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ON ESTIMATES FOR AXISYMMETRIC SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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1. Introduction

We consider the three-dimensional Navier-Stokes equations for viscous incompressible flows in the whole space

$$
\text{(NS)} \qquad \begin{cases} \partial_t v - \Delta v + (v\cdot\nabla)v + \nabla p &= 0, \qquad (x,t)\in\mathbb{R}^3\times(0,T),\\ \nabla\cdot v &= 0 \qquad (x,t)\in\mathbb{R}^3\times(0,T),\\ v|_{t=0} &= v_0 \qquad x\in\mathbb{R}^3. \end{cases}
$$

Here $v=v(x,t)=(v_1(x,t),v_2(x,t),v_3(x,t))$ and $p=p(x,t)$ denote unknown velocity field and pressure field, respectively. With the notation of $L^2_\sigma(\mathbb{R}^3)=\{u\in(L^2(\mathbb{R}^3))^3 \mid \nabla\cdot u=0$ in the sense of distributions$\}$, we recall that a weak solution to (NS) with initial velocity $v_0\in L^2_\sigma(\mathbb{R}^3)$ is defined as a vector field $v\in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3))\cap L^2(0,T;(H^1(\mathbb{R}^3))^3)$ satisfying

$$
\int_0^T\int_{\mathbb{R}}\big(v\cdot\partial_t\varphi + v\cdot(v\cdot\nabla)\varphi - \nabla v\cdot\nabla\varphi\big)dxdt = -\int_{\mathbb{R}^3} v_0(x)\cdot\varphi(x,0)dx,
$$

for all $\varphi\in(C_0^\infty(\mathbb{R}^3\times[0,T)))^3$ with $\nabla\cdot\varphi=0$. The existence of weak solutions to the Navier-Stokes equations for viscous incompressible flows is well-known; see Leray [6] and Hopf [4].

In this report we are interested in axisymmetric flows, that is, the velocity v is written in the form $v=v_r\mathbf{e}_r+v_\theta\mathbf{e}_\theta+v_3\mathbf{e}_3$, where

$$
\mathbf{e}_r=(\frac{x_1}{r},\frac{x_2}{r},0),\quad \mathbf{e}_\theta=(\frac{-x_2}{r},\frac{x_1}{r},0),\qquad \mathbf{e}_3=(0,0,1),\quad r=\sqrt{x_1^2+x_2^2},
$$

and

$$
v_r=v_r(r,x_3,t),\ v_\theta=v_\theta(r,x_3,t),\ v_3=v_3(r,x_3,t).
$$

The associated vorticity field $\omega=\nabla\times v=\omega_r\mathbf{e}_r+\omega_\theta\mathbf{e}_\theta+\omega_3\mathbf{e}_3$ is then given by

$$
\omega_r=-\partial_3 v_\theta,\ \omega_\theta=\partial_3 v_r-\partial_r v_3,\ \omega_3=\frac{1}{r}\partial_r(rv_\theta). \tag{1.1}
$$

Even in the presence of the axial symmetry, the regularity of weak solutions to (NS) is not known in general, if the swirling component of the velocity, i.e., v_θ, is not trivial. On the other hand, due to the results of Caffarelli-Kohn-Nirenberg [2], it is well-knwon that if there is any singularity of axisymmetric suitable weak solutions to (NS), it must be along the symmetry axis $r = 0$. So in the case of axisymmetric solutions there is a possibility to get uniform bounds of weighted L^p norms of vorticity if the weight is suitably set near $r = 0$. From this motivation we introduce the following weighted norms for axisymmetric functions

$$(1.2)\quad \|f\|_{L^p_k} = \left(\int_{\mathbb{R}} \int_0^1 |r^k f(r, x_3)|^p r dr dx_3 + \int_{\mathbb{R}} \int_1^\infty |f(r, x_3)|^p r dr dx_3 \right)^{\frac{1}{p}},$$

where $k \geq 0$ and $p \in [1, \infty)$. The case $p = \infty$ is defined in the same manner.

