Bergman核の問題

大沢健夫（名大多元数理）

Table of Contents

Introduction

The Bergman kernel, named after Stefan Bergman (1895-1977), is by definition the reproducing kernel of the space of L^2 holomorphic n-forms on any connected n-dimensional complex manifold. Its significance in complex geometry has been gradually understood through many spectacular works in the last century. For instance, C. Fefferman [F-1] analyzed the boundary behavior of the Bergman kernel on strongly pseudoconvex domains with C^∞ boundary, and proved that any biholomorphic map between such bounded domains in C^n extends smoothly to the closure. Recently, methods for analyzing the Bergman kernel brought new insights into algebraic geometry and differential geometry (cf. [Siu-2~5], [Brn-P], [D] and [Mab-1,2]). The purpose of this article is to review some of the results on the Bergman kernel with geometric backgrounds, presenting open questions on the way.

§1. Preliminaries — before and after the Bergman kernel

The circle division theory of C. F. Gauss (1777-1855), which was discovered on 1796/3/30, is a giant leap in mathematics and the first step towards complex geometry. In the early 19th century, it brought a new progress in the theory of elliptic integrals, which had been developed by L. Euler (1707-83) and A.-M. Legendre (1752-1833). Namely, generalizing

the work of Gauss, N.H.Abel (1802-29) was led at first to algebraic insolvability of equations of degree 5, subsequently discovered that the inverse functions of elliptic integrals are nothing but doubly periodic analytic functions in one complex variable (i.e. elliptic functions), and eventually arrived at a remarkable characterization of principal divisors in the theory of algebraic functions of one variable (Abel's theorem). The latter is now regarded as the starting point of algebraic geometry.

As a generalization of Abel's theory on elliptic functions, the theory of multiply periodic functions was developed in several variables by G. Jacobi (1804-51), K. Weierstrass (1815-97) and B. Riemann (1826-66).

On the other hand, in spite of an important contribution of H. Poincaré (1854-1912) on normal functions and a subsequent work of S. Lefschetz (1884-1972), it was not before the appearance of the celebrated theory of W. V. D. Hodge (1903-75) [Ho], of harmonic integrals on Kähler manifolds, that Abel's theorem on algebraic functions found a proper context in several variables. This delay is mainly because of the rack of the viewpoint of orthogonal projection in Hilbert spaces. Recall that it was only in 1899 that D. Hilbert (1862-1943) awoke Riemann's idea of Dirichlet's principle from a deep sleep (cf. [R] and [H]) and that the basic representation theorem of F. Riesz (1880-1956) was not available until 1907. Another historical remark is that such a systemization of abstract mathematics emerged only after detailed studies of orthogonal polynomials in the 19th century. Anyway, it culminated in a general method of orthogonal projection by H. Weyl (1885-1955). Weyl's method (cf. [W-1]) became the analytic base of the Hodge theory, which was later combined with analytic sheaf theory by Kodaira (1915-97) [K-1,2]. That Weyl anticipated a lot in this method had been modestly suggested in [W-2]. The Bergman kernel was born around 1922 (cf. [B] and [Bo]) in such a circumstance.

To be more explicit about the orthogonal projection and the Bergman kernel, let **D** be the unit disc centered at the origin in the complex plane with coordinate z, and let $L^2(\partial \mathbf{D})$ be the Hilbert space of L^2 complex-valued functions on $\partial \mathbf{D}$. Then the integral transform

$$f(z) \longrightarrow \frac{1}{2\pi\sqrt{-1}} \int_{\partial \mathbf{D}} \frac{f(\zeta)}{\zeta - \mathbf{z}} d\zeta$$

of A.-L. Cauchy (1789-1857) gives an orthogonal projection from $L^2(\partial \mathbf{D})$ onto the subspace of functions which are the boundary values of holomorphic functions on **D** in the L^2 sense (i. e. L^2 functions with vanishing Fourier coefficients in the negative powers of exp (i arg z)).

Replacing the integral along $\partial \mathbf{D}$ by the integral on **D**, one is naturally

led to the representation of the orthogonal projection from $L^2(\mathbf{D})$, the space of the complex-valued L^2 functions on $\mathbf{D}$, onto the subspace consisting of L^2 holomorphic functions on $\mathbf{D}$. The corresponding integral transform is

$$f(z) \longrightarrow \frac{1}{\pi} \int_D \frac{f(\zeta)}{(1 - \bar{\zeta}z)^2}\, d\lambda_\zeta.$$

Here $d\lambda_\zeta$ denotes the Lebesgue measure. The function $\pi^{-1}(1 - \bar{\zeta}z)^{-2}$ is the Bergman kernel of $\mathbf{D}$, where holomorphic 1-forms on $\mathbf{D}$ are naturally identified with holomorphic functions on $\mathbf{D}$.

Thus, from the viewpoint of orthogonal projection, the Bergman kernel is a brother of the Cauchy kernel. An advantage of the Bergman kernel is that it naturally encodes geometric information. Let us recall how it does.

Let M_j $(j = 1, 2)$ be two complex manifolds with Bergman kernels $\varkappa_{M_j}$, and let $\sigma : M_1 \longrightarrow M_2$ be a biholomorphic map. Then one has an equality

(1) $$\sigma^* \varkappa_{M_2} = \varkappa_{M_1}$$

which follows easily from the definition. We note that the equality (1) already suggests a link between the boundary behavior of Bergman kernels and biholomorphic maps.

To see it more explicitly, taking as M_1 any simply connected proper subdomain Ω of $\mathbf{C}$, let $z_0 \in \Omega$, $M_2 = \mathbf{D}$, $\sigma(z_0) = 0$ and $\sigma'(z_0) > 0$, based on Riemann's mapping theorem. Then, letting $\varkappa_\Omega = K_\Omega(\zeta, z)d\zeta d\bar{z}$, it follows immediately from (1) that

(2) $$\sigma(z) = \sqrt{\frac{\pi}{K_\Omega(z_0, z_0)}} \int_{z_0}^{z} K_\Omega(\zeta, z_0)d\zeta$$

holds true for any $z \in \Omega$. It is obvious from (2) that the boundary regularity of $K_\Omega(\zeta, z_0)$ implies that of σ. Efficiency of this reduction lies in that, as we shall see later, the regularity question on K_Ω can be transformed into a question on the canonical solution operator for the complex Laplacian. This observation might already suggest the reader the validity, and even the method of proof, of Fefferman's theorem which was mentioned in the introduction.

That's all for preliminaries. We shall now go into the substantial material, at first the boundary behavior of the Bergman kernel.

§2. Studies on the boundary behavior

From now on, let Ω be any bounded domain in $\mathbf{C}^n$ and let

$$\varkappa_\Omega = K_\Omega(z,w)2^{-n}dz_1 \wedge \cdots \wedge dz_n \otimes d\overline{w}_1 \wedge \cdots \wedge d\overline{w}_n,$$

where $z = (z_1, \ldots, z_n)$ and $w = (w_1, \ldots, w_n)$. $K_\Omega(z,w)$ will be referred to as the Bergman kernel function of Ω.

An alternate definition of $K_\Omega(z,w)$ is given by the formula

$$(3) \qquad K_\Omega(z,w) = \sum_{j=1}^{\infty} e_j(z)\overline{e_j(w)}$$

where $\{e_1, e_2, \ldots\}$ is any complete orthonormal system of the space, say $A^2(\Omega)$, of L^2 holomorphic functions on Ω with respect to the Lebesgue measure.

For simplicity we put

$$(4) \qquad K_\Omega(z) = K_\Omega(z,z).$$

Clearly $K_\Omega(z)$ is strictly plurisubharmonic and strictly positive. It is also easy to verify that $\log K_\Omega(z)$ is strictly plurisubharmonic. The complex Hessian of $\log K_\Omega(z)$, denoted by $\partial\bar{\partial}\log K_\Omega(z)$ by an abuse of notation, is called the Bergman metric of Ω. A fact of basic importance is that biholomorphic maps are isometries with respect to the Bergman metric.

For the case $\Omega = \mathbf{B}^n := \{z; \ |z| < 1\}$, where $|z|^2 := |z_1|^2 + \cdots + |z_n|^2$, one has

$$(5) \qquad K_\Omega(z,w) = \pi^{-n}n! \ (1-<z,w>)^{-n-1}$$

where $<z,w> := z_1\overline{w}_1 + \cdots + z_n\overline{w}_n$.

The expression (5) is an immediate consequence of (1) once the biholomorphic automorphisms of $\mathbf{B}^n$ are explicitly known. Although it is usually difficult to compute the Bergman kernels, it is obvious that the Bergman metrics on bounded homogeneous domains are complete. We note that there exists a complete Kähler metric on $\mathbf{B}^n - \{0\}$ and that the Bergman metric on $\mathbf{B}^n - \{0\}$ is not complete (cf. [G-1]).

One way of describing the boundary behavior of $K_\Omega(z,w)$ is to express the singularity of $K_\Omega(z)$ as $z \longrightarrow \partial\Omega$ in terms of the function $\delta_\Omega(z) := \inf\{\ |z-w|\ ; w\notin\Omega\}$ and geometric invariants on $\partial\Omega$. Here, by geometric invariants on $\partial\Omega$, we mean locally defined systems of functions satisfying covariance properties under biholomorphic maps, or more instrinsically,

under CR diffeomorphisms (cf. [Wk]).

For that purpose, the following formula is useful.

(6) $\quad K_{\Omega}(z) = \sup \{ \, |f(z)|^2 \, ; f(\zeta) \in A^2(\Omega) \text{ and } \|f\| = 1 \}.$

Here $\|f\|$ denotes the L^2 norm of f. Note that the supremum is attained by the function $K_{\Omega}(\zeta,z)/\sqrt{K_{\Omega}(z)}$. Basically, what can be done is to approximate this function from the geometric data by employing the techniques of producing L^2 holomorphic functions on Ω. As such a technique, there is a method, due to L. Hörmander [Hö-1], of solving the inhomogeneous Cauchy-Riemann equation $\bar{\partial}u=v$ with L^2 norm estimates (see also [Hö-2]). A similar method of A. Andreotti (1924-80) and E. Vesentini in [A-V-1,2] is also useful.

