数理解析研究所講究録
第 1731 巻 2011 年 162-174

PROBLEMS ON HOLOMORPHIC VECTOR BUNDLES ON COMPLEX MANIFOLDS

KŌTA YOSHIOKA

CONTENTS

0. INTRODUCTION.

Let X be a compact complex manifold. Holomorphic vector bundles on X contain various informations on analytic subsets of X. For a holomorphic vector bundle E on X, the zero set Z of a section is an analytic subset of X, and we can get the properties of Z by studying E. If the rank of E is 1, then Z is a divisor on X. Since line bundles and divisors are relatively easy objects, we are mainly interested in the vector bundles with rank > 1. Then we usually get an analytic subset with $\operatorname{codim} Z \geq 2$. Thus vector bundles with rank > 1 are related to analytic subsets of codimension > 1. For example, the Serre's construction gives a link between a codimension 2 subset Z and a vector bundle E of rank 2. By studying E, we can get informations of Z. In particular, if we know E is a direct sum of line bundles, then we can conclude that Z is a complete intersection of two divisors. We can also get informations on the Chow group of X.

In the theory of vector bundles, a fundamental question is the existence of vector bundles. Since a direct sum of line bundles gives a vector bundle, we are interested in indecomposable vector bundles. If this problem is solved, then the next fundamental

---

1991 *Mathematics Subject Classification.* 14D20.

The author is supported by the Grant-in-aid for Scientific Research (No. 22340010), JSPS.

problem is the classification of vector bundles. Due to the lack of the author's ability, we only treat the moduli spaces over projective surfaces.

Notation.

Let X be a compact complex manifold. $\mathrm{Pic}(X)$ is the Picard group of X, that is, the set of line bundles on X. Let $\mathrm{NS}(X) := \mathrm{im}(\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X, \mathbb{Z}))$ be the Neron-Severi group. It is the set of topological line bundles which have holomorphic structures.

For a projective manifold X, $\mathrm{CH}^*(X)$ denotes the Chow ring of X. Then there is a natural map $\mathrm{CH}^*(X) \to H^*(X)$ and the Chern classes of a coherent sheaf can be defined as element of $\mathrm{CH}^*(X)$.

For a coherent sheaf E, there is an analytic subset Z such that $E_{|X\setminus Z}$ is a locally free sheaf. We denote the rank of $E_{|X\setminus Z}$ by $\mathrm{rk}\, E$. If E is torsion free, then $\mathrm{codim}\, Z \geq 2$. Since a reflexive sheaf of rank 1 on a smooth manifold is locally free, $\det E := (\wedge^{\mathrm{rk}\, E} E_{|X\setminus Z})^{\vee\vee}$ is a line bundle. Since $H^2(X, \mathbb{Z}) \to H^2(X \setminus Z, \mathbb{Z})$ is an isomorphism, $c_1(\det E) = c_1(E)$.

1. Construction of holomorphic vector bundles.

For the existence and the classification of vector bundles, we need a good method of construction.

Problem* 1.** Find a good method to construct vector bundles.

We explain known methods for the construction.

Example 1.1 (Extension method). Serre construction.

$$0 \to \mathcal{O}_X \to E \to I_Z(L) \to 0.$$

E is locally free iff the extension class induces a surjective homnomorphism

$$\mathbb{C} \otimes \mathcal{O}_X \to \mathrm{Ext}^1(I_Z(L), \mathcal{O}_X) \otimes \mathcal{O}_X \to \mathcal{E}xt^1_{\mathcal{O}_X}(I_Z(L), \mathcal{O}_X).$$

Since

$$\mathcal{E}xt^1_{\mathcal{O}_X}(I_Z(L), \mathcal{O}_X) \cong \wedge^2 N_Z(L^\vee),$$

$N_Z(L^\vee) \cong \mathcal{O}_X$. In particular, if (i) $\wedge^2 N_Z$ can be extended to a line bundle L on X and (ii) $H^2(X, L^\vee) = 0$, then we have a vector bundle of rank 2 with a section whose zero is Z. Since $\omega_{X|Z} \cong \omega_Z \otimes (\wedge^2 N_Z)^\vee$, if ω_Z can be extended to a line bundle on X, then (i) holds. In particular, if $\omega_Z = \mathcal{O}_Z$, then (i) holds.

Example 1.2. Let Z be an abelian surface in $\mathbb{P}^4$, then there is a vector bundle E of rank 2 with a section whose zero is Z. E is the Horrocks-Mumford bundle.

