数理解析研究所講究録
第 1731 巻 2011 年 40-51

Weil-Petersson幾何の問題

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CONTENTS

NOTATIONS

$T_{g,n}$: the **Teichmüller space** of curves of genus g with n marked points $(2g-2+n>0)$

$C_{g,n}$: the **Teichmüller curve** over $T_{g,n}$ with the projection $\pi : C_{g,n} \to T_{g,n}$ which has n sections $\mathbf{P}_1, \ldots, \mathbf{P}_n$ corresponding to n marked points

$\Omega^1_{C_{g,n}}$(resp. $\Omega^1_{T_{g,n}}$) : the sheaf of holomorphic 1-forms on $C_{g,n}$ (resp. $T_{g,n}$)

$\omega_{C_{g,n}/T_{g,n}} := \Omega^1_{C_{g,n}}/\pi^*\Omega^1_{T_{g,n}}$: the sheaf of **relative differential forms** on $C_{g,n}$

$\lambda_l := \bigwedge^{\max} R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}\big((l-1)(\mathbf{P}_1+\cdots+\mathbf{P}_n)\big)$
: the **determinant line bundle** λ_l on $T_{g,n}$ $(l \in \mathbf{N})$

For a point $s \in T_{g,n}$,
$S := \pi^{-1}(s)$ a compact smooth curve
$S^0 := S - \{\mathbf{P}_1(s), \ldots, \mathbf{P}_n(s)\}$
$P_p := \mathbf{P}_p(s)$ $(p = 1, \ldots, n)$

$R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}\big((l-1)(\mathbf{P}_1+\cdots+\mathbf{P}_n)\big)|_s$
$= \Gamma(S, K_S^{\otimes l} \otimes \mathcal{O}_S(P_1+\cdots+P_n)^{\otimes(l-1)})$
$\simeq$\{meromorphic l differentials on S with possibly poles of order at most $l-1$ only at the marked points\}

§1. The index theorem for the family of curves
—Introduction to the Weil-Petersson metric

Pick a basis of local holomorphic sections $\phi_1, \dots, \phi_{d(l)}$ for $R^0\pi_*\omega^{\otimes l}_{C_{g,n}/T_{g,n}}\big((l-1)(\mathbf{P}_1 + \cdots + \mathbf{P}_n)\big)$, where

$$d(l) = \begin{cases} g & (l = 1) \\ (2l-1)(g-1) + (l-1)n & (l > 1). \end{cases}$$

$$\langle \phi_i, \phi_j \rangle := \iint_{S^0} \phi_i\, \overline{\phi_j}\, \rho_{S^0}^{-(l-1)} \; (i, j = 1, \dots, d(l))$$

the **Petersson product**, where ρ_{S^0} is the hyperbolic area element on S^0.

We set
$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2} := |\det(\langle \phi_i, \phi_j \rangle)|^{1/2}$

$\|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_Q := \|\phi_1 \wedge \cdots \wedge \phi_{d(l)}\|_{L^2}\, Z_{S^0}(l)^{-\frac{1}{2}}$

($l \geq 2$. For $l = 1$, employ $Z'_{S^0}(1)$ in place of $Z_{S^0}(1) = 0$.) Here, $Z_{S^0}(l)$ denotes the special value of $Z_{S^0}(\cdot)$ on S^0 at l integer.

$\lambda_l \to T_{g,n}$ is a Hermitian holomorphic line bundle equipped with the **Quillen metric** $\|\cdot\|_Q$. Here

$$Z_{S^0}(s) := \prod_{\{\gamma\}} \prod_{m=1}^{\infty} \left(1 - e^{-(s+m)L(\gamma)}\right)$$

is the **Selberg Zeta function** for S^0, Re $(s) > 1$, where γ runs over all oriented primitive closed geodesics on S^0, and $L(\gamma)$ denotes the hyperbolic length of γ. It extends meromorphically to the whole plane in s.

In the late 80's, we have discovered the following important formulas for the curvature forms of the determinant line bundles with respect to the Quillen metrics.