In the case where Ω is a strongly pseudoconvex domain, it was proved in [Hö-1] that

(7) $\quad \lim_{z \to z_0} K_{\Omega}(z)\delta_{\Omega}(z)^{n+1} = n!\pi^{-n}L(z_0)$

holds for any $z_0 \in \partial\Omega$, where

(8) $\quad L(z_0) = \lim_{\Omega \ni z \to z_0} (-1)^n \det \begin{pmatrix} 0 & \partial\delta/\partial z_j \\ \partial\delta/\partial \bar{z}_k & \partial^2\delta/\partial z_j \, \partial \bar{z}_k \end{pmatrix}, \qquad \delta = \delta_{\Omega}.$

Recall that Ω is called a strongly pseudoconvex domain if locally $\partial\Omega$ can be mapped to C^2 strictly convex hypersurfaces by appropriate choices of biholomorphic maps. $L(z_0)$ is a geometric invariant on $\partial\Omega$ in the above mentioned sense.

Actually, (7) holds for any bounded pseudoconvex domain with C^2 boundary. In fact, the left hand side of (7) vanishes if $L(z_0) = 0$. This can be seen from Cauchy's estimate applied on a sequence of polydiscs in Ω converging to z_0.

Hence, strong pseudoconvexity of $\partial\Omega$ at z_0, i. e. the condition that $\partial\Omega$ becomes strictly convex at z_0 after some biholomorphic coordinate change, is characterized by the condition that

(9) $\quad \liminf_{z \to z_0} K_{\Omega}(z)\delta(z)^{n+1} > 0$

holds true.

On the other hand, it is implicitly contained in [Oh-2] that the Levi flatness of $\partial\Omega$ on a neighbourhood $U \ni z_0$ ($U \subset \partial\Omega$), i.e. the property that $\partial\bar{\partial}\delta \,|\, \mathrm{Ker}\, \partial\delta = 0$ on U, is characterized by the condition that

$$(10) \qquad \limsup_{\zeta \to z} K_{\Omega}(\zeta)\delta(\zeta)^2 \; < \; \infty$$

holds for any $z \in U$.

For the general smooth pseudoconvex domains, "building blocks of the singularity" of $K_{\Omega}(z)$ have been studied case by case (cf. [Oh-1], [D-H-Oh], [D-H], [B-S-Y], [Km], [Ch-Km-Oh]).

Next we shall discuss the boundary behavior of the Bergman kernel on pseudoconvex domains from slightly more analytic viewpoint.

Although the motivation of Bergman's thesis was to introduce a new method in the theory of potentials and conformal mappings, it was soon recognized that analysis of the Bergman kernel would play an important role in several complex variables, too, for instance to solve the Levi problem (cf. [Hö-3]). (Recall that the Levi problem asks whether or not every pseudoconvex domain is holomorphically convex.)

Indeed, it is easy to see that Ω is holomorphically convex if

$$(11) \qquad \lim_{z \to \partial\Omega} K_{\Omega}(z) = \infty$$

holds true. The converse is false because the punctured disc $D - \{0\}$ is a counterexample. For the domains in C, it recently turned out that (11) is equivalent to certain growth property of the logarithmic capacity function on Ω (cf. [Zw-2]).

Concerning the Levi problem, which was the principal question in several complex variables for some time, Kiyoshi Oka (1901-78) first came up with a solution by the strategy of exhausting pseudoconvex domains by strongly pseudoconvex ones, constructing holomorphic functions on strongly pseudoconvex domains by patching locally defined ones by solving Cousin's problem, and approximating them by globally defined ones by a theorem of Runge type (cf.[O]). However, all these arguments are independent of the Bergman kernel.

A counterpart of Oka's theorem on compact manifolds was established by Kodaira by the method of harmonic integrals (cf. [K-1.2]).

After an important work of C. B. Morrey (1907-84) (cf. [M]), the method of Oka was extended by H. Grauert [G-2] to prove that strongly pseudoconvex domains in complex manifolds are holomorphically convex, and Kodaira's method was extended in [A-V-1,2], [Kh] and [Hö-1] to yield a powerful method of **directly** and **effectively** reaching the basic existence theorems in several complex variables. Especially, it is remarkable that [Hö-1] gave a simple alternate proof to Oka's theorem by establishing a quantitative solution to the additive Cousin problem by the method of L^2 estimates for the $\bar{\partial}$-operator. Here the $\bar{\partial}$-operator means

a closed linear operator from $L^2(\Omega)$ to $\overset{n}{\oplus}L^2(\Omega)$ defined by $\bar{\partial}f = (\partial f/\partial \bar{z}_1, ..$
$.., \partial f/\partial \bar{z}_n)$ on Dom $\bar{\partial} = \{f \; ; \; \partial f/\partial \bar{z}_j \in L^2(\Omega), j=1,2,...,n\}$.

Since $A^2(\Omega) = \text{Ker } \bar{\partial}, K_\Omega$ involves operator theoretic information on $\bar{\partial}$ as we shall see later.

The $\bar{\partial}$-operator is naturally extended to L^2 differential forms giving rise to a complex. Generalizing the situation to the L^2 spaces with respect to arbitrary measures, L^2 estimates are formulated as inequalities of the form

(12) $\qquad \|u\| \leq \text{const}\,(\|\bar{\partial}u\| + \|\bar{\partial}^*u\|)$,

where $\bar{\partial}^*$ denotes the Hilbert space adjoint of $\bar{\partial}$. L^2 estimates that work in several complex variables were planned by P. R. Garabedian (1927-2010) and D. C. Spencer (1912-2001) [G-S]. Based on the idea of orthogonal projection and pushed by the complete solution of the Levi problem for the domains over $\mathbb{C}^n$(cf. [O], [Br] and [Ng]), the plan was realized in the above mentioned papers.

An advantage of this method is that the passage to limits is quite easy, so that one has effective existence theorems on general pseudoconvex domains. (7) was obtained as an application of this method.

Inspired by the success of this approach, Skoda [S] and Ohsawa-Takegoshi [Oh-T] established respectively the L^2 variants of Oka's division theorem and extension theorem. The method of [Oh-T] was influenced by [D-F] and [Wi].

Skoda's L^2 division theorem was applied by Pflug [P] to show that (11) holds if Ω is a pseudoconvex domain satisfying the "generalized cone condition" (see [P] for the definition). Moreover it tuned out later that the same technique is available to show, under the assumption that $\partial(\Omega \cup \partial\Omega)$ $= \partial\Omega$, that Ω is pseudoconvex if and only if it carries a complete Kähler metric (cf. [D-P]).

On the other hand, by applying the L^2 extension theorem in [Oh-T], it was shown in [Oh-3] that (11) holds if Ω is *hyperconvex*, i.e. if Ω admits a bounded plurisubharmonic exhaustion function.

It is well known that a bounded domain in $\mathbb{C}$ is hyperconvex if and only if its boundary points are regular with respect to the Dirichlet problem (for the regularity of the boundary points in this sense, see [Kishi] for instance). We note that (11) holds on some non-hyperconvex domains, e. g. on $\{ (z,w) \in \mathbb{C}^2; \; |z| < 1, \; |w| < 1 \text{ and } |z| < |w| \}$, so that hyperconvexity is considered to be a more natural condition than (11).

Pluripotential theory, including the existence of pluricomplex Green function and Lelong-Jensen measure, has been developed on hyperconvex domains. Here, to be analyzed as the several variables version of the Laplace operator is the Monge-Ampère operator (cf. [Klm] and [Dm-1]). Recently, geometry of the Nevanlinna counting function is

discussed on hyperconvex domains (cf. [P-S]).

The condition (11) is very close to the completeness of the Bergman metric on hyperconvex manifolds. Such a link was first observed in [Kb] by identifying the Bergman metric with the pull-back of the Fubini-Study metric on the projectivization of the topological dual $A^2(\Omega)^*$ of $A^2(\Omega)$, say $P(A^2(\Omega)^*)$, by the canonically defined holomorphic embedding

(13) $\qquad \iota : \Omega \longrightarrow P(A^2(\Omega)^*)$

$\qquad\qquad$ ⋓ $\qquad\qquad\qquad$ ⋓

$\qquad\qquad z \longmapsto \{ m \in A^2(\Omega)^*-\{0\} \ ; \ m(f) = 0 \text{ if } f(z) = 0\}.$

By this identification, denoting the distance between $\iota(z)$ and $\iota(w)$ by $|z,w|$, one has

$$(14) \qquad |z,w| = \text{Arctan} \ \frac{\sqrt{K_\Omega(z)K_\Omega(w) - |K_\Omega(z,w)|^2}}{|K_\Omega(z,w)|} .$$

The following estimate, which is essentially equivalent to Kobayashi's criterion for the completeness of the Bergman metric, follows from (14).

(15) $\quad |z,w| \geq \min (1/2, \sup\{|f(z)|^2/K_\Omega(z) \ ; \ f \in A^2(\Omega), \ \|f\|=1 \text{ and } f(w)=0\}).$

(See also [Oh-8]).

Combining (15) with a recently developed technique of estimating integrals of type $\int_\Omega |u|^n (\partial\bar\partial v)^n$, Błocki-Pflug [B-P] and Herbort [Hb] independently proved that the Bergman metric is complete if Ω is hyperconvex. It is known that there exist non-hyperconvex domains in C whose Bergman metrics are complete (cf. [Zw-1, Theorem 5]).

The Bergman metric on a connected n-dimensional complex manifold M is defined in the same way as above via the map (13), by taking the space of L^2 holomorphic n-forms instead of $A^2(\Omega)$, as long as the map corresponding to ι is an immersion. A complex manifold is called hyperconvex if it carries a bounded strictly plurisubharmonic exhaustion function. It is easily seen by the L^2 method that every hyperconvex manifold carries a Bergman metric. In [Ch], the completeness result of [B-P] and [Hb] was generalized to hyperconvex manifolds.

In view of the fact that singularities of L^2 holomorphic functions are negligible if their Hausdorff dimension is not greater than $2n - 2$, it seems natural to ask the following.

1)   Let Ω be a proper subdomain of B^n. How "small" (in the sense of Hausdorff dimension, for instance) can $\partial\Omega \cap B^n$ be if the Bergman metric of Ω is complete ?