For smooth initial data, the problem in this direction was firstly discussed by Chae-Lee [3], and they obtained $\omega_\theta \in L^\infty(0, T; L^2_3)$. Their result was improved by Kim [5], in which it is proved that $\omega_\theta \in L^\infty(0, T; L^p_k)$ with $k = (5p-6)/p$, $2 \leq p < \infty$. Recently, the end point case $(k, p) = (5, \infty)$ was established by Burke-Zhang [1]. For ω_r and ω_3, it is proved that $\omega_r, \omega_3 \in L^\infty(0, T; L^2_2)$ in [5], and $\omega_r, \omega_3 \in L^\infty(0, T; L^\infty_{10})$ in [1]. Although it is not difficult to see that their arguments can be applied to axisymmetric weak solutions satisfying the strong energy inequality, as far as the author knows, it is not obtained for the case without the strong energy inequality.

In this report we introduce the results of [7], in which the known weighted estimates are improved and such estimates are obtained for any axisymmetric weak solutions.

Theorem 1.1 ([7]). *Let $v \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0, T; (H^1(\mathbb{R}^3))^3)$ be an axisymmetric weak solution to (NS) with initial data $v_0 \in L^2_\sigma(\mathbb{R}^3)$. Then for all $\delta \in (0, T)$ we have*

$$(1.3)\qquad \omega_\theta \in L^\infty(\delta, T; L^p_{k_1}), \qquad k_1 = \frac{7}{2} - \frac{4}{p}, \; 2 \leq p < \infty,$$

$$(1.4)\qquad \omega_\theta \in L^\infty(\delta, T; L^\infty_{7/2+\epsilon}), \qquad \text{for all } \epsilon > 0,$$

$$(1.5)\qquad \omega_r, \omega_3 \in L^\infty(\delta, T; L^p_{k_2}), \qquad k_2 = 3 - \frac{4}{p}, \; 2 \leq p < \infty,$$

$$(1.6)\qquad \omega_r, \omega_3 \in L^\infty(\delta, T; L^\infty_{3+\epsilon}), \quad \text{for all } \epsilon > 0.$$

In Theorem 1.1 we do not need to assume that v satisfies the energy inequality. Let us focus on the case $p = 2$ in Theorem 1.1. Then one will see that the estimates for ω_r and ω_3 are natural from the scaling point of view. Indeed, it is easy to see that $r\omega$ and v have "the same scaling", that is, both have an invariant property with respect to the same scaling $f_\lambda(r, x_3, t) = \lambda f(\lambda r, \lambda x_3, \lambda^2 t)$. So it is expected that we might show the components of ω belong to $L^\infty(\delta, T; L^2_1)$, $\delta \in (0, T)$, from the assumption $v \in L^\infty(0, T; L^2_\sigma(\mathbb{R}^3))$, and this is indeed valid at least for ω_r and ω_3 by Theorem 1.1. However, there is an essential difficulty in obtaining this regularity for ω_θ. Especially, it is still open if $\omega_\theta \in L^\infty(\delta,T;L^2_1)$ holds in general.

The reason why we can obtain better bounds for ω_r and ω_3 is that they are directly related with the swirling component of the velocity, i.e., v_θ (see 1.1), for which we have the L^∞ bound of rv_θ as follows.

Theorem 1.2 ([7]). *Let $v \in L^\infty(0,T;L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,T;(H^1(\mathbb{R}^3))^3)$ be an axisymmetric weak solution to (NS) with initial data $v_0 \in L^2_\sigma(\mathbb{R}^3)$. Then for all $\delta \in (0,T)$ we have*

$$v_\theta \in L^\infty(\delta,T;L^\infty_1). \tag{1.7}$$

For smooth axisymmetric solutions of the Navier-Stokes equations the a priori bound of $v_\theta \in L^\infty(0,T;L^\infty_1)$ when $v_\theta|_{t=0} \in L^\infty_1$ is already known; for example, see [3]. In fact, it is well-known that the equation for rv_θ enables us to apply the maximum principle to rv_θ at least when v is smooth. But some additional arguments are required under the weak regularity assumptions on v as in Theorem 1.2 in order to verify the estimates of the maximum principle type.

