughly speaking, we showed that if v belongs to

$$L^p_{loc}([0,\infty); W^{1,p}(\Omega)) \tag{4.4}$$

for $p > \frac{n+2}{2}$ (and $n = 2, 3$) uniformly with respect to ε, then (4.1) converges to (4.2) as $\varepsilon \to 0$ and (4.3) is satisfied. In the passing we mention that (4.2) is satisfied in the sense of Brakke [4]. We use this approximation in the following.

5. Two phase flow with surface energy interaction

Here we first describe the simpler model [10] than the one we would like to consider eventually. Suppose that we have two-phase fluids with the same density, viscosity and linear stress tensor. Let $v = v(x,t)$, $p = p(x,t)$ be the flow field and pressure, respectively, and assume that we have a hypersurface $\Gamma = \Gamma(t)$. We postulate that v, p and Γ satisfy

$$\begin{cases} v_t + v\cdot\nabla v = \Delta v - \nabla p + \lambda_1 H\nu\mathcal{H}^{n-1}\lfloor_\Gamma, \\ \operatorname{div} v = 0 \end{cases} \tag{5.1}$$

in the distributional sense, where $\lambda_1 > 0$ is a constant. We assume that v is continuous across Γ in some distributional sense, but ∇v and p typically have jump due to the mean curvature term. We also postulate that Γ moves according to

$$V_\Gamma = (v \cdot \nu)\nu + \lambda_2 H\nu \tag{5.2}$$

where $\lambda_2 > 0$ is a constant. The law of motion (5.2) is different from just flowing along the fluid ($\lambda_2 = 0$), and it is the mixture of the mean curvature flow and a simple transport. For sufficiently smooth flow, we have (with periodic boundary conditions)

Proposition 1.

$$\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|v|^2\,dx + \lambda_1 \mathcal{H}^{n-1}(\Gamma)\right) = -\int_\Omega |\nabla v|^2\,dx - \lambda_1\lambda_2 \int_\Gamma H^2\,d\mathcal{H}^{n-1}. \tag{5.3}$$

Proof. By the first variation formula [13] and (5.2) we have

$$\frac{d}{dt}\mathcal{H}^{n-1}(\Gamma) = -\int_\Gamma H\nu \cdot V_\Gamma\,d\mathcal{H}^{n-1} = -\int_\Gamma H(\nu\cdot v) + \lambda_2 H^2\,d\mathcal{H}^{n-1}.$$

Then by integration by parts, we can check (5.3) holds. □

Proposition 1 shows that this model combines the two well-known energy dissipation laws, one is the Navier-Stokes like dissipation, and the other is the mean curvature flow like dissipation. We next consider what the phase field approximation of (5.1) and (5.2) would be. According to (4.1) and (4.2), (5.2) can be approximated by

$$\phi_t + v\cdot\nabla\phi = \lambda_2\left(\Delta\phi - \frac{W'(\phi)}{\varepsilon^2}\right). \tag{5.4}$$

As for (5.1), $\operatorname{div} v = 0$ is left unchanged. By (3.2), the mean curvature term can be approximated by

$$\lambda_1 H\nu\mathcal{H}^{n-1}\lfloor_\Gamma \approx -\frac{\lambda_1}{\sigma}\varepsilon\nabla\phi\Delta\phi\,dx. \tag{5.5}$$

The reason that we dropped $W'(\phi)\nabla\phi$ in (3.2) is that we may include $W'(\phi)\nabla\phi = \nabla(W(\phi))$ in the pressure term by re-defining $p = p + W(\phi)$. The resulting set of equations would be

$$\begin{cases} v_t + v\cdot\nabla v = \Delta v - \nabla p - \frac{\lambda_1}{\sigma}\varepsilon\nabla\phi\Delta\phi, \\ \operatorname{div} v = 0 \end{cases} \tag{5.6}$$

with (5.4). It is straightforward to check the following:

Proposition 2.

$$\begin{aligned} &\frac{d}{dt}\left(\int_\Omega \frac{1}{2}|v|^2\,dx + \frac{\lambda_1}{\sigma}E_\varepsilon(\phi)\right) \\ &= -\int_\Omega |\nabla v|^2\,dx - \frac{\lambda_1\lambda_2}{\sigma\varepsilon}\int_\Omega \left(-\varepsilon\Delta\phi + \frac{W'(\phi)}{\varepsilon}\right)^2 dx. \end{aligned} \tag{5.7}$$

Obviously, one notices that there are one-to-one correspondences between quantities appearing in (5.3) and (5.7) via (2.2) and (3.3). Existence of weak solution for (5.4) and (5.6) can be proved using the Galerkin method and Leray-Schauder fixed point theorem [7]. Mugnai and Röger [11] investigated $\varepsilon \to 0$ limit problem and showed that the limit interface satisfies the law of motion (4.2) in the sense of L^2 velocity. It is interesting to investigate if (4.2) is satisfied in the sense of Brakke [4], but it is not known so far. We mention that we could have used our result [8] if v satisfies (4.4) with $p > \frac{n+2}{2}$. But the apriori energy estimate (5.3) gives only $p = 2$, which is equal or less than $\frac{n+2}{2}$, the equality holding for $n = 2$. We expect that for $n = 2$, the smallness of initial energy should allow us to push the proof, but it is still under investigation.