2)   Is there a proper subdomain D of a compact complex manifold M without nonconstant bounded plurisubharmonic functions such that the Bergman metric of D is complete ?

As for 1), case studies based on the analysis of Cauchy kernel should be possible at least for n = 1. As a related result, see [An].

Boundary behavior of the Bergman metric on strongly pseudoconvex domains was first described by K. Diederich (cf. [Di]). As well as the Bergman metric on the model domain B^n, the Bergman metrics on strongly pseudoconvex domains are complete Kähler metrics. A famous result of Lu Qi-Keng [L] says that Ω is biholomorphically equivalent to B^n if the Bergman metric on Ω is complete and of constant holomorphic sectional curvature. A natural question asked by S.-Y. Cheng [Chg] is whether or not Ω is equivalent to B^n if the Bergman metric on Ω is Kähler-Einstein. By Fu and Wong [F-W], this was answered affirmatively when Ω is simply connected and $n \leq 2$. Recently, it was pointed out by Nemirovski and Shafikov [N-S] that Cheng's conjecture follows from the Ramadanov conjecture (see Q5 below), so that the result of Fu and Wong holds without assuming that Ω is simply connected.

When Ω is not strongly pseudoconvex, more case studies seem to be necessary in order to find how the geometry of Ω and $\partial\Omega$ determines the Bergman kernel. For instance, in view of the fact that one can characterize the strong pseudoconvexity of Ω in terms of the boundary behavior of the Bergman metric (cf. [Kl] and [Di-Oh-1]), the author would like to ask the following question.

    Let Ω be a bounded pseudoconvex domain in C^n with C^2 boundary and let $z_0 \in \partial\Omega$. Is it true that $\partial\Omega$ is Levi flat near z_0 if and only if there exists a neighbourhood U of z_0 in $\partial\Omega$ such that

(16) $\varlimsup\limits_{\zeta \to z} |\xi|^{-2} |<\partial\bar{\partial}\log K_\Omega, \xi\otimes\xi> - \delta(\zeta)^{-2}|\xi\delta|^2| < \infty$

holds for any $z \in U$ and for any nonzero holomorphic tangent vector ξ of C^n at ζ ? Here $<,>$ denotes the natural pairing and $| \cdot |$ the length of vectors.

This should not be too difficult because it is already known by [D-F-H] and [C-1] that (16) does not hold if $\partial\Omega$ is of finite type at z.

A related question was raised in [Di-Oh-2] and [Oh-7] on the effective estimate of the distance function. Let us put it here in a more idealized form;

Q3 (conjecture)--

    Let Ω be a bounded pseudoconvex domain in C^n with C^1 boundary, let d(z,w) be the distance between z and w with respect to $\partial\bar\partial\log K_\Omega$, and let $z_0 \in \Omega$ be any point. Then

(17) $\lim\limits_{w\to\partial\Omega} d(z_0,w)/|\log \delta(w)| = 1.$

An estimate obtained in [Di-Oh-2] is weaker than (17), but still gives a quantitative completeness result for the Bergman metric. It was improved by Błocki [Bł]. We note that the infinitesimal variant of (17) asked in [Oh-7] was negatively solved (cf. [D-H-2]).

Note 1. If we restrict ourselves to a class of bounded homogeneous domains, it was shown by Nomura [Nm] that bounded symmetric domains can be characterized by a property of the Bergman kernel, e.g. the commutativity of the Laplacian with respect to the Bergman metric and the Berezin transform. (See also [En].) It is known that a complex manifold equipped with the Bergman metric is homogeneous if and only if it is equivalent to a bounded homogeneous domain (cf. [PS]). Exploiting the fact that every bounded homogeneous domain is equivalent to a domain on which a set of affine transformations acts transitively (cf. [V-G-PS]), it is easy to see that bounded homogeneous domains are hyperconvex (cf. [K-Oh]. See also [Dn-3] and [Is]). A longstanding open question is whether or not, for the n-dimensional bounded homogeneous domains, the L^2 $\bar\partial$-cohomology groups of type (p,q) with respect to the Bergman metric are all infinite dimensional if p + q = n. For the bounded symmetric domains, the assertion was verified by Gromov [Gm]. Recently, Ishi-Yamaji [I-Y] showed that the Bergman metric of a bounded homogeneous domain is the pull-back of that of a bounded symmetric domain by a canonically defined embedding.

Going back to the theory of Oka and Grauert, the Levi problem on complex manifolds makes sense only after suitable restrictions or

modifications because otherwise there exist counterexamples (cf. [G-3], [Siu-1] and [Oh-10]). Generally speaking, with all conceivably natural settings, the Levi problem is still far reaching on complex manifolds. Among the tough questions of this kind, the Shafarevich conjecture is most attracting. It asks whether or not the universal covering space of any compact Kähler manifold is holomorphically convex. It is remarkable that a recent partial answer to it by Robert Treger [Tr-1,2] is based on the analysis of the Bergman kernel.

Optimistically speaking, it is not only challenging but also profitable to explore methods to characterize the domains of existence of analytic functions on complex manifolds, because they will lead us to new boundary value problems with rich contents.

In this spirit, it may be also worthwhile to consider refined Levi problems on complex manifolds. For instance, let M be a complex manifold equipped with a volume form dV, let $\mathcal{D}$ be a bounded domain in M, and let Δ be the embedding of $\mathcal{D}$ into $\mathcal{D} \times \mathcal{D}$ as the diagonal.

Q4**

Does $\lim_{z \to \partial\mathcal{D}} \Delta^* \varkappa_{\mathcal{D}} / dV = \infty$ **hold if** $\mathcal{D}$ **is hyperconvex ?**

Let $\mathbf{CP}^n$ denote the complex projective space of dimension n. For $\mathbf{CP}^n$, it is known that every pseudoconvex proper subdomain is hyperconvex (cf. [Oh-S]). The above question is open even in such a restrictive situation. The main difficulty is that the hyperconvexity of the domain $\mathcal{D}$ *does not imply* the existence of a strictly plurisubharmonic function on a neighbourhood of $\partial\mathcal{D}$. More precisely, it is false in general (cf. [Di-Oh-4]) and not known even if $\mathcal{D}$ is a domain in $\mathbf{CP}^n$ (n $\neq$ 1). Nevertheless, it is known that the $\bar\partial$-equations for (n,q)-forms are solvable with L^2 norm estimates for all q on any pseudoconvex proper subdomain with C^2 boundary in $\mathbf{CP}^n$(cf. [C-S-W]. See also [H-I] and [Brn-Cha]). So, the solution for the case $M = \mathbf{CP}^n$ should not be too difficult and may clarify the essential part of Q4. Concerning the related questions, see also [Di-Oh-3] and [M-Oh].

More intricate relationship between $K_\Omega(z)$ and geometric invariants on $\partial\Omega$ can be explored when $\partial\Omega$ is C^∞ and everywhere strongly pseudoconvex. A groundbreaking result in this direction was a theorem of Fefferman [F-1] assuring that there exist two C^∞ functions φ and ψ

defined on a neighbourhood of $\partial\Omega$ such that

$$(18) \qquad K_\Omega(z) = \varphi(z)\delta(z)^{-n-1} + \psi(z)\log\delta(z)$$

holds near $\partial\Omega$. (7) implies that $\varphi(z_0) = n!\pi^{-n}L(z_0)$ for any $z_0 \in \partial\Omega$.

Geometric invariants besides $L(z_0)$ are involved in the coefficients of the asymptotic expansions of φ and ψ in δ (their expression can be made simpler after some rescaling), which have been investigated by Fefferman [F-2], Bailey-Eastwood-Graham [B-E-G] and Hirachi [Hr-1].

The following was a very famous question known as the Ramadanov conjecture (cf. [Rm]).

Q5(*✲✲)

Let Ω be a strongly pseudoconvex domain with C^∞ boundary, and let $z_0 \in \partial\Omega$. Suppose that there exists a neighbourhood $V \ni z_0$ in C^n such that $\psi = 0$ on V. Then, is $\partial\Omega$ spherical around z_0? Namely, is there a neighbourhood $W \ni z_0$ and a C^∞ diffeomorphism Φ from $\Omega \cap W$ onto $B^n \cap \{z ; \operatorname{Re} z > 1 - \varepsilon\}$ for some ε such that $\Phi | \Omega \cap W$ is holomorphic?

The answer is yes if $n \leq 2$ (cf. [BM], [G] and [Bu]) and turned out to be no if $n \geq 3$ (cf. [E-Z] and [Hr-2]). However, it is not known whether or not the conclusion holds if one strengthens the assumption to " $\psi = 0$ on a neighbourhood of $\partial\Omega$ ".

Fefferman applied (18) in [F-1] to analyze the geodesics with respect to the Bergman metric, which is in fact a very hard work.

Another effective way of describing the boundary behavior of $K_\Omega(z,w)$ is in terms of the operator theoretic properties of the orthogonal projection, say P_Ω, from the space $L^2(\Omega)$ onto $A^2(\Omega)$. P_Ω is called the Bergman projection.

The principal question in this setting is whether or not

$$(19) \qquad P_\Omega(C^\infty(\overline{\Omega})) \subset C^\infty(\overline{\Omega})$$

holds true, where $\overline{\Omega} := \Omega \cup \partial\Omega$ and $C^\infty(\overline{\Omega})$ denotes the set of C^∞ functions on $\overline{\Omega}$. (19) is called "condition R" by S. Bell [Bl].

It turned out that the property (19) is directly linked to the smooth extendibility of biholomorphic maps. It is actually very efficient, because by this method it is possible to generalize the results to proper holomorphic maps between the domains of finite type (cf. [W], [B-L], [B-

C] and [B-B-C]. See also [Oh-7]).

If $\Omega = \mathbf{B}^n$, (19) can be verified directly by using (5) (cf. [Cha] and [L-M]). In order to generalize this to more general classes of pseudoconvex domains with C^∞ boundary, a natural method is to convert (19) into the property of another operator N_Ω by using Kohn's formula

$$(20) \qquad P_\Omega = \mathrm{Id} - \bar{\partial}^* N_\Omega \, \bar{\partial}.$$

Here N_Ω denotes the inverse of $\bar{\partial}\bar{\partial}^*$ on the image of $\bar{\partial}$. N_Ω is called the Neumann operator. The Neumann operator exists because Ω is pseudoconvex (cf. [C-1]. See also [Oh-9]).