If $X = \mathbb{P}^r$, $r \geq 6$, then the assumption (i) and (ii) are satisfied for codimension 2 submanifold Z of X.

Example 1.3 (Basic operations). (i) Tensor products of vector bundles. e.g., $E \otimes F$, $S^n(E)$, $\wedge^n E$.

(ii) Pull-backs:$\pi^*(E)$ by $\pi : Y \to X$.

(iii) The (higher) direct images: For a proper morphism $\pi : X \to Y$, $R^i\pi_*(E)$ are coherent sheaves on Y. It is difficult to study the properties (e.g. the torsion freeness, the locally freeness and the stability) of $R^i\pi_*(E)$. If X, Y are smooth and π is finite, then $\pi_*(E)$ is a vector bundle, where E is a vector bundle on Y. Schwarzenberger showed that every vector bundle of rank 2 is the direct image of a vector bundle on a double cover of X.

Assume that X is a complex torus. Let $Y \to X$ be an etale cover of X and L a line bundle on Y. Then $\pi_*(L)$ is a projectively flat vector bundle on X. Conversely all simple and projectively flat vector bundles are obtained in this way.

Example 1.4 (Elementary transformation). Let E be a vector bundle on X. Let F be a vector bundle on a divisor D of X and $\phi : E \to F$ a surjective homomorphism. Then $E' := \ker\phi$ is a vector bundle on X. E' is the *elementary transformation* of E along F. This operation was introduced by Maruyama as a generalization of the elementary transformation of ruled surfaces. If $\dim X \leq 3$, then all vector bundles on projective manifolds are the elementary transforms of $\mathcal{O}_X^{\oplus r}$.

Sumihiro generalized the notion of the elementary transformation and proved that every vector bundle is obtained from a trivial vector bundle by his elemenraty transform. Unfortunately it is not so easy to construct non-trivial example of Sumihiro's elementary transform.

Example 1.5 (Fourier-Mukai transform). A Fourier-Mukai transform Φ is an equivalence of the derived categories of the categories of coherent sheaves: $\Phi : \mathbf{D}(Y) \to \mathbf{D}(X)$. By Orlov, there is an object $\mathbf{E} \in \mathbf{D}(X \times Y)$ such that

$$\Phi(y) = \mathbf{R}p_X(p_Y^*(y) \otimes \mathbf{E}), y \in \mathbf{D}(Y).$$

If $\mathbf{E} = \mathcal{O}_{\Gamma_f} \otimes p_Y^*(L)[n]$, $L \in \operatorname{Pic}(Y)$ and Γ_f is the graph of an isomorphism $f : Y \to X$, then $\Phi(E) = f_*(E \otimes L)[n]$. This is a trivial Fourier-Mukai transform. For an abelian variey, a $K3$ surface, or an elliptic surface, there are non-trivial Fourier-Mukai transforms. These are very useful to study coherent sheaves on these manifolds.

Although it is not explicit, we can construct vector bundles as a deformation of torsion free sheaves.

2. The existence of vector bundles.

2.1. **Some problems.** For the existence of holomorphic vector bundles, the problem is divided into two parts:

(i) The existence of topological vector bundles.
(ii) The existence of holomorphic structures on topological vector bundles.

We set

$$\operatorname{Vect}^r_{\mathrm{top}}(X) := \{E : \text{ topological vector bundle of rank } r\}. \tag{2.1}$$

We have a bijection:

$$\begin{array}{ccc} \operatorname{Vect}^{\dim X}_{\mathrm{top}}(X) & \to & \operatorname{Vect}^r_{\mathrm{top}}(X) \\ E & \mapsto & E \oplus \mathbb{C}^{r-\dim X}. \end{array} \tag{2.2}$$

Hence for the classification of topological vector bundles, it is sufficient to study topological vector bundles E with $\mathrm{rk}\, E \leq \dim X$.

Remark 2.1. Assume that X is a projective manifold with an ample divisor H. Then for a vector bundle E with $\mathrm{rk}\, E \geq \dim X$, there is an exact sequence

$$0 \to \mathcal{O}_X(-nH)^{\oplus(\mathrm{rk}\, E - \dim X)} \to E \to F \to 0,$$

where F is a vector bundle on X. Thus for the study of holomorphic vector bundles on projective manifolds, it is important to study vector bundles E with $\mathrm{rk}\, E \leq \dim X$.