Theorem 1 (Belavin-Knizhnik+Wolpert(1986)).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} \quad (n = 0).$$

Theorem 2 (Takhtajan-Zograf (1988, 1991)).

$$c_1(\lambda_l, \|\cdot\|_Q) = \frac{6l^2 - 6l + 1}{12\pi^2}\omega_{WP} - \frac{1}{9}\omega_{TZ} \; (n > 0).$$

Here, ω_{WP}, ω_{TZ} are the Kähler forms of the Weil-Petersson, the Takhtajan-Zograf metrics respectively.

Here remind us of the definitions of the Weil-Petersson and the Takhtajan-Zograf metrics. By the deformation theory of Kodaira-Spencer and the Hodge theory, for $[S^0] \in T_{g,n}$,

$$T_{[S^0]}T_{g,n} \simeq HB(S^0),$$

where $HB(S^0)$ is the space of harmonic Beltrami differentials on S^0.

By the Serre duality,

$$T^*_{[S^0]}T_{g,n} \simeq Q(S^0),$$

where $Q(S^0)$ is the space of holomorphic quadratic differentials on S^0 with finite the Petersson-norm, which is dual to $HB(S^0)$.

The inner product of the **Weil-Petersson metric** at $T_{[S^0]}T_{g,n}$ is defined to be

$$\langle \alpha, \beta \rangle_{WP}([S^0]) := \iint_{S^0} \alpha\overline{\beta}\ \rho_{S^0},$$

where α, β are in $HB(S^0) \simeq T_{[S^0]}T_{g,n}$.

The inner products of the **Takhtajan-Zograf metrics** are defined to be

$$\langle \alpha, \beta \rangle_p([S^0]) := \iint_{S^0} \alpha\overline{\beta}E_p(\cdot, 2)\ \rho_{S^0}$$

$(p = 1, \ldots, n)$. Here, $E_p(\cdot, 2)$ is the Eisenstein series associated with the p-th marked point with index 2. Moreover, we set

$$\langle \alpha, \beta \rangle_{TZ}([S^0]) := \sum_{p=1}^{n} \langle \alpha, \beta \rangle_p([S^0]).$$

The **Eisenstein series** associated with the p-th marked point with index 2 is defined to be

$$E_p(z, 2) := \sum_{A \in \Gamma_p \backslash \Gamma} \left\{\mathrm{Im}(\sigma_p^{-1}A(z))\right\}^2, \text{ for } z \in \mathbf{H},$$

where $\mathbf{H}$ is the upper-half plane, Γ is a uniformizing Fuchsian group and Γ_p is the parabolic subgroup associated with the p-th marked point, and $\sigma_p \in \mathrm{PSL}(2, \mathbf{R})$ is a normalizer.

$E_p(z, 2)$ assumes the infinity at the p-th marked point and vanishes at the other marked points. In addition, the Eisenstein series satisfy

$$\Delta E_p(z, 2) = 2E_p(z, 2),$$

where Δ is the negative hyperbolic Laplacian on S^0. $E_p(z, 2)$ is a positive subharmonic function on S^0.

$\mathrm{Mod}_{g,n}$ denotes the **mapping class group** of curves of genus g with n marked points. Then the **moduli space** $\mathcal{M}_{g,n}$ of curves of genus g with n marked points is described as $\mathcal{M}_{g,n} = T_{g,n}/\mathrm{Mod}_{g,n}$. λ_l and all metrics we defined are compatible with the action of $\mathrm{Mod}_{g,n}$, thus they all naturally descend down to $\mathcal{M}_{g,n}$ as orbifold line sheaves and orbifold metrics respectively.

There are several basic results for the second cohomology groups of the moduli spaces of curves and the Weil-Petersson and the Takhtajan-Zograf Kähler forms.