By such an argument, (19) can be verified for the domains of finite type. The point is that subelliptic estimates hold on them (cf. [C-1]).

On the other hand, it is known that (19) is satisfied by certain domains of infinite type. For instance, (19) holds whenever Ω is a complete Reinhardt pseudoconvex domain with smooth boundary (cf. [B-B]).

By [K-N], it is known that (19) is a consequence of the compactness of N_Ω. If Ω is convex, Fu and Straube [F-S] proved that the compactness of N_Ω is equivalent to the condition that $\partial\Omega$ does not contain any complex curve. For the proof, the boundary behavior of $K_\Omega(z)$ is analyzed. In this context, domains for which (19) does not hold are also of considerable interest (cf. [Ba] and [Chr]).

Thus, as a state of art, we understand a general tendency that the existence of a complex curve in the boundary destroys the regularity properties of P_Ω and N_Ω. So, it may be worthwhile to extend Fu-Straube's theorem to more general domains. A candidate is the class of lineally convex domains. Recall that Ω is said to be lineally convex if every point $z_0 \in \partial\Omega$ is contained in a complex hyperplane $H = H(z_0)$ which does not intersect with Ω.

Q6-*--

 Suppose that Ω is lineally convex. Is it true that N_Ω is compact if and only if $\partial\Omega$ does not contain any complex curve ?

----

§3. Asymptotic expansion in tensor powers

We shall now review some results on the asymptotics of the generalized Bergman kernels for tensor powers of positive line bundles, as the power tends to infinity. Motivation for considering such a question is related to algebraic geometry [Serre-1, 2], the heat kernel asymptotics

[Dn-1] and a supersymmetric field theory [Wi] (see [Dm-3] for instance).

Let M be a connected complex manifold of dimension n and let E be a holomorphic line bundle over M. The canonical line bundle of M will be denoted by ω_M. Given a fiber metric h of E, we denote by $A^2(E \otimes \omega_M)$ the space of L^2 holomorphic sections of $E \otimes \omega_M$ with respect to h. The reproducing kernel of $A^2(E \otimes \omega_M)$ will be denoted by $\varkappa_h$.

Let Δ be the diagonal embedding from M into $M \times M$. Then $\Delta^* \varkappa_h$ is a section of $E \otimes \omega_M \otimes \overline{E \otimes \omega_M}$. $\varkappa_h$ and $\Delta^* \varkappa_h$ are called the weighted Bergman kernels, where we shall allow h to be locally of the form $e^{-\varphi}$ for a locally integrable function φ. Such a generalized fiber metric is called a **singular fiber metric**. Similarly as the Bergman kernel function, the weighted Bergman kernel function is defined whenever a trivialization of the canonical bundle exists and is fixed. We shall denote it by K_φ if $h = e^{-\varphi}$. Generally, the product $h \cdot \Delta^* \varkappa_h$, being a section of $\omega_M \otimes \overline{\omega_M}$, can be written as $\varrho_h dV$, where dV is a volume form and ϱ_h is a nonnegative function. ϱ_h measures the size of the Bergman kernel with respect to dV.

As in the case of the Bergman kernel function, the value of ϱ_h at z_0 is characterized as the supremum of the squared length at z_0 of L^2 holomorphic sections of $E \otimes \omega_M$ with L^2 norm one.

Similarly as in the proof of (7), Bouche [Bou] proved that

$$(21) \qquad \lim_{m \to \infty} \varrho_{h^m}^{1/m} = 1$$

holds if M is compact and h is C^∞ and of positive curvature, by extending a work of Tian [Ti], where the Hodge metric is approximated by $1/m$ times the curvature form of $(\Delta^* \varkappa_{h^m})^{-1}$. The model case for (21) is the anti-tautological line bundle over $\mathbb{CP}^n$ equipped with the fiber metric induced from the euclidean metric of $\mathbb{C}^{n+1}$. Although the method is similar as in the estimate of the Bergman kernel, what is approximated is reversed here. Namely, the fiber metric is approximated by the m-th roots of the Bergman kernels.

In the same spirit, Demailly [Dm-2] applied the L^2 extension theorem to approximate any plurisubharmonic function and its Lelong number in terms of the weighted Bergman kernels. Recall that, for any plurisubharmonic function φ on a domain Ω and for any point $z_0 \in \Omega$, the Lelong number $\nu(\varphi, z_0)$ of φ at z_0 is defined by

$$\nu(\varphi, z_0) := \liminf_{z \to z_0} \frac{\varphi(z)}{\log |z - z_0|}.$$

Theorem 1. (cf.[Dm-2]) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exist constants C_1 and C_2 depending only on n and the diameter of Ω such that the following hold for any plurisubharmonic function φ on Ω and for any positive integer m.

$$(22) \qquad \varphi(z) - C_1/m \leq (2m)^{-1}\log K_{2m\varphi}(z) \leq \sup_{|\zeta - z| < r} \varphi(\zeta) + m^{-1}\log(C_2/r^n)$$

if $z \in \Omega$ and $r < \delta(z)$.

$$(23) \qquad \nu(\varphi, z_0) - n/m \leq \nu((2m)^{-1}\log K_{2m\varphi}(z), z_0) \leq \nu(\varphi, z_0), \quad z_0 \in \Omega.$$

Since (23) is a comparison between $2m\varphi$ and $\log K_{2m\varphi}$ near $\varphi = -\infty$, one may naturally ask its counterpart near $\varphi = \infty$.

Q8-**

Let φ be a plurisubharmonic function on a bounded pseudoconvex domain Ω in $\mathbb{C}^n$. Is there a constant C such that $\int_{0 \leq \varphi < R} (i\partial\bar{\partial}\varphi(z))^n < C\int_{0 \leq \varphi < R} (i\partial\bar{\partial}\log K_\varphi(z))^n$ for all R ?

Catlin [C-2] and Zelditch [Z] reversed again the role of h and $\varkappa_h$ in the approximation and established the asymptotics of $\Delta^*\varkappa_h{}^m$ in m as a counterpart of (18). The spirit is to construct the orthogonal projection explicitly from the geometric data. It will be stated below, where the factor ω_M is not explicitly involved, for simplicity.

Let M and (E,h) be as above and let dV be any C^∞ volume form on M. By $A(E)$ we denote the space of holomorphic sections of E. The length of $s \in A(E)$ with respect to h will be denoted by $|s|_h$.
Then we put

$$\|s\|^2 = \int_M |s|_h^2 \, dV,$$

$$\varrho(dV, h)(x) = \sup \{ |s(x)|^2 / \|s\|^2 ; s \in A(E) - \{0\} \},$$

$$\beta(dV, h) = \varrho(dV, h)dV/\dim A(E), \quad \text{whenever } A(E) \neq \{0\}$$

and

$$\Theta = \text{the curvature form of } h.$$

$\beta(dV, h)$ is a probability measure on M which is canonically associated to dV and (E, h). The behavior of $\beta(dV, h^m)$ in m is in question.

Theorem 2. (cf. [C-2] and [Z])   In the above situation suppose that h is C^∞ and of positive curvature, then there exist C^∞ functions b_j on M such that

$$(24) \qquad \beta(dV, h^m) = \sum_{j=0}^{\infty} b_j \, m^{n-j} dV, \qquad b_0 dV = MA(h)/V$$

holds asymptotically in m, where $MA(h) := (\sqrt{-1}\Theta)^n$ and $V := \int_M (\sqrt{-1}\Theta)^n$.

Catlin's proof depends on the analysis of the Bergman kernel of the disc bundle associated to E^*, the dual of E, and Zelditch's on that of the Szegö kernel. Their approaches are both natural because, letting T be the unit disc bundle associated to E^*, which is a tubular neighbourhood of the zero section, $\rho(dV, h^m)$ are naturally identified with the diagonalized reproducing kernels of (relatively small) subspaces of $L^2(T)$ or $L^2(\partial T)$ consisting of holomorphic functions of restricted type on T. However, their common tool is the microlocal method of Boutet de Monvel and Sjöstrand [BM-S] which extends [F-1]. For an elementary proof of Theorem 2, see [B-B-S].

In view of such a close relationship between (18) and (24), it seems natural to ask a counterpart of the Ramadanov conjecture in this context.

For instance, if $M = \mathbf{CP}^1$ and E is the anti-tautological bundle, it is easy to see that the fiber metric h is determined by b_0 up to a constant factor. When $M = \mathbf{CP}^n$ and $n > 1$, it becomes more difficult to formulate the question. Of course it will be even more difficult when M and E are not fixed in advance. We note that, according to the work of Ziqin Lu [Lu], $\beta(dV, h^m)$ looks like the stress energy in Einstein's equation.

In such a way, the Bergman kernel is related to algebraic geometry and differential geometry through (pluri-)potential theory. It is remarkable that Theorem 2 was applied by Donaldson [D] to the stability theory of projectively embedded manifolds (see also [Mab-1,2]). Motivated by Donaldson's work, Theorem 2 was extended to a more general context of symplectic manifolds and orbifolds (cf. [D-L-M]). But let us wait for another opportunity to enter this fancy topic.

§4. Variations in analytic families

Returning to the formula (2), it suggests, as well as Fefferman's theorem, that any C^∞ family of biholomorphic maps say $\{\alpha_t\}_{0<t<1}$, from a C^∞ family of C^∞ strongly pseudoconvex domains $\{\Omega_t\}_{0<t<1}$ in $\mathbf{C}^n$ to another C^∞ family $\{\Omega'_t\}_{0<t<1}$ in $\mathbf{C}^n$, extends to the boundaries also smoothly in t.

In fact, in virtue of the approach of Bell-Ligocka [B-L], the smooth

extendibility of $\{\alpha_t\}$ is reduced to the smoothness in t of the Bergman projections, which can be verified, via Kohn's formula, by checking the corresponding property of the family of Neumann operators (cf. [G-K]).

Hence, (18) relates the Bergman kernel of Ω not only to geometric invariants of $\partial\Omega$, but also to their variations. So does (24) similarly. Thus the Bergman kernel is linked not only to the Levi problem, but also to the moduli problem. In this sense, variational questions for the Bergman kernels on complex analytic families are particularly interesting. Let us review some results in this direction.