6. Non-Newtonian two-phase flow with surface interaction

We next discuss one-step more complicated situation, where we have different non-Newtonian stress tensors and viscosity on each phase, but still the same density. We would like to apply the result of [8] and we find the non-Newtonian flow provides the correct setting, giving a better apriori regularity for v than the Navier-Stokes flow. Let τ_+ and τ_- be the stress tensors for fluids occupying Ω_+ and Ω_-, respectively. Assume that, for simplicity,

$$\tau_\pm(S) = \alpha_\pm(1+|S|^2)^{\frac{p-2}{2}} S \tag{6.1}$$

for symmetric $n \times n$ matrix $S = (S_{i,j})_{1\le i,j\le n}$, where we substitute $S = e(v)$, the symmetric part of ∇v. The constants $\alpha_\pm > 0$ are given. Furthermore we assume

$$p > \frac{n+2}{2}, \qquad n = 2, 3. \tag{6.2}$$

In particular we have $\tau_\pm(S) : S = \sum_{1\le i,j\le n} (\tau_\pm(S))_{i,j} S_{i,j} \ge \alpha_\pm |S|^p$. We jump right in to the phase field approximation now since the limit problem can be guessed easily from the discussion in Section 5. For ϕ we define

$$\tau(\phi, S) = \frac{\tau_+(S) - \tau_-(S)}{2}\phi + \frac{\tau_+(S) + \tau_-(S)}{2} \tag{6.3}$$

so that $\tau(1, S) = \tau_+(S)$ and $\tau(-1, S) = \tau_-(S)$. Then consider the following problem:

$$\begin{cases} v_t + v\cdot\nabla v = \operatorname{div}\tau(\phi, e(v)) - \nabla p - \frac{\lambda_1}{\sigma}\varepsilon\nabla\phi\Delta\phi, \\ \operatorname{div} v = 0, \\ \phi_t + v\cdot\nabla\phi = \lambda_2\left(\Delta\phi - \frac{W'(\phi)}{\varepsilon^2}\right). \end{cases} \tag{6.4}$$

The regular solution of (6.4) satisfies the energy law similar to (5.7), the difference being the replacement of $|\nabla v|^2$ by $\tau(\phi, e(v)) : e(v)$. Due to the assumptions (6.1) and (6.2), for this problem we have a uniform bound on the norm of (4.4) independent of ε. Thus we can apply the result of [8]. Here we just mention that we can show that the limit problem $\varepsilon \to 0$ defines a well-behaving weak solution with general initial data in the energy class and periodic boundary conditions. The detail will appear in [9]. We mention that the case of $\lambda_2 = 0$ has attracted much attention (see [1, 2, 14]).

7. Different density case

Finally in this section we describe the problem mentioned in Section 1. The problem is slightly different from the original Shen-Yang model in the definition of ρ but it is a minor difference. The guiding principle to deal with the density difference is the correct energy dissipation law. To do so define

$$\Phi(s) = \sigma^{-1} \int_{-1}^{s} \sqrt{2W(t)}\, dt,$$

$\rho(\phi) = \rho_+ \Phi(\phi) + \rho_-(1 - \Phi(\phi))$ so that $\rho(1) = \rho_+$ and $\rho(-1) = \rho_-$. We simply write ρ for $\rho(\phi)$. Even though it is more difficult to guess what the limit problem is than the previous cases, we still start out with the phase field approximation. Consider the following problem :

(7.1)

$$\begin{cases} \rho(v_t + v \cdot \nabla v) + \frac{1}{2}(\rho_t + v \cdot \nabla \rho) v = \operatorname{div}(\tau(\phi, e(v))) - \nabla p - \frac{\lambda_1}{\sigma} \varepsilon \nabla \phi \Delta \phi, \\ \operatorname{div} v = 0, \\ \phi_t + v \cdot \nabla \phi = \lambda_2 \left(\Delta \phi - \frac{1}{\varepsilon^2} W'(\phi) \right) \end{cases}$$

with a set of suitable boundary and initial conditions. Note that the first equation of (7.1) reduces to (1.1) on each bulk pure phase since ϕ and ρ are nearly constant.