Theorem 3 (Weng (2001)).
We have an isometric decomposition of the determinant line bundle with appropriate hermitian metrics $(2g-2+n>0, n>0)$.

$$\lambda_l^{\otimes 12} \simeq \Delta_{WP}^{\otimes 6l^2-6l+1} \otimes \Delta_{TZ}^{-1},$$

$$c_1(\Delta_{WP}) = \frac{\omega_{WP}}{\pi^2}, \quad c_1(\Delta_{TZ}) = \frac{4}{3}\omega_{TZ}.$$

Δ_{WP}, Δ_{TZ}*: the Weil-Petersson line bundle, the Takhtajan-Zograf line bundle respectively.*

Theorem 4 (Wolpert (1986), Takhtajan-Zograf (1991)).
For $g>2$,

$$H^2(\mathcal{M}_g, \mathbf{Z}) \simeq \mathbf{Z} \simeq \left\langle \left[\frac{\omega_{WP}}{\pi^2}\right] \right\rangle,$$

$$H^2(\mathcal{M}_{g,1}, \mathbf{Z}) \simeq \mathbf{Z}^2 \simeq \left\langle \left[\frac{\omega_{WP}}{\pi^2}\right], \left[\frac{4}{3}\omega_{TZ}\right] \right\rangle.$$

Here, $\mathcal{M}_g = \mathcal{M}_{g,0}$.

Theorem 5 (Weng (2001), Wolpert (2007), Albin-Rochon (2009)).
For $2g-2+n>0, n>0$,

$$c_1(\Delta_p) = \left[\frac{4}{3}\omega_p\right].$$

Here, Δ_p *denotes the line bundle associated with the p-th marked point over* $T_{g,n}$. ω_p *denotes the Kähler form of the Takhtajan-Zograf metric associated with the p-th marked point.*

Theorem 6 (Weng (2001), Wolpert (2007) + Harer).
For $g>2, n>0$,

$$H^2(\mathcal{M}_{g,n}, \mathbf{Z}) \simeq \mathbf{Z}^{n+1}$$
$$\simeq \left\langle \left[\frac{\omega_{WP}}{\pi^2}\right], \left[\frac{4}{3}\omega_1\right], \ldots, \left[\frac{4}{3}\omega_n\right] \right\rangle.$$

Let $\overline{\mathcal{M}}_{g,n}$ denote the **Deligne-Mumford compactification** of $\mathcal{M}_{g,n}$. We have known the relations of the L^2–cohomology of $\mathcal{M}_{g,n}$ with respect to the Weil-Petersson metric and the second cohomology of $\overline{\mathcal{M}}_{g,n}$.

Theorem 7 (Saper (1993)).
For $g>1, n=0$,

$$H^*_{(2)}(\mathcal{M}_g, \omega_{WP}) \simeq H^*(\overline{\mathcal{M}}_g, \mathbf{R}).$$

Here, the left hand side is the L^2-*cohomology with respect to the Weil-Petersson metric.*

The proof of Theorem 7 is based on the asymptotic behavior of the Weil-Petersson metric near the boundary of the moduli space which we will review now.

Consider the asymptotic behavior of the W-P metric and the T-Z metric near the boundary of $\mathcal{M}_{g,n}$. Here we set

$D := \overline{\mathcal{M}}_{g,n} \backslash \mathcal{M}_{g,n}$: the compactification divisor

$R_0 \in D$: a stable curve of genus g with n marked points and k nodes (we regard the marked points as deleted from the surface.)

Each node q_i $(i = 1, 2, \dots, k)$ has a neighborhood

$$N_i = \{(z_i, w_i) \in \mathbf{C}^2 \mid |z_i|, |w_i| < 1, z_i w_i = 0\}.$$

R_t denotes the smooth surface gotten from R_0 after cutting and pasting N_i under the relation $z_i w_i = t_i$, $|t_i|$ small. Then, D is locally described as $\{t_1 \cdots t_k = 0\}$.

D has locally the pinching coordinate $(t, s) = (t_1, \dots, t_k, s_{k+1}, \dots, s_{3g-3+n})$ around $[R_0]$. Set $\alpha_i = \partial/\partial t_i, \beta_\mu = \partial/\partial s_\mu \in T_{(t,s)}(T_{g,n})$. We define the Riemannian tensors for the Weil-Petersson metric

$$g_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{WP}(t, s),$$
$$g_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{WP}(t, s),$$
$$g_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{WP}(t, s),$$

$(i, j = 1, 2, \dots, k,\ \mu, \nu = k+1, \dots, 3g-3+n)$.