Let $\mathcal{M}$ be a connected complex manifold, let U be a domain in $\mathbf{C}^m$ and let $\pi : \mathcal{M} \longrightarrow U$ be a surjective holomorphic map such that $d\pi$ is everywhere of maximal rank. The family of the Bergman kernels on the fibers of π is called the relative Bergman kernel on $\mathcal{M}$. For any $\zeta \in U$ we put $\varkappa_\zeta = \Delta^* \varkappa_{\pi^{-1}(\zeta)}$. We shall assume, for simplicity, that $\varkappa_\zeta$ is not everywhere zero. Then the collection $\{\varkappa_\zeta^{-1}\}_{\zeta \in U}$ is naturally regarded as a singular fiber metric of $\omega_{\mathcal{M}/U}$, where we put $\omega_{\mathcal{M}/U} := \omega_{\mathcal{M}} \otimes (\pi^* \omega_U)^*$. We put $\beta_{\mathcal{M}/U} = \{\varkappa_\zeta^{-1}\}_{\zeta \in U}$.

There are two model cases :

1) $\mathcal{M} = \mathbf{B}^{n+m}$, $U = \mathbf{B}^m$ and $\pi(z) = z''$, where $z = (z', z'')$. In this case, the curvature form of $\beta_{\mathcal{M}/U}$ is

$$- \partial\bar{\partial}(\log(1 - |z''|^2)^n + \log(1 - |z'|^2)^{n+1}).$$

Obviously it is positive on $\mathcal{M}$.

2) $\mathcal{M} = \underset{Z}{\bigsqcup} \mathbf{C}^n / \Gamma_Z$ (disjoint union), where Z runs through the set $End^+\mathbf{C}^n := \{ Z \in End\,\mathbf{C}^n; \det \mathrm{Im}\,Z > 0 \}$ and Γ_Z stands for the lattice in $\mathbf{C}^n$ generated by the columns of the $n\times n$ unit matrix and those of Z. Here, $End\,\mathbf{C}^n$, the set of complex endomorphisms of $\mathbf{C}^n$, is naturally identified with the set of $n\times n$ matrices whose entries are complex numbers. Then we put $U = End^+\mathbf{C}^n$ and $\pi(q) = Z$ for any $q \in \mathbf{C}^n / \Gamma_Z$. In this case, the curvature form of $\beta_{\mathcal{M}/U}$ is $-\partial\bar{\partial}\log(\det \mathrm{Im}\,Z)$, which is easily seen to be **not** semipositive on $\mathcal{M}$ if $n > 1$.

As is well known, $-\log(\det \mathrm{Im}\,Z)$ becomes strictly plurisubharmonic when it is restricted to the set of those Z for which $^t Z = Z$ and $\mathrm{Im}\,Z$ is positive definite (cf. [Sg]).

Q8 ✶ ---

Compute the signature of $\partial\bar{\partial}\log(\det \mathrm{Im}\, Z)$.

It is surprising that nothing general about $\beta_{M/U}$ was known in the last century, although the semipositivity properties of the direct image sheaf $\pi_* \omega_{M/U}$ had been known in the context of variation of Hodge structures and its application to the classification theory of algebraic varieties (cf. [Gr] and [Fj]). (See also [Oh-1].) The first result in this direction, extending the model case 1), was obtained by Maitani-Yamaguchi [M-Y] in the case where $\mathcal{M}$ is a Stein manifold of dimension 2. Namely, by combining the analysis of variation of Green functions (cf. [L-Y]) with a characterization of the Bergman kernel by N. Suita (1933-2002) as the second derivative of the Green function (cf. [Sui]), they proved that $\beta_{M/U}$ is of semipositive curvature in this situation. (See also [Mt].)

It is an interesting coincidence that Suita's work was motivated by an open question raised in a treatise by K. Oikawa (1927-92) and L. Sario, where the comparison between the Bergman kernel and the capacity of the boundary of an arbitrary domain W in $\mathbb{C}$ was asked (cf. [O-S] p.342, 7). Here, the capacity $c_\beta(z)$ of the boundary $\beta = \partial W$ is defined by

$$(25) \qquad \log c_\beta(z) = \lim_{w \to z} \left(g_W(z,w) - \log|z - w| \right),$$

where g_W denotes the (negative valued) Green function of W ($c_\beta := 0$ if $g_W \equiv 0$). It is clear that $\pi K_W = c_\beta^2$ holds if $W = \mathbf{D}$. In [O-S], it was shown that $K_W \equiv 0$ if and only if $c_\beta \equiv 0$. Suita showed that

$$(26) \qquad \pi K_W(z) > c_\beta(z)^2 \quad \text{for any } z \in W$$

holds if W is an annulus.

✶✶✶
Q9 (Suita's conjecture) ---

$$(27) \qquad \pi K_W(z) > c_\beta(z)^2 \quad \text{holds for any } z \in W$$

if $g_W \not\equiv -\infty$ and W is not equivalent to the unit disc.

The reader may notice that Q9 could have been included in §2. In fact, after proving the divergence of $K_\Omega(z)$ at $\partial\Omega$ for hyperconvex Ω in [Oh-2], the author refined [Oh-T] in [Oh-5] and meanwhile found its application to Q9 (cf. [Oh-6,7]), in which an inequality $750\pi K_W > c_\beta^2$ was obtained. The latter was improved by [Brn-1] and [Bł]. According to [Bł],

$2\pi\,K_W > c_\beta^2$ holds in the situation of Q9.

After [M-Y], Berndtsson [Brn-2] and Tsuji [Tj] succeeded in generalizing the result to Stein manifolds of arbitrary dimension, by directly exploiting the reproducing property of the Bergman kernel. On the other hand, the method of [M-Y] was extended to explore variational properties of families of harmonic functions with prescribed singularities and Dirichlet or Neumann type boundary conditions (cf. [Hm] and [Hm-M-Y]).

Recently, Berndtsson-Păun [Brn-P] obtained a result which is also related to 2). Motivated by applications to algebraic geometry, they consider a surjective projective morphism say $p : X \longrightarrow Y$ between complex manifolds X and Y, and a holomorphic line bundle (L, h) over X endowed with a singular fiber metric h. Let $I(h)$ denote the multiplier ideal sheaf of h (cf. [Dm-3]). Let $Y^0 \subset Y$ be the Zariski open set of points that are not critical values of p in Y, and let $X^0 \subset X$ be the inverse image of Y^0 with respect to p. As y varies in Y^0, the relative Bergman metric on $\omega_{X/Y}\otimes L$ over X^0 is defined similarly as $\beta_{M/U}$, only it is allowed to be identically ∞.

Theorem 3. In the above situation, assume the following.

i) the curvature current of (L, h) is semipositive on X.

ii) $H^0(X_y, \omega_{X_y} \otimes L\otimes I(h)) \neq 0$ for some $y \in Y^0$, where $X_y = p^{-1}(y)$.

Then the relative Bergman kernel metric of the bundle $\omega_{X/Y}\otimes L \,|\, X^0$ is not identically ∞. It has semipositive curvature current and extends across $X - X^0$ to a metric with semipositive curvature current on all of X.

For the proof, the assumption that p is projective is crucial. The point is that every point $y \in Y$ admits a neighbourhood V such that $p^{-1}(V)$ contains a divisor whose complement is Stein. Theorem 3 has interesting applications to pluricanonical maps via an inequality for the "restricted volume" (cf. Theorem 0.3 in [B-P]. See also [Tk]). For the asymptotics of the restricted volume, see [Hs].

Q10**✱✱**--

For which morphism is Theorem 3 valid ?

Analyzing the model case 2) from this viewpoint seems interesting.

Any extension of the model case 2) towards this direction will be quite interesting and fruitful. Berndtsson [Brn-2] has proved Nakano-semipositivity of the direct images for Kähler morphisms and applies the result in [Brn-3] to study variations of Kähler metrics. Moreover, the deep work of Fang-Lu-Yoshikawa [F-L-Y] on the family of Calabi-Yau threefolds seems to be closely related to this question.

Note. If $\mathcal{M}$ is a Hartogs domain over U, there is a formula which relates the weighted Bergman kernels on U to $\varkappa_{\mathcal{M}}$ (cf. [Li]), which is useful to derive explicit formulae (cf. [Ym]). Computation for Hartogs domains is done also in [M-Y] for the relative Bergman kernels.

§5. Bergman Kernel and L^2 Extension

As before, let Ω be a domain in $\mathbf{C}^n$ and let $A^2(\Omega)$ be the Hilbert space of L^2 holomorphic functions on Ω with respect to the Lebesgue measure. Let $z = (z, \ldots, z)$ be the coordinate of $\mathbf{C}^n$.

For any pseudoconvex domain Ω in $\mathbf{C}^n$, for any plurisubharmonic function φ on Ω, and for any nonnegative number ε, we put

$$A^2_{\varphi,\varepsilon}(\Omega) = \{\, f \mid f \text{ is holomorphic on } \Omega \text{ and } \int_\Omega e^{-\varphi}(1+|z_n|^2)^{-1-\varepsilon}|f|^2 < \infty \,\}$$

and, by letting $\Omega' = \{\, z \in \Omega \mid z_n = 0 \,\}$, put

$$A^2_\varphi(\Omega') = \{\, f \mid f \text{ is holomorphic on } \Omega' \text{ and } \int_{\Omega'} e^{-\varphi}|f|^2 < \infty \,\}.$$

Then we have

Theorem 4. (cf. [Oh-T], [Oh-3])　Suppose that Ω is pseudoconvex. Then, for any $\varepsilon > 0$, there exists a bounded linear operator

$$I_\varepsilon : A^2_\varphi(\Omega') \longrightarrow A^2_{\varphi,\varepsilon}(\Omega)$$

whose norm does not exceed a constant C_ε depending only on ε, such that $I_\varepsilon(f)|\Omega' = f$ holds for any $f \in A^2(\Omega')$.

Obviously Theorem 4 does not hold for $\varepsilon = 0$. The best constant for C_ε is not yet known (cf. [Bł]). As was mentioned in §3 and §4, Theorem 4 was applied to plurisubharmonic functions and to the Bergman kernels .