The key step of the proofs of Theorem 1.1 and 1.2 is to analyze the axisymmetirc solutions to the linear second-order parabolic equations with a divergence free drift term in the cylindrical coordinates. The divergence free drift term, written as $u_r\partial_r + u_3\partial_3$ in the cylindrical coordinates, is assumed to have the regularity $u_r \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3))$ and $u_3 \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;H^1(rdrdx_3))$. Here $L^p(rdrdx_3), H^1(rdrdx_3)$, and $\tilde{H}^1(rdrdx_3)$ are defined as

$$\begin{aligned}
L^p(rdrdx_3) &= \{f = f(r,x_3) \in L^p(\mathbb{R}^2_+) \mid \|f\|_{L^p_0} < \infty\}, \ \mathbb{R}^2_+ = (0,\infty)\times\mathbb{R}, \\
H^1(rdrdx_3) &= \{f \in L^2(rdrdx_3) \mid \partial_r f, \ \partial_3 f \in L^2_0\}, \\
\tilde{H}^1(rdrdx_3) &= \{f \in H^1(rdrdx_3) \mid \frac{f}{r} \in L^2_0\}.
\end{aligned}$$

The norms of these function spaces are set in the natural way. The important fact in the axisymmetric framework is that three-dimensional nature of the problem appear only near the symmetry axis $r = 0$, and the equation is essentially two-dimensional if it is away from the symmetry axis. Focusing on this property, we will establish the regularity estimates for axisymmetric solutions to the linear parabolic equations with a divergence free drift term. In particular, our estimates reflect two-dimensional nature of the problem away from the symmetry axis; see Lemma 2.1 and Lemma 2.2. The detailed proofs of these lemmas are given in [7]. The key tool in the proofs is an interpolation inequality, which is special to three-dimensional axisymmetric functions; see Lemma 2.4 below.

2. Linearized problem and key lemma

In this section we state key lemmas on the axisymmetirc solutions to the linear second-order parabolic equations with a divergence free drift term. Then a brief outline of the proofs of Theorem 1.1 and Theorem 1.2 will be also given.

We first observe that the vorticity ω defined by (1.1) and the swirling component of the velocity, v_θ, satisfy the following equations

$$(2.1)\quad \partial_t\omega_r+v_r\partial_r\omega_r+v_3\partial_3\omega_r=\partial_r^2\omega_r+\frac{1}{r}\partial_r\omega_r+\partial_3^2\omega_r-\frac{1}{r^2}\omega_r+\partial_r v_r\omega_r+\omega_3\partial_3 v_r,$$

$$(2.2)\quad \partial_t\omega_\theta+v_r\partial_r\omega_\theta+v_3\partial_3\omega_\theta=\partial_r^2\omega_\theta+\frac{1}{r}\partial_r\omega_\theta+\partial_3^2\omega_\theta-\frac{1}{r^2}\omega_\theta+\frac{1}{r}v_r\omega_\theta+\frac{1}{r}\partial_3 v_\theta^2,$$

$$(2.3)\quad \partial_t\omega_3+v_r\partial_r\omega_3+v_3\partial_3\omega_3=\partial_r^2\omega_3+\frac{1}{r}\partial_r\omega_3+\partial_3^2\omega_3+\partial_3 v_3\omega_3+\omega_r\partial_r v_3.$$

$$(2.4)\quad \partial_t v_\theta+v_r\partial_r v_\theta+v_3\partial_3 v_\theta=\partial_r^2 v_\theta+\frac{1}{r}\partial_r v_\theta+\partial_3^2 v_\theta-\frac{1}{r^2}v_\theta-\frac{v_r}{r}v_\theta.$$