Furthermore, we define the Riemannian tensors for the Takhtajan-Zograf metric

$$h_{i\bar{j}}(t, s) := \langle \alpha_i, \alpha_j \rangle_{TZ}(t, s),$$
$$h_{i\bar{\mu}}(t, s) := \langle \alpha_i, \beta_\mu \rangle_{TZ}(t, s),$$
$$h_{\mu\bar{\nu}}(t, s) := \langle \beta_\mu, \beta_\nu \rangle_{TZ}(t, s),$$

$(i, j = 1, 2, \dots, k,\ \mu, \nu = k+1, \dots, 3g-3+n)$.

The following theorem is a pioneering result for the asymptotic behavior of the W-P metric near the boundary of the moduli space.

Theorem 8 (Masur (1976)). *As $t_i, s_\mu \to 0$,*

$$i)\quad g_{i\bar{i}}(t, s) \approx \frac{1}{|t_i|^2(-\log|t_i|)^3} \quad for\ i \le k,$$
$$ii)\quad g_{i\bar{j}}(t, s) = O\left(\frac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$$
$$for\ i, j \le k, i \ne j,$$
$$iii)\quad g_{i\bar{\mu}}(t, s) = O\left(\frac{1}{|t_i|(-\log|t_i|)^3}\right)$$
$$for\ i \le k, \mu \ge k+1,$$
$$iv)\quad g_{\mu\bar{\nu}}(t, s) \longrightarrow g_{\mu\bar{\nu}}(0, 0) \quad for\ \mu, \nu \ge k+1.$$

Recently, we updated Masur's result by improving Wolpert's formula for the asymptotic of the hyperbolic metric for degenerating Riemann surfaces.

Theorem 9 (O. and Wolpert (2008)). *We can improve iv) in Theorem 8 as follows;*

$$iv)'\ g_{\mu\bar{\nu}}(t,s) = g_{\mu\bar{\nu}}(0,s) + \frac{4\pi^4}{3}\sum_{i=1}^{k}(\log|t_i|)^{-2}\Big\langle \beta_\mu, (E_{i,1}+E_{i,2})\beta_\nu \Big\rangle_{WP}(0,s) + O\Big(\sum_{i=1}^{k}(\log|t_i|)^{-3}\Big)$$

as $t \to 0$, *for* $\mu, \nu \geq k+1$.

Here, $E_{i,1}, E_{i,2}$ *denote a pair of the Eisenstein series with index* 2 *associated with the* i*-th node of the limit surface* R_0.

That is, the Takhtajan-Zograf metrics have appeared from degeneration of the Weil-Petersson metric! On the other hand, we have a result for asymptotics of the Takhtajan-Zograf metric near the boundary of the moduli space.

Theorem 10 (O.-To-Weng (2008)). *As* $(t,s) \to 0$, *we observe the followings:*
i) For any $\varepsilon > 0$, *there exists a constant* $C_{1,\varepsilon}$ *such that*

$$h_{i\bar{i}}(t,s) \leq \frac{C_{1,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4-\varepsilon}} \quad \text{for } i \leq k;$$

For any $\varepsilon > 0$, *there exists a constant* $C_{2,\varepsilon}$ *such that*

$$h_{i\bar{i}}(t,s) \geq \frac{C_{2,\varepsilon}}{|t_i|^2(-\log|t_i|)^{4+\varepsilon}} \quad \text{for } i \leq k$$

and the node q_i *adjacent to punctures;*

ii) $h_{i\bar{j}}(t,s) = O\left(\dfrac{1}{|t_i||t_j|(\log|t_i|)^3(\log|t_j|)^3}\right)$
for $i, j \leq k, i \neq j$;

iii) $h_{i\bar{\mu}}(t,s) = O\left(\dfrac{1}{|t_i|(-\log|t_i|)^3}\right)$
for $i \leq k, \mu \geq k+1$;

iv) $h_{\mu\bar{\nu}}(t,s) \longrightarrow h_{\mu\bar{\nu}}(0,0)$ *for* $\mu, \nu \geq k+1$.

Open problems

1.** Determine $H^*_{(2)}(\mathcal{M}_{g,n}, \omega_{TZ})$ for general (g,n), originally asked by To and Weng. For that, we need more informations on precise asymptotics of degenerating Eisenstein series.