As before, let M be a complex manifold of dimension n, let E be a

holomorphic vector bundle over M, and let ω_M be the canonical line bundle of M. Let dV be a C^∞ volume form on M and let h be a C^∞ fiber metric of E. For any reduced analytic set $S \subset M$ equipped with a measure μ, $A^2(S, E \otimes \omega_M, h \otimes (dV)^{-1}, d\mu)$ will stand for the space of L^2 holomorphic sections of $E \otimes \omega_M$ over S with respect to $h \otimes (dV)^{-1}$ and μ. Since $A^2(M, E \otimes \omega_M, h \otimes (dV)^{-1}, dV)$ is independent of dV, we shall denote it by $A^2(M, E \otimes \omega_M, h)$ for brevity.

Given locally integrable functions $\psi : M \longrightarrow [-\infty, \infty)$, the spaces $A^2(S, E \otimes \omega_M, e^{-\psi} h \otimes (dV)^{-1}, d\mu)$ and $A^2(M, E \otimes \omega_M, e^{-\psi} h)$ are defined similarly. We shall call $e^{-\psi} h$ a singular fiber metric of E. Given any singular fiber metric $\hat{h}$ of E, an L^2 extension operator for $(E \otimes \omega, \hat{h} \otimes (dV)^{-1})$ from (S, μ) is defined to be a bounded linear operator

$$I : A^2(S, E \otimes \omega_M, \hat{h} \otimes (dV)^{-1}, d\mu) \longrightarrow A^2(M, E \otimes \omega_M, \hat{h})$$

satisfying $I(f)|S = f$ for any f. Let $\Phi : M \longrightarrow [-\infty, 0)$ be any continuous function. We shall say that Φ is of logarithmic type along S if the following are satisfied.

$$\Phi^{-1}(-\infty) = S.$$

$$\Phi|(M \setminus S) \text{ is } C^\infty.$$

$e^{-\Phi}$ is not integrable on an open subset $U \subset M$ whenever $U \cap S \neq \emptyset$.

Given a function Φ which is of logarithmic type along S, we say that μ is a residual majorant of (dV, Φ) if the inequality

$$\limsup_{r \to \infty} \int_{-r < \Phi < -r+1} \rho e^{-\Phi} dV \leq \int_S \rho \, d\mu$$

holds for any nonnegative continuous function ρ with compact support on M.

We say that (E,h) is Φ-positive if there exists a positive number τ_0 such that $(E, he^{-(1+\tau)\Phi})$ are Nakano semipositive on $M \setminus S$ for any $\tau \in [0, \tau_0]$. We shall denote the supremum of such τ_0 by $\tau(h, \Phi)$.

Let T be a closed subset of M. We say that T is L^2-negligible if, for any point $p \in T$ and for any neighbourhood $W \ni p$, every L^2 holomorphic n-form on $W \setminus T$ is holomorphically extendible to W.

In these terms, the main result of [Oh-5] is expressed as follows.

Theorem 5. Let M be a complex manifold with a C^∞ volume form dV, let E be a holomorphic vector bundle over M with a C^∞ fiber metric h, let S be a reduced analytic subset of M equipped with a measure μ, and let $\Phi : M \longrightarrow [-\infty, 0)$ be a continuous function which is of logarithmic type along S. Suppose that μ is a residual majorant of (dV, Φ), h is Φ-positive, and that there exists an L^2-negligible set $T \subset M$ such that $M \setminus T$ is Stein and $S \cap T$ is nowhere dense in S. Then, for any plurisubharmonic function ψ on M, there exists an L^2 extension operator for $(E \otimes \omega_M, e^{-\psi} h \otimes (dV)^{-1})$ from (S, μ) whose norm is bounded by a constant depending only on $\tau(h, \Phi)$.

In [Oh-5, Theorem 4], the result is stated for a more restricted class of Φ, but it is easy to see that the proof of this generalized version is completely similar.

The point of Theorem 5 as well as Theorem 4 is that the norm of the L^2 extension operator is estimated by a relatively simple geometric quantity. Therefore it seems to make sense to ask the following.

Q11 **

Find a reasonable generalization of Theorem 5 for the $\bar{\partial}$ closed forms of type $(0,q)$ for $q \geq 1$.

For a nice but partial answer, see [Kz] for instance.

Finally, let's see how one can derive a division theorem from an extension theorem in such a way that Theorem 5 yields an L^2 division theorem.

Let E^* denote the dual bundle of E, let $P(E^*)$ be the projectivization of E^*, i.e. $P(E^*) = \cup (E^* - \{0\})/(C - \{0\})$, and let $\varpi : P(E^*) \longrightarrow M$ be the bundle projection.
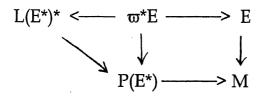
Recall that, in the presence of such a fiber structure, the sheaf cohomology groups of $P(E^*)$ and those of M are related by the Leray spectral sequence. Based on this, one has a canonical isomorphism between the E-valued cohomology groups of M and cohomology groups of $P(E^*)$ with values in a certain line bundle. More precisely, one has the following.

Theorem 6. (cf. [LP]) Let $L(E^*)$ denote the tautological line bundle over $P(E^*)$, i.e. $L(E^*) = \underset{x \in M}{\cup} L(E_x^*)$, where $L(E_x^*)$ denotes the tautological line bundle over $P(E_x^*)$. Then there is a natural isomorphism

(28) $\qquad H^{p,q}(M, E) \cong H^{p,q}(P(E^*), L(E^*)^*).$

Here $H^{p,q}(\cdot, \cdot)$ denotes the Dolbeault cohomology group of type (p,q).

We note that $L(E^*)^*$ and E are related by the following commutative diagram.

$$L(E^*)^* \longleftarrow \varpi^*E \longrightarrow E$$
$$\searrow \qquad \downarrow \qquad\quad \downarrow$$
$$P(E^*) \longrightarrow M$$

Here the morphism $\varpi^*E \longrightarrow L(E^*)^*$ is defined over $y \in P(E^*)$ as the natural projection to the quotient space of $E_{\varpi(y)}$ by the kernel of y.

Now, by applying (28), one can transform a division problem on M to an extension problem on $P(E^*)$ as follows.

Let $\gamma : E \longrightarrow Q$ be a surjective morphism between holomorphic vector bundles E and Q over M. A (generalized) division problem asks for conditions for the induced morphisms from $H^{p,q}(M, E)$ to $H^{p,q}(M, Q)$ to be surjective. In view of (28), this surjectivity is equivalent to that of

$$H^{p,q}(P(E^*), L(E^*)^*) \longrightarrow H^{p,q}(P(Q^*), L(Q^*)^*),$$

which is nothing but the extendibility because $P(Q^*)$ is naturally identified with a complex submanifold of $P(E^*)$ by γ^* and a canonical isomorphism between $L(E^*)^* | Q$ and $L(Q^*)^*$ is induced by γ.

Thus, by interpreting the conditions in Theorem 5 in this situation, we shall obtain an L^2 division theorem.

In fact, given $E \longrightarrow Q$ as above, any C^∞ fiber metric h of E and a point $v \in P(E^*)$, let $\delta_h(v)$ denote the fiberwise distance from v to $P(Q^*)$ with respect to the Fubini-Study metric associated to h, normalized in such a way that $\sup \{\delta_h(v); v \in P(E^*) \} = 1$ for every $x \in M$. In this situation we have the following.

Theorem 7. Let (E,h) and Q be as above. Assume that there exists an L^2-negligible set $T \subset M$ such that $M - T$ is Stein and, with respect to the fiber metric of $L(E^*)^*$ induced from h, $L(E^*)^*$ is $\log\delta$-positive. Then the natural homomorphism

$$A^2(M, E \otimes \omega_M, h) \longrightarrow A^2(M, Q \otimes \omega_M, h_Q)$$

is surjective. Here h_Q denotes the fiber metric of Q induced from h.

Corollary 1. (cf. [Oh-7]) Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Then there exists a constant C depending only the diameter of Ω such that, for any plurisubharmonic function φ on Ω and for any holomorphic function f on Ω satisfying

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)-2n\log|z|} \, d\lambda < \infty$$

there exists a vector valued holomorphic function $g = (g_1, \ldots, g_n)$ on Ω satisfying

$$f(z) = \sum_{j=1}^n z_j \, g_j(z)$$

and

$$\int_\Omega |g(z)|^2 e^{-\varphi(z)-2(n-1)\log|z|} \, d\lambda \le C \int_\Omega |f(z)|^2 e^{-\varphi(z)-2n\log|z|} \, d\lambda .$$

Here $d\lambda$ denotes the Lebesgue measure.

Remark. It does not seem to be easy to derive Corollary 1 just by applying the result of [S].

References

[A-V-1] Andreotti, A., Vesentini, E. Sopra un teorema di Kodaira, Ann.Scuola Norm. Sup.Pisa, (3) **15**, (1961), 283-309.

[A-V-2] ————, Carleman estimates for the Laplace-Beltrami equation in complex manifolds, Publ. Math. I.H.E.S. **25** (1965), 81-130.

[An] Andrievskii, V., The highest smoothness of the Green function implies the highest density of a set, Ark. Mat. **42** (2004), 217-238.

[A] Atiyah, M., Elliptic operators, discrete groups and von Neumann algebras, Astérisque **32-33**, (1976), 43-72.

[B-E-G] Bailey, Toby N.; Eastwood, Michael G.; Graham, C. Robin Invariant theory for conformal and CR geometry. Ann. of Math. (2) 139 (1994), no. 3, 491--552.

[Ba] Barrett, D., Biholomorphic domains with inequivalent boundaries, Invent. Math. **85** (1986), 373-377.

[B-B-C] Bedford, E., Bell, S. and Catlin, D., Boundary behavior of proper holomorphic mappings. Michigan Math. J. 30 (1983), no. 1, 107--111.

[Bl] Bell, S., Biholomorphic mappings and the $\bar{\partial}$-problem, Ann. of Math. **114** (1981), 103-113.

[B-B] Bell, S. and Boas, H. , Regularity of the Bergman projection in weakly pseudoconvex domains, Math. Ann. **257** (1981), 23-30.