We have the Chen class map

$$\begin{array}{ccc} \mathrm{Vect}^r_{\mathrm{top}}(X) & \to & \oplus_{i=1}^{\dim X} H^{2i}(X, \mathbb{Z}) \\ E & \mapsto & (c_1(E), c_2(E), ..., c_{\dim X}(E)) \end{array}$$

So the following natural question appears.

Problem* 2.** Characterize the Chern classes of holomorphic vector bundles.

This problem is not solved yet even for non-projective surfaces.

Assume that X is a projective manifold. We are interested in constructing indecomposable vector bundles. If X is a projective manifold, then Maruyama constructed many stable vector bundles of rank $\geq \dim X$. So we are interested in the following question.

Problem* 3.** Let X be a projective manifold. Find an indecomposable vector bundles of rank r with $r < \dim X - 1$.

Remark 2.2. In [Ma, Prop. A.1], Maruyama constructed a stable bundle E for any $c_1(E)$ and $(c_2(E), H^{\dim X - 2}) \gg 0$, where H is the ample divisor. So Maruyama's result does not imply the characterization of Chern classes.

For a torsin free sheaf E on X, there is a proper birational map $\pi : Y \to X$ such that $\pi^*(E)^{\vee\vee}$ is a vector bundle on Y. In this sense, we are interested in vector bundles on manifolds which does not have any birational contraction into a smooth manifold. If $\pi_1(X)$ is non-trivial, then we have (projectively) flat vector bundles. So we also assume that X is simply connected. In particular, we are interested in the following problem.

Problem* 4.** (i) Is there a non-split vector bundle of rank 2 on $\mathbb{P}^n$, $n \geq 5$?
(ii) Is there an indecomposable vector bundle of rank 2 on $\mathbb{P}^4$ except the Mumford-Horrocks vector bundle?

These problems are related to the properties of codimension 2 submanifolds via the Serre construction.

2.2. **The case where** $\dim X = 2$.

Proposition 2.3 (Wu). *Assume that* $\dim X = 2$. *We have a bijection:*

$$\begin{array}{ccc} \mathrm{Vect}^r_{\mathrm{top}}(X) & \to & H^2(X, \mathbb{Z}) \times H^4(X, \mathbb{Z}) \\ E & \mapsto & (c_1(E), c_2(E)) \end{array} \tag{2.3}$$

Thus the topological vector bundles are classified by the Chern classes. Next we want to know when a topological vector bundle has a holomorphic structure. If X is an algebraic surface, then by using Serre's construction, we have a simple answer to this problem.

Theorem 2.4 (Schwarzenberger). *Let E be a topological vector bundle on X. If X is an algebraic surface, then E has a holomorphic structure iff $c_1(E) \in \mathrm{NS}(X)$.*

Proof. Assume that there is a holomorphic line bundle L with $c_1(L) = c_1(E)$. Let H be an ample divisor on X. We want to consider a torsion free sheaf F fitting in the exact sequence:

$$0 \to \mathcal{O}_X(-nH) \to F \to I_Z \otimes L(nH) \to 0.$$

For $n \ll 0$, $H^2(X, L^\vee(-2nH)) \cong H^0(X, L(2nH + K_X))^\vee = 0$. By the local-global spectral sequence, the restriction map

$$\mathrm{Ext}^1(I_Z \otimes L(nH), \mathcal{O}_X(-nH)) \to H^0(X, \mathcal{E}xt^1_{\mathcal{O}_X}(I_Z \otimes L(nH), \mathcal{O}_X(-nH)))$$

is surjective. If Z consists of distinct m points $p_1, p_2, ..., p_m$, then for a general extension, F is a locally free sheaf with $(c_1(F), c_2(F)) = (c_1(L), -(c_1(L)+nH, nH)+m)$. Since $c_2(E)+(c_1(L)+nH, nH) > 0$ for $n \ll 0$, we may set $m := c_2(E)+(c_1(L)+nH, nH)$. Then $F \oplus \mathcal{O}_X^{\oplus(\mathrm{rk}\,E-2)}$ gives a holomorphic structure on E. □

Remark 2.5. By the proof, obviously E is not stable. Indeed the Bogomolov inequality implies that there is no stable vector bundle with $2rc_2(E) - (r-1)(c_1(E)^2) < 0$.

Assume that X is not algebraic.