2.* Is it possible that the index theorem for punctured surfaces could be derived from the one for compact surfaces through degeneration? —Bismut-Bost (1990) studied a related problem.

3.*** Is the curvature of the Takhtajan-Zograf metric negative?

4.*** If the answer to the question 3. is YES, study $-\mathrm{Ric}\,\omega_{TZ}$.
– Recently, K. Liu, X. Sun & S.-T. Yau (2004, 2005, 2008-) find good geometry of the moduli of curves using $-\mathrm{Ric}\,\omega_{WP}$, which we will survey later.

5.*** Does the Takhtajan-Zograf Kähler form have a global representation formula?
– The Weil-Petersson Kähler form has a global representation formula in terms of the Fenchel-Nielsen global coordinates, which reveals the symplectic nature of the Teichmüller space. (S.A. Wolpert (1982, 1983, 1985))

§2. Several metrics on the moduli space

We will review properties of other metrics on the moduli space and their relations to the W-P metric. Two metrics $\omega_{g_1}, \omega_{g_2}$ on a manifold (orbifold) are called **equivalent**, if for a positive constant C

$$C^{-1}\omega_{g_1} \leq \omega_{g_2} \leq C\omega_{g_1}.$$

Liu-Sun-Yau, McMullen et als. proved that the Teichmüller space has various equivalent metrics.

McMullen (2000) defined the **McMullen metric**

$$\omega_M := \omega_{WP} - i\delta \sum_{l_\gamma < \varepsilon} \partial\bar{\partial}\, Log \frac{\varepsilon}{l_\gamma},$$

where the sum is taken over primitive short geodesics γ on the curve, and $\varepsilon, \delta > 0$ are suitable small constants, and Log is a suitably modified logarithmic function.

McMullen (2000) used ω_M to give an affirmative answer to the conjecture by Gromov that $\mathcal{M}_{g,n}$ is Kähler hyperbolic. Remember the definition of Kähler-hyperbolicity.

(X, g): a Kähler manifold (orbifold).
An n-form α is d(bounded) if $\alpha = d\beta$ for some bounded $(n-1)$-form β.

(X, g) is **Kähler hyperbolic** if:
1. On the universal cover $\tilde{X}$, the Kähler form of the pull-back metric $\tilde{g}$ is d(bounded);
2. (X, g) is complete and of finite volume;
3. The sectional curvature of (X, g) is bounded;
4. The injectivity radius of $(\tilde{X}, \tilde{g})$ is bounded below.

Since the Ricci curvature of the W-P metric is shown to be bounded above by a negative constant, we can define the **Ricci metric**

$$\omega_\tau := -\mathrm{Ric}\,(\omega_{WP}).$$

Moreover, Liu-Sun-Yau (2004) has defined the **perturbed Ricci metric**

$$\omega_{\tilde{\tau}} := -\mathrm{Ric}\,(\omega_{WP}) + C\omega_{WP},$$

where C is a positive constant.

Theorem 11 (McMullen, Liu-Sun-Yau, et als.). *We can observe basic properties of the metrics on the moduli spaces.*

- $\omega_{WP}, \omega_{TZ}, \omega_M, \omega_\tau, \omega_{\tilde{\tau}}$ *are Kähler metrics.*
- $\omega_M, \omega_\tau, \omega_{\tilde{\tau}}$ *are complete, but* ω_{WP}, ω_{TZ} *are incomplete on* $\mathcal{M}_{g,n}$.
- *The holomorphic sectional, Ricci and scalar curvatures of* ω_{WP} *are bounded from negative constants.*
- *The bisectional and sectional curvatures of* ω_{WP} *are negative.*

- *The curvature of ω_{WP} is not bounded below.*
- *The holomorphic sectional, the bisectional and the Ricci curvatures of $\omega_{\tau}, \omega_{\tilde{\tau}}$ are bounded from above and below.*
- *For nice C, the holomorphic sectional and the Ricci curvatures of $\omega_{\tilde{\tau}}$ are negatively pinched.*
- *$\omega_M, \omega_{\tau}, \omega_{\tilde{\tau}}$ are equivalent each other.*
- *$\omega_M, \omega_{\tau}, \omega_{\tilde{\tau}}$ have Poincaré growth and thus $\mathcal{M}_{g,n}$ has finite volumes with respect to those metrics.*
- *The injectivity radii of $T_{g,n}$ with respect to $\omega_M, \omega_{\tau}, \omega_{\tilde{\tau}}$ are bounded from below.*

Furthermore, $\mathcal{M}_{g,n}$ has some other metrics!