Let us introduce the weight function $\Phi_k(r)$ as

$$(2.5)\quad \Phi_k(r)=\chi_1(r)r^k+\chi_1^c(r),\qquad r\geq 0,$$

where χ_1 is a smooth nonnegative cut-off function such that $\chi_1(r)=1$ for $0\leq r\leq 1$ and $\chi_1(r)=0$ for $r\geq 2$, and $\chi_1^c(r)=1-\chi_1(r)$. Then by considering $\Phi_k\omega$ and $\Phi_k v_\theta$ and using (2.1)-(2.4), Theorem 1.1 and Theorem 1.2 are essentially reduced to the regularity problem for solutions to the linear problem

(2.6)
$$\begin{cases} \partial_t w-Lw=f+\dfrac{g}{r}+\partial_r h_r+\partial_3 h_3, & (r,x_3)\in(0,\infty)\times\mathbb{R},\ t>0,\\ w|_{r=0}=0,\quad w|_{t=0}=w_0, \end{cases}$$

where

$$(2.7)\quad Lw=\partial_r^2 w+\frac{1+l_1}{r}\partial_r w+\partial_3^2 w+(b-\frac{l_2}{r^2})w-(u_r+a_r)\partial_r w-(u_3+a_3)\partial_3 w.$$

Here $l_1\in\mathbb{R}$ and $l_2\geq 0$ are given numbers, and a_r, a_3, b, u_r, u_3, f, g, h_r, h_3, w_0 are given functions which possess suitable regularities. We also assume that (u_r,u_3) satisfies the divergence free condition: $u_r/r+\partial_r u_r+\partial_3 u_3=0$. Then the formal adjoint operator L^* of L with respect to the inner product of $L^2(rdrdx_3)$ under the zero boundary condition at $r=0$ is given by

$$\begin{aligned}(2.8)\,L^*w &= \partial_r^2 w+\frac{1-l_1}{r}\partial_r w+\partial_3^2 w+(\frac{a_r}{r}+\partial_r a_r+\partial_3 a_3+b-\frac{l_2}{r^2})w\\ &\quad +(u_r+a_r)\partial_r w+(u_3+a_3)\partial_3 w.\end{aligned}$$

Note that (2.1)-(2.3) have to be considered in the very weak sense, for we have only $\omega \in L^2(0,T;L^2(\mathbb{R}^3))$ at the first stage. For this reason we introduce a definition of very weak solutions to (2.6).

We say that $w \in L^2(0,T;L^2_{loc}(rdrdx_3))$ is a very weak solution to (2.6) with initial data $w_0 \in L^1_{loc}(rdrdx_3)$ if

$$\begin{aligned} -\int_0^T\int_{\mathbb{R}^3} w(\partial_t\varphi + L^*\varphi)dxdt &= \int_{\mathbb{R}^3} w_0\varphi(\cdot,0)dx + \int_0^T\int_{\mathbb{R}^3}(f+\frac{g}{r})\varphi dxdt \\ &\quad -\int_0^T\int_{\mathbb{R}^3}(\frac{h_r}{r}\varphi + h_r\partial_r\varphi + h_3\partial_3\varphi)dxdt, \end{aligned} \tag{2.9}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2_+ \times [0,T))$. In (2.9) the notation $\int_{\mathbb{R}^3} dx$ is interpreted as $\int_{\mathbb{R}^2_+} rdrdx_3$ and L^* is the operator defined by (2.8). On the other hand, as usual, we say that $w \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;H^1(rdrdx_3))$ is a weak solution to (2.6) if

$$\begin{aligned} &\int_0^T\int_{\mathbb{R}^3}\{-w\partial_t\varphi + (u_r\partial_r w + u_3\partial_3 w)\varphi + (\partial_r w\partial_r\varphi + \partial_3 w\partial_3\varphi)dxdt \\ &= \int_{\mathbb{R}^3} w_0\varphi(\cdot,0)dx + \int_0^T\int_{\mathbb{R}^3}\{\frac{l_1}{r}\partial_r w - a_r\partial_r w - a_3\partial_3 w + (b - \frac{l_2}{r^2})w\}\varphi dxdt \\ &\quad + \int_0^T\int_{\mathbb{R}^3}(f+\frac{g}{r})\varphi dxdt - \int_0^T\int_{\mathbb{R}^3}(\frac{h_r}{r}\varphi + h_r\partial_r\varphi + h_3\partial_3\varphi)dxdt, \end{aligned} \tag{2.10}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2_+ \times [0,T))$. Clearly, if w is a weak solution, then it is a very weak solution.