Theorem 2.6 (Banica-LePotier). *Let E be a holomorphic vector bundle of rank r on a non-algebraic surface. Then*

$$\Delta(E) := c_2(E) - \frac{r-1}{2r}(c_1(E)^2) \geq 0.$$

Problem* 5. Let X be a non-algebraic compact complex surface and $\xi \in \mathrm{NS}(X)$. Let E be a topological vector bundle on X of rank r and $c_1(E) = \xi$. Find a condition on $c_2(E)$ such that E has a holomorphic structure.

Proposition 2.7. *Let X be a non-algebraic complex surface.*

(i) *If X is a complex torus, then the condition is $\Delta(E) \geq 0$ (*[T2],[KY]*).*
(ii) *If X is a $K3$ surface, then the condition is also known, although it is very complicated (*[T-T],[KY]*).*
(iii) *If X is a primary Kodaira surface, then the condition is $\Delta(E) \geq 0$ (*[ABT]*).*
(iv) *If X is a Hopf surface, then the condition is $\Delta(E) \geq 0$ (*[B-L]*).*

Remark 2.8. If X is a Hopf surface, then X is diffeomorphic to $S^1 \times S^3$. So $H^2(X, \mathbb{Z}) = 0$. Thus $c_1(E) = 0$ and $\Delta(E) = c_2(E)$.

Brînzănescu and Moraru ([BM1],[BM2],[BM3]) studied rank two vector bundles on non-Kähler elliptic surfaces by using the relative Fourier-Mukai transforms.

Problem* 6. Generalize the results of Brînzănescu and Moraru to higher rank cases.

Definition 2.9. Let E be a holomorphic vector bundle on X.

(i) E is irreducible, if there is no subsheaf F with $\operatorname{rk} F < \operatorname{rk} E$.

(ii) E is filtrable, if there is a filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_r = E$$

such that F_i are subsheaves with $\operatorname{rk} F_i = i$.

Remark 2.10. If X is projective, then all torsion free sheaves are filtrable.

For a non-algebraic surface, Banica and LePotier proved that there are irreducible vector bundles if $\Delta(E) \gg 0$. If X is a complex torus of algebraic dimension 0, then there is an irreducible vector bundle E iff $v(E) \neq v_0 + nv_1$, $v_0, v_1 \in \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}$, $\langle v_0^2 \rangle = \langle v_1^2 \rangle = 0$ and $\langle v_0, v_1 \rangle = 1$, where $v(E)$ is the Mukai vector of E (see subsection 3.2).

Problem* 7. Find a condition for the existence of irreducible vector bundles.

For non-Kähler elliptic surfaces, the relative Fourier-Mukai transforms are useful to this problem.

2.3. **The case where** $\dim X \geq 3$.

Proposition 2.11. *Assume that* $\dim X = 3$.

(i) *We have a bijection:*

$$\mathrm{Vect}^3_{\mathrm{top}}(X) \to \left\{ (c_1, c_2, c_3) \,\middle|\, \begin{array}{l} c_i \in H^{2i}(X, \mathbb{Z}), i = 1, 2, 3 \\ c_3 \equiv c_1 c_2 + c_1(X) c_2 \mod 2 \end{array} \right\}. \tag{2.4}$$

(ii) *For* $X = \mathbb{P}^3$, *we also have the Chern class map*

$$\mathrm{Vect}^2_{\mathrm{top}}(\mathbb{P}^3) \to \left\{ (c_1, c_2) \,\middle|\, \begin{array}{l} c_i \in H^{2i}(\mathbb{P}^3, \mathbb{Z}), i = 1, 2 \\ c_1 c_2 \equiv 0 \mod 2 \end{array} \right\} \tag{2.5}$$

is surjective. If $c_1 \equiv 1 \mod 2$, *then it is bijective. If* $c_1 \equiv 0 \mod 2$, *then the fiber is classified by the* α*-invariant. If* E *is a holomorphic structure and* $c_1(E) = 0$, *then* $\alpha(E) = h^0(E(-2)) + h^1(E(-2)) \mod 2$.

Remark 2.12. Bănică and Putinar [B-P] showed that a topological vector bundle E with $\operatorname{rk} E = 3$ on a projective manifold has a holomorphic structure iff the Chern classes are represented by algebraic cycles.