By Cheng-Yau, there is a unique complete **Kähler-Einstein metric** ω_{KE} on $T_{g,n}$ whose Ricci curvature is -1. The canonical bundle of $T_{g,n}$ naturally induces the **Bergman metric** ω_B on $T_{g,n}$. Both ω_{KE}, ω_B are invariant under the action of $\mathrm{Mod}_{g,n}$, thus naturally descend to the metrics on $\mathcal{M}_{g,n}$ denoted by the same symbols.

Here we set

Δ_R: the disk centered at 0 with radius R in $\mathbf{C}$

$\mathrm{Hol}(A, B)$: the space of holomorphic maps from a domain A to a domain B

The **Carathéodory** and the **Kobayashi norms** of $v \in T_{[S^0]}T_{g,n}$ are defined to be

$$\|v\|_C := \sup_{f \in \mathrm{Hol}(T_{g,n}, \Delta_1)} \|f_* v\|_{\Delta_1, \mathrm{hyp}},$$

$$\|v\|_K := \inf_{f \in \mathrm{Hol}(\Delta_R, T_{g,n}), f(0)=[S^0], f'(0)=v} \frac{2}{R}.$$

Royden showed that, on $T_{g,n}$, the Kobayashi metric coincides with the **Teichmüller metric**. Recently we have

Theorem 12 (Liu-Sun-Yau (2004-5)).
On $\mathcal{M}_{g,n}$, $\omega_M, \omega_{\tau}, \omega_{\tilde{\tau}}, \omega_{KE}, \omega_B$, the Teichmüller-Kobayashi metric and the Carathéodory metric are all equivalent.
The curvature of ω_{KE} is bounded and the injectivity radius of ω_{KE} is bounded from below.

The proof of the second statement in Theorem 12 is based on the Kähler-Ricci flow.

Open problems

6.$***$ Does the Kobayashi metric g_K coincide with the Carathéodory metric g_C ?
–It is already known that $g_C \leq g_K$ in general, and $g_C = g_K$ on some loci (Kra (1981)).

7.$*$ Give a new proof for the Kähler hyperbolicity of $\mathcal{M}_{g,n}$ using other metrics than ω_M, ω_B.
–The original proof was much involved with Teichmüller theory.

8.$**$ Investigate curvature of ω_B, ω_M.
–There seems to exist less results on them.

9.$**$ make a better metric on $\mathcal{M}_{g,n}$!.

§3. Applications of metrics to the geometry of the moduli space

We will survey applications of metrics by Liu-Sun-Yau to the geometry of the moduli space.

Theorem 13 (Liu-Sun-Yau (2008+preprint)). *The metrics on the logarithmic cotangent bundle $T^*_{\overline{\mathcal{M}}_{g,n}}(\log D)$ over $\overline{\mathcal{M}}_{g,n}$ induced from $\omega_{WP}, \omega_\tau, \omega_{\tilde{\tau}}$ are good in the sense of Mumford. Thus the Chern forms of those metrics, as currents, are equal to the Chern classes of $T^*_{\overline{\mathcal{M}}_{g,n}}(\log D)$.*

Here we will summarize some definitions and remarks needed to state Theorem 13.

For the local pinching coordinates (t, s) around a nodal curve in D, a local holomorphic frame of $T^*_{\overline{\mathcal{M}}_{g,n}}(\log D)$ is

$\left(\frac{dt_1}{t_1}, \cdots, \frac{dt_k}{t_k}, ds_{k+1}, \cdots, ds_m\right)$.

On the other hand, the logarithmic tangent bundle $T_{\overline{\mathcal{M}}_{g,n}}(-\log D)$ has a local frame $\left(t_1\frac{\partial}{\partial t_1}, \cdots, t_k\frac{\partial}{\partial t_k}, \frac{\partial}{\partial s_{k+1}}, \cdots, \frac{\partial}{\partial s_m}\right)$. Here $m = 3g-3+n$.