The following lemma for very weak solutions to (2.6) is the key of the proofs of Theorem 1.1. For simplicity we consider the case $2 \le p < \infty$ in the theorem.

Below we always assume that there are $m_1,\ m_2 \in (3/2,\infty]$ such that

$$\begin{cases} u_r \in L^\infty(0,T;L^2_\sigma(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3)), \\ u_3 \in L^\infty(0,T;L^2_\sigma(rdrdx_3)) \cap L^2(0,T;H^1(rdrdx_3)), \\ w_0 \in L^2(rdrdx_3), \\ a_r,\ a_3 \in L^\infty(0,T;L^{2m_1}(rdrdx_3)),\ b \in L^\infty(0,T;L^{m_2}(rdrdx_3)), \\ f \in L^2(0,T;L^{\frac{6}{5}}(rdrdx_3)), \quad g,\ h_r,\ h_3 \in L^2(0,T;L^2(rdrdx_3)). \end{cases} \tag{2.11}$$

Lemma 2.1 ([7]). *Let (2.11) holds and $l_2 > 0$. Assume further that $a_r/r,\ \partial_r a_r,\ \partial_3 a_3 \in L^\infty(0,T;L^{m_1}(rdrdx_3))$. Let $w \in L^2(0,T;L^2(rdrdx_3))$ be a very weak solution to (2.6) satisfying*

$$\frac{w}{r} \in L^2(0,T;L^2(rdrdx_3)). \tag{2.12}$$

Then $w \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3))$.

If in addition there are $p \in (2,\infty)$, $q_1 \in (1,p]$, $q_i \in (1,p/2]$, $i = 2,3,4$, such that

$$r^{\frac{1}{p}-\frac{1}{q_1}} f \in L^{\frac{pq_1}{pq_1+q_1-p}}(0,T;L^{q_1}(rdrdx_3)),\ r^{\frac{1}{p}-\frac{1}{2q_2}} g \in L^{\frac{2pq_2}{pq_2+2q_2-p}}(0,T;L^{2q_2}(rdrdx_3)),$$
$$r^{\frac{1}{p}-\frac{1}{2q_3}} h_r \in L^{\frac{2pq_3}{pq_3+2q_3-p}}(0,T;L^{2q_3}(rdrdx_3)),\ r^{\frac{1}{p}-\frac{1}{2q_4}} h_3 \in L^{\frac{2pq_4}{pq_4+2q_4-p}}(0,T;L^{2q_4}(rdrdx_3)),$$

then $w \in L^\infty(\delta,T;L^p(rdrdx_3))$ for all $\delta > 0$,

The key lemma for Theorem 1.2 is stated as follows.

Lemma 2.2 ([7]). *Let (2.11) holds and $l_2 \geq 0$. Assume further that a_r/r, $\partial_r a_r$, $\partial_3 a_3 \in L^\infty(0,T;L^{m_1}(rdrdx_3))$. Let $w \in L^2(0,T;L^2(rdrdx_3))$ be a very weak solution to (2.6) satisfying*

$$\frac{w}{r} \in L^2(0,T;L^2(rdrdx_3)). \tag{2.13}$$

Then $w \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3))$.
If in addition there are $\kappa_i > 0$ and $q_i \in (1,\infty)$, $i = 1,2,3,4$, such that

$$r^{-\frac{1}{q_1}} f \in L^{\frac{q_1}{q_1-1}}(0,T;L^{q_1+\kappa_1}(rdrdx_3)),\quad r^{-\frac{1}{2q_2}} g \in L^{\frac{2q_2}{q_2-1}}(0,T;L^{2q_2+\kappa_2}(rdrdx_3)),$$
$$r^{-\frac{1}{2q_3}} h_r \in L^{\frac{2q_3}{q_3-1}}(0,T;L^{2q_3+\kappa_3}(rdrdx_3)),\quad r^{-\frac{1}{2q_4}} h_3 \in L^{\frac{2q_4}{q_4-1}}(0,T;L^{2q_4+\kappa_4}(rdrdx_3)),$$

then $w \in L^\infty(\delta,T;L^\infty(rdrdx_3))$ for all $\delta > 0$,

Remark 2.3. It is possible to generalize the regularity assumptions on a_r, a_3, b. But we omit it since it is not essential in this work.