If there is a fibration $\pi : X \to T$, we can study vector bundles on X by using the structure of the fibration. If a general fiber of π is a projective line, then we can use Grothendieck's classification of vector bundles on $\mathbb{P}^1$. If a general fiber has a trivial canonical bundle, then we may use the theory of relative Fourier-Mukai transforms [BrMa].

Problem* 8. Let X be a smooth projective 3-fold with a fibration $X \to T$ such that a general fiber is an elliptic surface, an abelian surface or a $K3$ surface. Study vector bundles on X.

3. Classification of vector bundles.

3.1. **Moduli of stable sheaves.** We cannot expect a good holomorphic structure on the set of all vector bundles. One of the reason is the behavior of the automorphism groups of vector bundles under deformations. So we need some requirements on the structure of vector bundles. Simpleness and the stability are introduced to get a nice space.

Definition 3.1. A coherent sheaf E is simple, if $\mathrm{Hom}(E, E) \cong \mathbb{C}$.

Theorem 3.2. *The set of simple sheaves has a structure of algebraic space, which is non-Hausdorff in general.*

Definition 3.3. Let g be a Gauduchon metric and ω_g the associated $(1,1)$-form, that is, $\bar{\partial}\partial\omega_g^{d-1} = 0$. A holomorphic vector bundle E on X is ω_g-stable, iff for any subsheaf F of E with $\mathrm{rk}\, F < \mathrm{rk}\, E$,

$$\frac{(c_1(F), \omega_g^{d-1})}{\mathrm{rk}\, F} < \frac{(c_1(E), \omega_g^{d-1})}{\mathrm{rk}\, E}.$$

If X is a projective manifold and g the Hodge metric associated to an ample divisor H, then this notion is the μ-stability of E with respect to H.

For a projective manifold, we have a refined notion of stability called Gieseker-Maruyama stability.

Theorem 3.4 (Gieseker-Maruyama). *The set $M_H(v)$ of (S-equivalence classes of) semi-stable sheaves E with a topological invariant v has a structure of projective scheme.*

For a torsion free sheaf E, there is a unique filtration

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_s = E$$

such that $E_i := F_i/F_{i-1}$ are semi-stable sheaves with

$$\frac{\chi(E_1(nH))}{\mathrm{rk}\, E_1} > \frac{\chi(E_2(nH))}{\mathrm{rk}\, E_2} > \cdots > \frac{\chi(E_s(nH))}{\mathrm{rk}\, E_s}, (n \gg 0).$$

So the classification of vector bundles is reduced to the classification of stable sheaves modulo the classification of successive extensions.

For the classification of vector bundles, the following problem is important.

Problem* 9.** Construct (general) members of the moduli spaces explicitly.

If we have an explicit family of stable sheaves, then we will know (the birational type of) the moduli space. Conversely, if we know the structure of the moduli spaces well, then we may also construct a family of stable sheaves. Unfortunately this problem is not easy. Let E be a holomorphic vector bundle of rank r on a projective manifold X and H an ample divisor. Then we have an exact sequence

$$0 \to \mathcal{O}_X(-nH)^{\oplus(r-1)} \to E \to I_Z(n(r-1)H) \otimes \det E \to 0,$$

where Z is a subscheme of $\dim Z < \dim X - 1$. So we can expect to construct stable sheaves (or more generally a flat family of stable sheaves) as extensions of $I_Z \otimes L_1$

by $L_2^{\oplus(r-1)}$, $L_1, L_2 \in \mathrm{Pic}(X)$, but this information is not so useful unless $\dim X = 1$. For example, assume that $\dim X = 2$. Then

$$
\begin{aligned}
&\mathrm{Ext}^1(I_Z(nH) \otimes \det E, \mathcal{O}_X(-nH)) \\
&\cong \mathrm{Ext}^1(\mathcal{O}_X(-nH), I_Z(nH + K_X) \otimes \det E)^\vee \\
&\cong H^1(X, I_Z(2nH + K_X) \otimes \det E)^\vee .
\end{aligned}
\tag{3.1}
$$

Since $\chi(I_Z(2nH+K_X)\otimes \det E) = \chi(E(nH+K_X)) - (r-1)\chi(\mathcal{O}_X(K_X)) > 0$ for $n \gg 0$, $\mathrm{Ext}^1(I_Z(nH) \otimes \det E, \mathcal{O}_X(-nH)) \neq 0$ implies that Z is a special configuration of points of X. This makes the construction of a family of vector bundles difficult. Indeed the following holds.