[B-C] Bell, S. and Catlin, D., Boundary regularity of proper holomorphic mappings. Duke Math. J. 49 (1982), no. 2, 385--396.

[B-L] Bell, S. and Ligocka, E., A simplification and extension of Fefferman's theorem on biholomorphic mappings, Invent. Math. **57** (1980), no. 3, 283-289.

[B] Bergman, S., Über die entwicklung der harmonischen Funktionen der Ebene und des Raumes nach

Orthogonalfunktionen, Math. Ann. 86 (1922), 238-271.

[B] Bergman, S., The Kernel Function and Conformal Mapping. Mathematical Surveys, No. 5. *American Mathematical Society, New York, N. Y.*, 1950. vii+161 pp.

[Br-Bk] Berman, R. and Boucksom, S., Equidistribution of Fekete points on complex manifolds, arXiv: 0807.0035v1 [math.CV]

[Brn-Cha] Berndtsson, B., Charpentier, P.: A Sobolev mapping property of the Bergman kernel. Math. Zeitschrift **235** (2000) , 1-10.

[Brn-1] Berndtsson, B., Subharmonicity properties of the Bergman kernel and some other functions associated to pseudoconvex domains. Ann. Inst. Fourier (Grenoble) 56 (2006), no. 6, 1633--1662.

[Brn-2] ———, Curvature of vector bundles associated to holomorphic fibrations, Ann. of Math. Vol. 169 (2009), 531-560

[Brn-3] ———, Positivity of direct image bundles and convexity on the space of Kähler metrics, J. Differential Geom. Volume 81, Number 3 (2009), 457-482.

[Brn-P] Berndtsson, B. and Paun, M., Bergman kernels and the pseudoeffectivity of relative canonical bundles. Duke Math. J. 145 (2008), no. 2, 341--378.

[Bł] Błocki, Z., Some estimates for the Bergman kernel and metric in terms of logarithmic capacity. Nagoya Math. J. 185 (2007), 143--150.

[B-P] Błocki,Z. and Pflug, P., Hyperconvexity and Bergman completeness, Nagoya Math. J., **151** (1998), 221-225.

[B-S-Y] Boas, H. and Straube, E. and Yu, Ji Ye, Boundary limits of the Bergman kernel and metric. Michigan Math. J. 42 (1995), no. 3, 449--461.

[Bo] Bochner, S., Über orthogonale Systeme analytischer Funktionen, math. Z. 14 (1922), 180-207.

[Bou] Bouche, Th., Convergence de la métrique de Fubini-Study d'un fibré linéaire positif, Ann. Inst. Fourier **40** (1990), 117-130.

[BM] Boutet de Monvel, L., Le noyau de Bergman en dimension 2. (French) [The Bergman kernel in dimension 2] Séminaire sur les Équations aux Dérivées Partielles 1987--1988, Exp. No. XXII, 13 pp., École Polytech., Palaiseau, 1988.

[Bu] Burns, D., unpublished computation.

[C-S-W] J. Cao, M.-C. Shaw and L. Wang, Estimates for the $\bar{\partial}$-Neumann problem and nonexistence of Levi-flat hypersurfaces in CP^n , Math. Zeit 248 (2004), 183-221; Erratum, 223-225.

[C-1] Catlin, D., Necessary conditions for subellipticity of the $\bar{\partial}$-Neumann problem, Ann. of Math. **117** (1983), 147-171.

[C-2] ———, The Bergman kernel and a theorem of Tian, Analysis and geometry in several complex variables (Katata, 1997), Trends Math., Birkhäuser Boston, Boston, MA, 1999, pp. 1-23.

[Cha] Formules explicites pour les solutions minimales de l'équation $\bar{\partial}u=f$ dans la boule et dans le polydisque de C ,Ann. Inst. Fourier **30** (1980), no. 4, 121--154.

[Ch] Chen, B.-Y., Bergman completeness of hyperconvex manifolds, Nagoya Math. J. **175** (2004), 165 –170.

[Ch-Km-Oh] Chen, B.-Y., Kamimoto, J. and Ohsawa, T., Behavior of the Bergman kernel at infinity. Math. Z. 248 (2004), no. 4, 695--708.

[Chg] Cheng, S.-Y., Open Problems, Conference on Nonlinear Problems in Geometry held in Katata, September 3-8, 1979, p.2, Tohoku university, Department of mathematics, Sendai, 1979.

[Chr] Christ, M., Global C irregularity of the $\bar{\partial}$-Neumann problem for worm domains, J. Amer. Math. Soc. **9** (1996), 1171-1185.

[D-L-M] Dai, X., Liu, K. and Ma, X., On the asymptotic expansion of Bergman kernel, C. R. Acad. Sci. Paris, Ser. I 339 (2004), 193-198.

[Dm-1] Demailly, J.-P., Mesures de Monge-Ampère et mesures pluriharmoniques, Math. Z. **194** (1987), 519--564.

[Dm-2] ———, Regularization of closed positive currents and intersection theory. J. Algebraic Geom. 1 (1992), no. 3, 361--409.

[Dm-3] ———, Complex analytic and differential geometry, typeset on Saturday June 9, 2007.

[Di] Diederich, K., Das Randverhalten der Bergmanschen Kernfunktion und Metrik in streng pseudokonvexen Gebieten, Math. Ann. 187 (1970), 9-36.

[D-F-H] Diederich, K. , Fornaess, J.-E. and Herbort, G., Boundary behavior of the Bergman metric. Complex analysis of several variables (Madison, Wis., 1982), 59--67

[D-H-1] Diederich, K. and Herbort, G., Geometric and analytic boundary invariants on pseudoconvex domains. Comparison results. J. Geom. Anal. 3 (1993), no. 3, 237--267.

[D-H-Oh] Diederich, K., Herbort, G. and Ohsawa, T., The Bergman kernel on uniformly extendable pseudoconvex domains, Math. Ann. 273 (1986), 471--478.

[D-P] Diederich, K. and Pflug, P., Über Gebiete mit vollständiger Kählermetrik. (German) [On domains with a complete Kähler metric] Math. Ann. 257 (1981), no. 2, 191--198.

[Di-Oh-1] Diederich, K. and Ohsawa, T., Une caractérisation des points de stricte pseudoconvexité par la métrique de Bergman. (French) [A characterization of strictly pseudoconvex points by the Bergman metric] C. R. Acad. Sci. Paris Sér. I Math. 317 (1993), no. 6, 555--558.

[Di-Oh-2] ———, An estimate for the Bergman distance on pseudoconvex domains. Ann. of Math. (2) 141 (1995), no. 1, 181--190.

[Di-Oh-3] ———, On pseudoconvex domains in P^n Tokyo J. Math. 21 (1998), 353--358.

[Di-Oh-4] ———, On the displacement rigidity of Levi flat hypersurfaces---the case of boundaries of disc bundles over compact Riemann surfaces. Publ. Res. Inst. Math. Sci. 43 (2007), no. 1, 171--180.

[D] Donaldson, S. K., Scalar curvature and projective embeddings, I J. Differential Geom. 59 (2001), 479-522

[Dn-1] Donnelly, H., Elliptic operators and covers of Riemannian manifolds, Math. Z. 223 (1996), no. 2, 303-308.

[Dn-2] ———, L^2 cohomology of the Bergman metric for weakly pseudoconvex domains. Illinois J. Math. 41 (1997), no. 1, 151--160.

[En] Engliš, M., A characterization of symmetric domains. J. Math. Kyoto Univ. 46 (2006), no. 1, 123--146.

[F-L-Y] Fang, H., Lu, Z. and Yoshikawa, K., Analytic torsion for Calabi-Yau threefolds, J. Differential Geom. 80 (2008), 175--259.

[F-1] Fefferman, C., The Bergman kernel and biholomorphic mapping of pseudoconvex domains, Invent. math. 26 (1974), 1-65.

[F-2] ———, Parabolic invariant theory in complex analysis. Adv. in Math. 31 (1979), no. 2, 131--262.

[F-H] Fefferman, C. and Hirachi, K., Ambient metric construction of Q-curvature in conformal and CR-geometries, Math. Res. Lett. 10 (2003), 819-832.

[F-S] Fu, Siqi and Straube, E., Compactness of the $\overline{\partial}$-Neumann problem on convex domains. J. Funct. Anal. 159 (1998), no. 2, 629--641.

[F-W] Fu, Siqi and Wong, B., On strictly pseudoconvex domains with Kähler-Einstein Bergman metrics, Math. Res. Lett. 4 (1997), 697-703.

[G] Graham, R., Scalar boundary invariants and the Bergman kernel. Complex analysis, II (College Park, Md., 1985--86), 108--135, Lecture Notes in Math., 1276, Springer, Berlin, 1987.

[G-1] Grauert, H., Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik , Math. Ann., 131 (1956), 38--75.

[G-2] ———, On Levi's problem and the embedding of real-analytic manifolds, Ann. Math. 68 (1958) 460-472.

[G-3] ———, Bemerkenswerte pseudokonvexe Mannigfaltigkeiten, Math. Z. 81 (1963) 377--391.

[G-K] Greene, R. and Krantz, S., Deformation of complex structures, estimates for the $\bar{\partial}$ equation, and stability of the Bergman kernel. Adv. in Math. 43 (1982), no. 1, 1--86.

[H-I] Henkin, G. M., Iordan, A.: Regularity of $\bar{\partial}$ on pseudoconcave compacts and applications. (see also Erratum: Asian J. Math. 7, (2003) No. 1, pp. 147–148) Asian J. Math. 4, 855–884 (2000)

[Hb] Herbort, G., The Bergman metric on hyperconvex domains, Math. Z. 232 (1999), 183–196.

[H] Hilbert, D., Über das Dirichletsche Prinzip, Math. Ann. 59 (1904), 161-186, Ges. Abh. III, pp.15-37.

[Hr] Hirachi, K., Logarithmic singularity of the Szegő kernel and a global invariant of strictly pseudoconvex domains. Ann. of Math. (2) 163 (2006), no. 2, 499--515.

[Hs] Hisamoto, T., Restricted Bergman kernel asymptotics, master thesis.