Lemma 2.1 and Lemma 2.2 are proved through two steps. Under the assumptions of lemmas we first establish the existence of weak solutions to (2.6) satisfying the desired regularities. In particular, for the L^∞ estimates in Lemma 2.2 we use the Nash-Moser iteration arguments. Next we prove the uniqueness of very weak solutions to (2.6). In this second step the condition $w/r \in L^2(0,T;L^2(rdrdx_3))$ is essential, which is satisfied by $\Phi_k\omega$ and $\Phi_k v_\theta$ at least when $k \geq 1$. These two steps clearly ensure the regularity of very weak solutions to (2.6).

In the proofs of the above lemmas the following Sobolev embedding theorem, which is special to three-dimensional axisymmetric functions, plays important roles.

Lemma 2.4. *Let $f \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3))$. Then $r^{1/2} f \in L^{2p/(p-2)}(0,T;L^p(\mathbb{R}^2_+))$ for each $2 \leq p < \infty$, and the inequality*

$$\|r^{\frac{1}{2}} f\|_{L^{\frac{2p}{p-2}}(0,T;L^p(\mathbb{R}^2_+))} \leq C\|f\|^{1-\frac{2}{p}}_{L^2(0,T;\tilde{H}^1(rdrdx_3))}\|f\|^{\frac{2}{p}}_{L^\infty(0,T;L^2(rdrdx_3))}, \tag{2.14}$$

holds.

Proof of Lemma 2.4. Here we give a sketch of the proof of Lemma 2.4. By the integration by parts, we observe that for all $f \in \tilde{H}^1(rdrdx_3)$,
(2.15)
$$\int_{\mathbb{R}^2_+} |\partial_r(r^{\frac{1}{2}}f(t,r,x_3))|^2 drdx_3 = \int_{\mathbb{R}^2_+} |\partial_r f(r,x_3)|^2 rdrdx_3 + \frac{1}{4}\int_{\mathbb{R}^2_+} |\frac{f(r,x_3)}{r}|^2 rdrdx_3.$$

Hence from the Gagliardo-Nirenberg inequality we have
(2.16)
$$\|r^{\frac{1}{2}}f\|_{L^p(\mathbb{R}^2_+)} \le C\|\tilde{\nabla}(r^{\frac{1}{2}}f)\|^{1-\frac{2}{p}}_{L^2(\mathbb{R}^2_+)}\|r^{\frac{1}{2}}f\|^{\frac{2}{p}}_{L^2(\mathbb{R}^2_+)} \le C\|f\|^{1-\frac{2}{p}}_{\tilde{H}(rdrdx_3)}\|f\|^{\frac{2}{p}}_{L^2(rdrdx_3)},$$
for each $p \in [2,\infty)$. Here $\tilde{\nabla} = (\partial_r, \partial_3)$. Then it is easy to see that $r^{1/2}f \in L^{2p/(p-2)}(0,T;L^p(\mathbb{R}^2_+))$ if $f \in L^\infty(0,T;L^2(rdrdx_3)) \cap L^2(0,T;\tilde{H}^1(rdrdx_3))$ and (2.14) holds. This completes the proof.

By using Lemma 2.1 - 2.4 we obtain the main theorems in the following manner. The whole proof will be given in [7].