Theorem 3.5 (Mukai, J. Li, O'Grady). (i) *Let X be a projective surface with $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$. Then the moduli of simple sheaves has a holomorphic symplectic structure.*
(ii) *Let X be a minimal surface of general type with $p_g > 0$. Under suitable assumptions, the moduli spaces of stable sheaves is of general type.*

Remark 3.6. If $\dim X = 1$, then $Z = \emptyset$ implies that we have "enough" families of vector bundles. Thus we can construct all member of small deformations of E.

In order to compute the Kodaira dimension, we need to study the canonical bundle of a desingularization of the moduli space.

Problem 10. (i) Study the singularities of the moduli spaces.
(ii) Let X be a minimal surface of general type with $p_g = 0$. Study the birational geometry of the moduli spaces. In particular, compute the Kodaira dimension.

For other problems, we pick up 3 problems.

Problem* 11. Let E be a stable sheaf and M the moduli of stable sheaves containing E. Let $\theta : K(X) \to \mathbb{Z}$ be the homomorphism such that $\theta(F) = \chi(E \otimes F)$. If θ is surjective, then M is a fine moduli space, that is, there is a universal family. Is the surjectivity necessary?

Problem 12.** Compute the topological invariants (e.g. the Betti numbers) of the moduli spaces $M_H(v)$ for $\Delta \gg 0$. In particular, show that $b_1(M_H(v)) = 2b_1(X)$ and $b_2(M_H(v)) = b_2(X) + 1 + \binom{2b_1(X)}{2}$ for $\Delta \gg 0$.

Remark 3.7. The claim are known for $r = 2$ by J. Li ([Li2]). If these assertions are correct, then Problem* 11 has an affermative answer.

Problem 13.** Study the holomorphic Euler characteristic of line bundles on the moduli spaces.

This problem is related to LePotier's strange duality conjecture, and also the Donaldson type invariant of X [GNY].

3.2. **Moduli spaces of stable sheaves on an abelian or a $K3$ surface.** Let X be a $K3$ surface or an abelian surface defined over $\mathbb{C}$. We define a lattice structure $\langle\ ,\ \rangle$ on $H^{ev}(X,\mathbb{Z}) := \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$ by

$$
\begin{aligned}
\langle x, y\rangle &:= -\int_X x^\vee \cup y \\
&= \int_X (x_1 \cup y_1 - x_0 \cup y_2 - x_2 \cup y_0),
\end{aligned}
\tag{3.2}
$$

where $x_i \in H^{2i}(X,\mathbb{Z})$ (resp. $y_i \in H^{2i}(X,\mathbb{Z})$) is the $2i$-th component of x (resp. y) and $x^\vee = x_0 - x_1 + x_2$. It is now called the *Mukai lattice.* Mukai lattice has a weight-2 Hodge structure such that the (p,q)-part is $\bigoplus_i H^{p+i,q+i}(X)$. For a coherent sheaf E on X,

$$
\begin{aligned}
v(E) &:= \mathrm{ch}(E)\sqrt{\mathrm{td}_X} \\
&= \mathrm{rk}(E) + c_1(E) + (\chi(E) - \epsilon\,\mathrm{rk}(E))\varrho_X \in H^{ev}(X,\mathbb{Z})
\end{aligned}
\tag{3.3}
$$

is called the *Mukai vector* of E, where $\epsilon = 0, 1$ according as X is an abelian surface or a $K3$ surface and ϱ_X is the fundamental class of X. Since the Mukai vector determine the underlying topological structure of E, we use the Mukai vector as the topological invariant v of $M_H(v)$.

Problem* 14 (Duality of $K3$ surfaces)**.** Let (X, H) be a polarized $K3$ surface. Let Y be a $K3$ surface which is a fine moduli of μ-stable vector bundles on X. Then there is a natural polarization on Y. Let $\mathcal{E}$ be a universal family. Show the μ-stability of $\mathcal{E}_{|Y\times\{x\}}$ by a differential geometric way. This will be a conceptual proof.

Remark 3.8. There is an algebraic proof by using the theory of Fourier-Mukai transforms. This method also works for the moduli of stable sheaves, but is not so natural.

Problem 15.** Describe a general member of the moduli space for the following cases.