We cover a neighborhood of the boundary D by finitely many polydiscs $(m = 3g-3+n)$ $\{U_\alpha = (\Delta^m, (t_1, \cdots, t_k, s_{k+1}, \cdots, s_m))\}_{\alpha\in A}$ such that $V_\alpha = U_\alpha \setminus D = (\Delta^*)^k \times \Delta^{m-k}$. Namely, $U_\alpha \bigcap D = \{t_1 \cdots t_k = 0\}$. Set $V = \bigcup_{\alpha\in A} V_\alpha$.

On each V_α, we have the local Poincaré metric

$$\omega_{P,\alpha} = \frac{\sqrt{-1}}{2}\left(\sum_{i=1}^{k} \frac{dt_i \wedge d\bar{t}_i}{|t_i \log t_i|^2} + \sum_{i=k+1}^{m} ds_i \wedge d\bar{s}_i\right).$$

Let η be a smooth local p-form defined on V_α.

• η has **Poincaré growth** if there is a constant $C_\alpha > 0$ depending on η such that

$|\eta(v_1, \cdots, v_p)|^2 \leq C_\alpha \prod_{i=1}^{p} \|v_i\|^2_{\omega_{P,\alpha}}$ for any point $z \in V_\alpha$ and any $v_i \in T_z V_\alpha$.

• η is **good** if η and $d\eta$ has Poincaré growth.

Let $\overline{E}$ be a holomorphic vector bundle of rank r on $\overline{\mathcal{M}}_{g,n}$ and and $E = \overline{E}|_{\mathcal{M}_{g,n}}$.

An Hermitian metric h on E is **good in the sense of Mumford** if: for all $z \in V$, assuming $z \in V_\alpha$, and all basis $(e_1, \cdots, e_r)$ of $\overline{E}$ over U_α,

• For some $C, d > 0$, $h_{i\bar{j}} = h(e_i, e_j)$ satisfy $|h_{i\bar{j}}|$, $(\det h)^{-1} \leq C\left(\sum_{i=1}^{k} \log|t_i|\right)^{2d}$;

• The local 1-form $(\partial h \cdot h^{-1})_\alpha$ is good on V_α.

Recently we found some new aspects of L^2- cohomology of several metrics on the moduli spaces.

Theorem 14 (Liu-Sun-Yau (preprint)).
We can observe

$H^*_{(2)}((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})) \simeq H^*(\overline{\mathcal{M}}_g, T_{\overline{\mathcal{M}}_g}(-\log D))$,

$H^{0,q}_{(2)}((\mathcal{M}_g, \omega_\tau), (T_{\mathcal{M}_g}, \omega_{WP})) = 0$
unless $q = 3g-3$.

Thus $(\overline{\mathcal{M}}_g, D)$ is infinitesimally rigid, which was originally proved by Hacking.

Theorem 15 (Ji-Liu-Sun-Yau (preprint)).
The Gauss-Bonnet theorem holds on $\mathcal{M}_g$ equipped with $\omega_\tau, \omega_{\tilde\tau}, \omega_{KE}$:

$$\int_{\mathcal{M}_g} c_{3g-3}(\omega_\tau) = \int_{\mathcal{M}_g} c_{3g-3}(\omega_{\tilde\tau}) = \int_{\mathcal{M}_g} c_{3g-3}(\omega_{KE}) = \chi(\mathcal{M}_g) = \frac{B_{2g}}{4g(g-1)}.$$

Here $\chi(\mathcal{M}_g)$ is the orbifold Euler characteristic and B_{2g} is the Bernoulli number.

Open problems

10.$**$ Does it still hold true that the metrics on $T^*_{\overline{\mathcal{M}}_{g,n}}(\log D)$ over $\overline{\mathcal{M}}_{g,n}$ induced from ω_{KE}, ω_B are good in the sense of Mumford?

§4. The Weil-Petersson geometry of the universal Teichmüller space

We survey Takhtajan-Teo's results on the universal Teichmüller space.