Proposition 3. *The regular solution of* (7.1) *satisfies the following energy law:*

$$\begin{aligned} &\frac{d}{dt} \left(\int_\Omega \frac{1}{2} \rho |v|^2 \, dx + \frac{\lambda_1}{\sigma} E_\varepsilon(\phi) \right) \\ &= - \int_\Omega \tau(\phi, e(v)) : e(v) + \frac{\lambda_1 \lambda_2}{\varepsilon \sigma} \left(-\varepsilon \Delta \phi + \frac{W'(\phi)}{\varepsilon} \right)^2 dx. \end{aligned} \tag{7.2}$$

The proof is the consequence of direct computations. It is rather remarkable that the energy is still dissipative. From what we know already, when $\varepsilon \to 0$, (7.2) heuristically represents:

$$\begin{aligned} &\frac{d}{dt} \left\{ \int_{\Omega_+(t)} \frac{1}{2} \rho_+ |v|^2 + \int_{\Omega_-(t)} \frac{1}{2} \rho_- |v|^2 + \lambda_1 \mathcal{H}^{n-1}(\Gamma(t)) \right\} \\ &= - \int_{\Omega_+(t)} \tau_+(e(v)) : e(v) - \int_{\Omega_-(t)} \tau_-(e(v)) : e(v) - \lambda_1 \lambda_2 \int_{\Gamma(t)} H^2 \, d\mathcal{H}^{n-1}. \end{aligned}$$

Some heuristic argument using $\tanh(\cdot)$ shows that the jump condition across $\Gamma(t)$ for problem (7.1) as $\varepsilon \to 0$ reads as

$$\lambda_2 \rho_{\mathrm{gap}} (H \cdot \nu) v = (\tau_+(e(v)_+) - \tau_-(e(v)_-)) \cdot \nu - (p_+ - p_-)\nu + \lambda_1 H$$

all evaluated on $\Gamma(t)$ and where $\rho_{\mathrm{gap}} = (\rho_+ - \rho_-)/2$. Here p_+, $e(v)_+$ and p_-, $e(v)_-$ are limiting values of p, $e(v)$ approaching from $\Omega_+(t)$ and $\Omega_-(t)$, respectively. In the distributional sense, the limiting problem as $\varepsilon \to 0$ is (with $\rho = \rho(\phi)$ with $\phi = 1$ on $\Omega_+(t)$ and $\phi = -1$ on $\Omega_-(t)$)

$$(7.3)\qquad \begin{cases} \rho(v_t + v \cdot \nabla v) = \operatorname{div}(\tau(\phi, e(v))) - \nabla p \\ \qquad\qquad + (-\lambda_2 \rho_{\mathrm{gap}} (H \cdot \nu) v + \lambda_1 H)\, \mathcal{H}^{n-1} \lfloor_{\Gamma(t)}, \\ \operatorname{div} v = 0, \\ V_\Gamma = (v \cdot \nu)\nu + \lambda_2 H. \end{cases}$$

It is interesting to observe how the difference of density affects the jump conditions. Under the stated assumptions on $\tau_\pm$ and some suitable initial data, we can prove the existence results of the weak solution for the limiting problem (7.3) by using [8]. We only mention that we need to consider 'oriented varifold' to characterize the limit interface since we need to define $(H \cdot \nu)$ as in (7.3) in a weak form. The detail of the results are now in preparation.

References

[1] Abels, H., *On generalized solutions of two-phase flows for viscous incompressible fluids*, Interfaces Free Bound. **9** (2007), 31–65.

[2] Ambrose, D. M., Lopes Filho, M. C., Nussenzveig Lopes, H. J., Strauss, W. A., *Transport of interfaces with surface tension by 2D viscous flows*, Interfaces Free Bound. **12** (2010), 23–44.

[3] Anderson, D. M., McFadden, G. B., Wheeler, A. A., *Diffuse-interface methods in fluid mechanics*, Appl. Math. Lett. **30** (4) (1998), 139-165.

[4] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, NJ, (1978).

[5] Gurtin, M. E., Polignone, D., Viñals, J., *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Methods Appl. Sci., **6**(6) (1996), 815–831.

[6] Jacqmin, D., *Diffuse interface model for incompressible two-phase flows with large density ratios*, J. Comput. Phys. **155**(1) (2007), 96–127.

[7] Lin, F.-H., Liu, C., *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure. Appl. Math. **48** (1995), 501–537.

[8] Liu, C., Sato, N., Tonegawa, Y., *On the existence of mean curvature flow with transport term*, Interfaces Free Bound. **12** (2010), 251–277.

[9] Liu, C., Sato, N., Tonegawa, Y., *Two-phase flow problem coupled with mean curvature flow*, in preparation.

[10] Liu, C., Shen, J., *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Physica D, **179**(3-4) (2003), 211–228.

[11] L. Mugnai, M. Röger, *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, preprint.

[12] Shen, J., Yang, X., *A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities*. SIAM J. Sci. Comput. **32** (2010), 1159–1179.

[13] Simon, L., *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[14] Plotnikov, P. I., *Generalized solutions of a problem on the motion of a non-Newtonian fluid with a free boundary*. Siberian Math. J. **34** (1993), 704–716.

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