(i) X is an abelian surface.
  (a) The Mukai vector v is not written as $v = v_0 \pm n v_1$ where $\langle v_0^2\rangle = \langle v_1^2\rangle = 0$ and $\langle v_0, v_1\rangle = \pm 1$.
  (b) The Mukai vector v is written as $v = v_0 \pm n v_1$, $\langle v_0^2\rangle = \langle v_1^2\rangle = 0$ and $\langle v_0, v_1\rangle = \pm 1$, but $\rho(X) \geq 2$.

(ii) X is a $K3$ surface.
  (a) The Mukai vector v is not written as $v = v_0 \pm n v_1$ where $\langle v_0^2\rangle = -2$, $\langle v_1^2\rangle = 0$ and $\langle v_0, v_1\rangle = \pm 1$.
  (b) The Mukai vector v is written as $v = v_0 \pm n v_1$, $\langle v_0^2\rangle = -2$, $\langle v_1^2\rangle = 0$ and $\langle v_0, v_1\rangle = \pm 1$.

(1) For the case (b), the choice of (v_0, v_1) is not unique. Since $\langle v_1^2\rangle = 0$ and $\langle v_0, v_1\rangle = \pm 1$, $Y := M_H(v_1)$ is a surface and has a universal family. Hence we have a Fourier-Mukai transform $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$. Then it is expected that for a special choice of (v_0, v_1), Φ induces a birational correspondence from $M_H(v)$ to the moduli of rank 1 sheaves on Y. From this correspondence, we will get a description of a general member of the moduli spaces (cf. Remark 3.10). If X is an abelian surface,

then Orlov proved that every Fourier-Mukai transform is induced by the moduli of stable sheaves. In particular, it is determined by the pair (v_0, v_1). So the remaining problem is to choose the pair (v_0, v_1).

On the other hand, if X is a $K3$ surface, then we don't have a classification of the Fourier-Mukai transforms. In particular, the Fourier-Mukai transform is not determined by the pair (v_0, v_1).

Example 3.9. (i) We note that $M_H(v_1)$ depends on the choice of H. So there are many Fourier-Mukai transforms associated to v_1, if $\rho(X) \geq 2$.

(ii) Let $\mathcal{E}$ be the universal family on $X \times M_H(v_1)$. In general $\Phi(E)$, $E \in M_H(v_0)$ is not a sheaf up to shift functor. Then the family of complexes $\{\Phi(E) | E \in M_H(v_1)\}$ gives a Fourier-Mukai transform which does not comes from the moduli of stable sheaves.

(iii) Let C be a smooth (-2)-curve on X. Then the complex

$$\mathcal{E} := \mathrm{Cone}(\mathcal{O}_C(a) \boxtimes \mathcal{O}_C(a)^{\vee} \to \mathcal{O}_{\Delta})$$

gives a Fourier-Mukai transform, where $\mathcal{O}_C(a)^{\vee}$ is the dual of $\mathcal{O}_C(a)$ in $\mathbf{D}(X)$.

Problem 16.** Let X be a $K3$ surface. Assume that two Mukai vectors $v \in H^{ev}(X, \mathbb{Z})$ and $\pm w \in H^{ev}(Y, \mathbb{Z})$ are related by a Fourier-Mukai transform $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$. Is there a Fourier-Mukai transform $\Phi' : \mathbf{D}(X) \to \mathbf{D}(Y)$ such that $\Phi'(v) = \pm w$ and Φ' induces a birational map $M_H(v) \cdots \to M_{H'}(w)$, where H and H' are general ample divisors on X and Y.

Problem 17.** Classify Fourier-Mukai transforms on $K3$ surfaces.

An explicit construction of Φ will give (ii) (b). For $\Phi = \mathrm{id}_X$, Problem** 16 is reduced to the following problem.

Problem* 18. Does the birational type of $M_H(v)$ depend on a general H?

Remark 3.10. For an abelian surface, a similar problem to Problem** 16 was proved in [Y2]. Moreover if $\mathrm{NS}(X) = \mathbb{Z}$, then (i) (b) was treated in [YY]. As a consequence, we described a general member of the moduli space in terms of projectively flat bundles (that is, semi-homogeneous vector bundles). Since projectively flat bundles are most fundamental and also simple vector bundles, our description is a good one.

For related problems to Problem** 17, we pick up 3 problems.

Problem* 19. Assume that X is a $K3$ surface. Construct many examples of Fourier-Mukai transforms whose kernel are not sheaves, and study their properties.

Problem 20.** Assume that X is a $K3$ surface. Introduce a stability condition on complexes, and construct the moduli space as a projective scheme.