$\mathbb{D} := \{z \in \mathbf{C} |\ |z| < 1\}$, $\mathbb{D}^* := \{z \in \mathbf{C} |\ |z| > 1\}$

$L^\infty(\mathbb{D}^*) := \left\{\mu(z)\frac{d\bar z}{dz} \text{ measurable on } \mathbb{D}^* \mid \|\mu\|_{\mathbb{D}^*} < \infty\right\}$

Here $\|\mu\|_{\mathbb{D}^*} := \sup_{\mathbb{D}^*} |\mu(z)|$.

Let $L^\infty(\mathbb{D}^*)_1$ be the unit open ball in $L^\infty(\mathbb{D}^*)$. Extend $\mu \in L^\infty(\mathbb{D}^*)_1$ to be 0 outside $\mathbb{D}^*$. Consider the unique q.c. mapping $w^\mu : \mathbf{C} \to \mathbf{C}$ which satisfies the Beltrami equation $w^\mu_{\bar z} = \mu\, w^\mu_z$,the condition $f(0) = 0, f'(0) = 1, f''(0) = 0$.

For $\mu, \nu \in L^\infty(\mathbb{D}^*)_1$, set $\mu \sim \nu$ if $w^\mu|_{\mathbb{D}} = w^\nu|_{\mathbb{D}}$.

The **universal Teichmüller space** is defined as a set of equivalence classes of normalized q.c. mappings

$$T(1) := L^\infty(\mathbb{D}^*)_1 / \sim .$$

We set $A_\infty(\mathbb{D}) := \{\phi \text{ holomorphic on } \mathbb{D} |\ \|\phi\|_\infty < \infty\}$, $\|\phi\|_\infty := \sup_{\mathbb{D}} |(1-|z|^2)^2\phi(z)|$.

The **Bers embedding** $\beta : T(1) \hookrightarrow A_\infty(\mathbb{D})$ is defined as follows. The **Schwarzian derivative** of a conformal map f is given by

$$\mathcal{S}(f) := \frac{f_{zzz}}{f_z} - \frac{3}{2}\left(\frac{f_{zz}}{f_z}\right)^2.$$

For $\mu \in L^\infty(\mathbb{D}^*)_1$, set $\beta([\mu]) = \mathcal{S}(w^\mu|_{\mathbb{D}})$. Here $[\mu]$ is the equivalent class of μ for $\sim$.

$T(1)$ has a Banach structure naturally induced from $A_\infty(\mathbb{D})$ which is not a Hilbert structure. Takhtajan-Teo have given $T(1)$ a Hilbert structure to define the Weil-Petersson metric. They proved that the tangent space of $T(1)$ at $[0]$ can be identified with a Hilbert space $H^{-1,1}(\mathbb{D}^*) := \{\mu = \rho^{-1}\bar\phi \mid \phi \text{ holomorphic on } \mathbb{D}^*,\ \|\mu\|_2 < \infty\}$. Here $\|\mu\|_2^2 := \iint_{\mathbb{D}^*} |\mu|^2\rho$, ρ : hyperbolic on $\mathbb{D}^*$.

The inner product of the W-P metric at $[0]$ of $T(1)$ is defined to be

$$\langle \mu, \nu \rangle_{WP} := \iint_{\mathbb{D}^*} \mu\bar\nu\rho, \quad \text{for } \mu, \nu \in H^{-1,1}(\mathbb{D}^*) \simeq T_{[0]}T(1).$$

The Weil-Petersson metric ω_{WP} on $T(1)$ is real-analytic and Kählerian. Takhtajan-Teo gave the following surprising observation.

Theorem 16 (Takhtajan-Teo (2006)).
$T(1)$ is a Kähler-Einstein manifold with negative constant Ricci curvature,

$$Ric\, \omega_{WP} = -\frac{13}{12\pi}\, \omega_{WP}.$$

The sectional and the holomorphic sectional curvatures of ω_{WP} are negative.

Open problems

11.** Formulate the index theorem for $T(1)$.

12.* Define and study other metrics on $T(1)$.

13.** Is it true that the Weil-Petersson metrics on the infinite-dimensional Teichmüller spaces other than $T(1)$ are Kähler-Einstein?

14.* Is the Weil-Petersson metric on $T(1)$ complete or not?

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