[H] Hodge, W.V.D. (1941) The theory and applications of harmonic integrals, Cambridge University Press (1941)

[Hö-1] Hörmander, L., L^2 estimates and existence theorems for the $\bar{\partial}$ operator, Acta Math. 113 (1965), 89-152.

[Hö-2] ———, An introduction to Complex Analysis in several variables, 1st edition, Elsevier Science Pub., New York, 1966, 3rd revised edition, North-Holland Math. library, Vol 7, Amsterdam (1990)

[Hö-3] ———, A history of existence theorems for the Cauchy-Riemann complex in L^2 spaces. J. Geom. Anal. 13 (2003), 329-357.

[Is] Ishi, H., preprint.

[Is-Y] Ishi, H. and Yamaji, S., a seminar talk.

[Kj] Kajdan, D., Arithmetic varieties and their fields of quasi-definition, Actes du Congrès Internationale Mathématiciens, 1970, pp.321-325.

[Km] Kamimoto, J., Newton polyhedra and the Bergman kernel. Math. Z. 246 (2004), no. 3, 405--440.

[Kishi] Kishi, M., 岸正倫　ポテンシャル論　（数学全書 7 ）森北出版　1974.

[Kl] Klembeck, P. F., Kähler metrics of negative curvature, the Bergmann metric near the boundary, and

the Kobayashi metric on smooth bounded strictly pseudoconvex sets. Indiana Univ. Math..J. 27 (1978), 275-282.

[Klm] Klimek, M., Pluripotential theory. London Mathematical Society Monographs. New Series, 6. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1991. xiv+266 pp.

[Kb] Kobayashi, S., Geometry of bounded domains, Trans. AMS, 92 (1959), 267–290.

[Kh] Kohn, J., Harmonic integrals on strongly pseudo-convex manifolds, I, II, Ann. of math. (2)78 (1963), 112-148; ibid. 79 (1964), 450-472.

[K-N] Kohn, J. and Nirenberg, L., Non-coercive boundary value problems. Comm. Pure Appl. Math. 18 1965 443--492.

[K-Oh] Kai, C. and Ohsawa, T., A note on the Bergman metric of bounded homogeneous domains, Nagoya, math. J. 186 (2007), 157-163.

[K-1] Kodaira, K., On cohomology groups of compact analytic varieties with coefficients in some analytic faisceaux, Proc. Nat. Acad. Sci. U. S. A., 39 (1953), 865-872.

[K-2] ———, On Kähler varieties of restricted type (an intrinsic characterization of algebraic varieties). Ann. of Math. 60, (1954). 28--48.

[Kz] Koziarz, V., Extensions with estimates of cohomology classes, to appear in Manuscripta Math., arXiv:1006.4957v1 [math.CV].

[LP] Le Potier, J., Théorèmes d'annulation en cohomologie, C. R. Acad. Sci. Paris 276 (1973), 535-537.

[L-Y] Levenberg, N., Yamaguchi, H., The metric induced by the Robin function. Mem. AMS. 448 (1991), 1-155.

[L-M] Lieb, I. and Michel, J., The Cauchy-Riemann complex. Integral formulae and Neumann problem. Aspects of Mathematics, E34. Friedr. Vieweg \& Sohn, Braunschweig, 2002. x+362 pp.

[Li] Ligocka, E., Forelli-Rudin constructions and weighted Bergman projections, Studia Math. 94, (1989), 257-272.

[L] Lu, Q.-K., On Kaehler manifolds with constant curvature. Acta Math. Sinica 16 269--281 (Chinese); translated as Chinese Math.--Acta 8 1966 283--298.

[Lu] Lu, Z.,

[Mab] Mabuchi, T., Asymptotic structures in the geometry of stability and extremal metrics. (English summary) Handbook of geometric analysis. No. 1, 277–300, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.

[M-Y] Maitani, F. and Yamaguchi, H., Variation of Bergman metrics on Riemann surfaces. Math. Ann. 330 (2004), no. 3, 477--489.

[M-Oh] Matsumoto, K. and Ohsawa, T., On the real analytic Levi flat hypersurfaces in complex tori of dimension two. Ann. Inst. Fourier (Grenoble) 52 (2002), no. 5, 1525--1532.

[M] Morrey, C. B., The analytic embedding of abstract real-analytic manifolds.Ann. of math.(2)68 (1958), 159-201.

[M] Mumford, D., Curves and their Jacobians, University of Michgan Press, Ann Arbor,1975.

[Nm] Nomura, T., Berezin transforms and Laplace-Beltrami operators on homogeneous Siegel domains. Differential Geom. Appl. 15 (2001), no. 1, 91--106.

[N-S] Nemirovskiĭ, S. Yu.; Shafikov, R. G. Conjectures of Cheng and Ramadanov. (Russian) Uspekhi Mat. Nauk 61 (2006), no. 4(370), 193--194; translation in Russian Math. Surveys 61 (2006), no. 4, 780--782

[Oh-1] Ohsawa, T., Boundary behavior of the Bergman kernel function on pseudoconvex domains. Publ. Res. Inst. Math. Sci. 20 (1984), no. 5, 897--902.

[Oh-2] ———, On the Bergman kernel of hyperconvex domains, Nagoya math. J. 129 (1993), 43–52.

[Oh-3] ———, 多変数複素解析 岩波書店 （現代数学の展開） 1997.（Analysis of Several Complex Variables, Translation of mathematical monographs vol. 211 AMS, 2002)

[Oh-4] ———, An essay on the Bergman metric and balanced domains, Reproducing kernels and their applications, ed. by S. Saitoh et al., Kluwer 1999, pp. 141–148.

[Oh-5] ———, On the extension of L² holomorphic functions V —Effects of generalization, Nagoya Math. J. 161 (2001), 1–21.

[Oh-6] ———, ∂̄-Neumann 問題, 多変数複素解析三講 Nagoya mathematical Lectures （多元数理講義録） vol. 4 (2002), 39-54.

[Oh-7] ———, A precise L² division theorem, Complex geometry (Göttingen, 2000), 185-191, Springer, Berlin, 2002.

[Oh-8] ———, 複素解析幾何と∂̄方程式 培風館 2007.

[Oh-S] Ohsawa, T. and Sibony, N., Bounded p.s.h. functions and pseudoconvexity in Kähler manifold. Nagoya Math. J. 149 (1998), 1--8.

[Oh-T] Ohsawa, T. and Takegoshi, K., On the extension of L^2 holomorphic functions, Math. Z. **195** (1987), 197-204.

[O] Oka, K., Collected papers. Translated from the French by R. Narasimhan. With commentaries by H. Cartan. Edited by R. Remmert. Springer-Verlag, Berlin, 1984. xiv+223 pp.

[P] Pflug,P., Quadratintegrable holomorphe Funktionen und die Serre-Vermutung, (German) Math. Ann. **216** (1975), no. 3, 285-288.

[PS] Pyateskii-Shapiro, I. I. Automorphic functions and the geometry of classical domains. Translated from the Russian. Mathematics and Its Applications, Vol. 8 Gordon and Breach Science Publishers, New York-London-Paris 1969 viii+264 pp.

[P-S] Poletsky, E. and Stessin, M., Hardy and Bergman spaces on hyperconvex domains and their composition operators, arXiv:math/0703238v1 [math.CV] .

[Rm] Ramadanov, I.- P., A characterization of the balls in C by means of the Bergman kernel, C. R. Acad. Bulgare Sci. 34 (1981), no. 7, 927--929.

[R] Reid, C., Hilbert, Springer-Verlag New York Inc.; illustrated edition (1970/06). （ヒルベルト・現代数学の巨峰　彌永健一訳　岩波現代文庫　2010）

[Rh] Rhodes, J.A., Sequences of metrics on compact Riemann surfaces, Duke Math. J. **72** (1993), 725-738.

[Serre] Serre, J.-P., Faisceaux algébriques cohérents. (French) Ann. of Math. (2) 61, (1955). 197--278.

[Siu] Siu, Y.-T., Pseudoconvexity and the problem of Levi. Bull. Amer. Math. Soc. 84 (1978), 481--512.

[S] Skoda,H., Applications des techniques L^2 à la théorie des idéaux d'une algèbre de fonctions holomorphes avec poids, Ann. Sci. Ec. Norm. Sup. **5** (1972), 545-579.

[Sui] Suita, N., Capacities and kernels on Riemann surfaces. Arch. Rational Mech. Anal. **46**, (1972), 212-217.

[Tk] Takayama, S., On the invariance and lower semi-continuity of plurigenera of algebraic varieties, J. Algebraic Geom. **16** (2007), 1-18.

[Ti] Tian, G., On a set of polarized Kähler metrics on algebraic manifolds, J. Differential Geom. **32** (1990), 99-130.

[Tj] Tsuji, H., Curvature semipositivity of relative pluricanonical systems; math.AG/0703729.

[V-G-PS] Vinberg, È. B.; Gindikin, S. G.; Pjateckiĭ-Šapiro, I. I. Classification and canonical realization of complex homogeneous bounded domains. (Russian) Trudy Moskov. Mat. Obšč. 12 1963 359--388.

[Wb] Webster, S. M., Biholomorphic mappings and the Bergman kernel off the diagonal. Invent. Math. 51 (1979), no. 2, 155--169.

[Wi] Witten, E., Supersymmetry and Morse theory. J. Differential Geom. 17 (1982), no. 4, 661--692.

[W-1] Weyl, H., The method of orthogonal projection in potential theory, Duke Math. J. Volume 7, Number 1 (1940), 411-444.

[W-2] ———, David Hilbert and his mathematical work, Bull. Amer. Math. Soc. 50 (1944), 612 -654.

[Ym] Yamamori, A., あるHartogs領域のベルグマン核の明示公式について　数理解析研究所講究録 1694 RIMS共同研究　ポテンシャル論とベルグマン核　2010, pp. 151-159.

[Z] Zelditch, S., Szegö kernels and a theorem of Tian, Intern. Math. Res. Notices 6 (1998), 317-331.

[Zw-1] Zwonek, W., An example concerning Bergman completeness. Nagoya Math. J. **164** (2001), 89--101.

[Zw-2] ———, Wiener's type criterion for Bergman exhaustiveness, Bull. Polon. Acad. Math. 50 (3), (2002), 297-311.