(1) We first establish the $L^\infty(\delta,T;L^2_{3/2})$ bound of ω_θ by making use of Lemma 2.1. Indeed, after direct calculations we see that $\Phi_k\omega_\theta$ satisfies the following equation in the very weak sense:

$$(2.17)\quad \partial_t w + v_r\partial_r w + v_3\partial_3 w = \partial_r^2 w + \frac{1-2k}{r}\partial_r w + \partial_3^2 w - a\partial_r w + (b - \frac{l}{r^2})w + F,$$
where
(2.18)
$$F = \frac{l+k^2-1}{r^2}\Phi_k\omega_\theta + v_r\omega_\theta(\frac{1+k}{r}\Phi_k + r^k\chi_1' + (\chi_1^c)' - \frac{k}{r}\chi_1^c) + \frac{\Phi_k}{r}\partial_3(v_\theta^2).$$

Here $l > 0$ and a, b are smooth bounded functions depending only on r defined by

$$(2.19)\quad a(r) = \frac{2}{\Phi_k}(r^k\chi_1' + (\chi_1^c)' - \frac{k}{r}\chi_1^c),$$

$$(2.20)\quad b(r) = -\frac{1}{\Phi_k}(r^k\chi_1^{(2)} + (\chi_1^c)^{(2)} + r^{k-1}\chi_1' + \frac{1-2k}{r}(\chi_1^c)' + \frac{k^2}{r^2}\chi_1^c)$$
$$+\frac{2\Phi_k'}{\Phi_k^2}(r^k\chi_1' + (\chi_1^c)' - \frac{k}{r}\chi_1^c).$$

The key observation here is that if $k \ge 3/2$ then we can write F in the form $f + g/r + \partial_r h_r + \partial_3 h_3$ with the regularities stated in (2.11).

(2) Next we establish $L^\infty(\delta,T;L_1^\infty)$ bound of v_θ from Lemma 2.2. We see that $\Phi_k v_\theta$ satisfies the following equation in the weak sense
(2.21)
$$\partial_t w + v_r\partial_r w + v_3\partial_3 w = \partial_r^2 w + \frac{1-2k}{r}\partial_r w + \partial_3^2 w + (b - \frac{1-k^2}{r^2})w - a\partial_r w + H,$$

where a and b are functions defined by (2.19) and (2.20), and

$$H = v_r v_\theta \big(\frac{k-1}{r} \Phi_k + r^k \chi_1' + (\chi_1^c)' - \frac{k}{r} \chi_1^c \big). \tag{2.22}$$

In fact, the result of (1) implies that H satisfies the conditions in Lemma 2.2 if $k \geq 1$.

(3) Thirdly we establish the $L^\infty(\delta, T; L_1^2)$ bound of ω_r, ω_3 from Lemma 2.1. We can check that $\Phi_k \omega_r$ and $\Phi_k \omega_3$ respectively satisfy the following equations in the very weak sense

$$\partial_t w - Lw = G_r, \qquad \partial_t w - Lw = G_3, \tag{2.23}$$

where L is the operator defined by (2.7) and

$$\begin{aligned} G_r &= \frac{l+k^2-1}{r^2} \Phi_k \omega_r + v_r \omega_r \big(\frac{k}{r} \Phi_k + r^k \chi_1' + (\chi_1^c)' - \frac{k}{r} \chi_1^c \big) + \Phi_k (\partial_r v_r \omega_r + \omega_3 \partial_3 v_r), \\ G_3 &= \frac{l+k^2}{r^2} \Phi_k \omega_3 + v_r \omega_3 \big(\frac{k}{r} \Phi_k + r^k \chi_1' + (\chi_1^c)' - \frac{k}{r} \chi_1^c \big) + \Phi_k (\partial_3 v_3 \omega_3 + \omega_r \partial_r v_3). \end{aligned}$$

Then the special structures of G_r and G_3 enable us to use the result of (2) effectively, and we can show that if $k \geq 1$ then G_r and G_3 are written in the form $g/r + \partial_r h_r + \partial_3 h_3$ with the regularities stated in (2.11).

(4) Finally we establish from Lemma 2.1 the $L^\infty(\delta, T; L_k^p)$ bounds of ω_θ, ω_r, ω_3, by the bootstrap arguments.

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