Problem* 21. Assume that X is a $K3$ surface. Find a nice condition to preserve the stability of $E \in M_H(v_0)$.

Remark 3.11. Bridgeland introduced stability conditions on the objects of $\mathbf{D}(X)$. Inaba [In] introduced a stability condition which has a projective moduli. So it is interesting to find a non-trivial example of Inaba's stability condition.

(2) A more difficult but interesting case is (a). In this case, the technique of the Fourier-Mukai transforms is not sufficient and need other ideas.

4. Related problems.

4.1. **The Chow group.**

Theorem 4.1 (Beauville-Voisin [BV]). *Let X be a $K3$ surface. Let R_X be a subgroup of $\mathrm{CH}^*(X)$ generated by $e^D, D \in \mathrm{NS}(X)$ and ϱ_X is a point class lying on a rational curve. Then $R_X = \mathbb{Z} \oplus \mathrm{NS}(X) \oplus \mathbb{Z}\varrho_X$.*

Since vector bundles on projective surfaces are related to codimension 2 subsets, we can study the Chow group by vector bundles. Huybrechts proved the following interesting result.

Theorem 4.2 (Huybrechts [H]). *Let $\Phi : \mathbf{D}(X) \to \mathbf{D}(Y)$ be a Fourier-Mukai transform. Then $\Phi(R_X) = R_Y$ if $\rho(X) \geq 2$.*

Problem 22. (i) $\Phi(R_X) = R_Y$ for all X?
(ii) Let E be a rigid and simple vector bundle on X. Does $\mathrm{ch}(E)$ belong to R_X?

Huybrechts showed that (ii) implies (i).

4.2. **Twisted sheaves.** Let $\pi : Y \to X$ be a projective bundle over X. Then there is an analytic open covering $X = \cup_i U_i$ such that $Y_{|\pi^{-1}(U_i)} \cong \mathbb{P}(E_i)$, where E_i are locally free sheaves on U_i. We may assume that there are isomorphisms $\phi_{ij} : E_{i|U_i \cap U_j} \cong E_{j|U_i \cap U_j}$. In general $E := (\{E_i\}, \{\phi_{ij}\})$ does not satisfy the patching condition, but satisfy $\phi_{ki}\phi_{jk}\phi_{ij} = \alpha_{ijk}\,\mathrm{id}_{E_{i|U_i \cap U_j \cap U_k}}$, where $\alpha := \{\alpha_{ijk}\}$ is a 2-cocycle of $\mathcal{O}_X^\times$. For a covering $\{U_i\}$ and a 2-cocycle $\alpha := \{\alpha_{ijk}\}$, we call $E := (\{E_i\}, \{\phi_{ij}\})$ the α-twisted sheaf. We can define Gieseker's stability for α-twisted sheaves and constructed their moduli spaces [Y1]. Almost all problems in section 3 are generalized to these cases. Let E be a topological vector bundle on a $K3$ surface with $\langle v(E)^2 \rangle \geq -2$. In order to have a holomorphic structure, $c_1(E)$ is of type $(1,1)$. On the other hand, the associated projective bundle $\mathbb{P}(E)$ always has a holomorphic structure. Even if E has a holomorphic structure, under a deformation of X, E does not always deform to a holomorphic vector bundle. On the other hand, we have a holomorphic deformation of $\mathbb{P}(E)$, if $H^0(E^\vee \otimes E) = \mathbb{C}$. This is a benefit to consider projective bundles or twisted sheaves.

References

[ABT] Aprodu, M., Brînzănescu, V., Toma, M., *Holomorphic vector bundles on primary Kodaira surfaces,* Math. Z. **242** (2002), 63–73

[A-H] Atiyah, M. F., Hirzebruch, F., *Analytic cycles on complex manifolds,* Topology **1** (1962), 25–45

[B-L] Bănică, C., Le Potier, J., *Sur l'existence des fibrés vectoriels holomorphes sur les surfaces non-algébriques,* J. Reine Angew. Math. **378** (1987), 1–31

[B-P] Bănică, C., Putinar, M., *On complex vector bundles on projective threefolds,* Invent. Math. **88** (1987), no. 2, 427–438

[BV] Beauville, A., Voisin, C., *On the Chow ring of a $K3$ surface,* J. Algebraic Geom. **13** (2004), no. 3, 417–426

[Br4] Bridgeland, T., *Stability conditions on K3 surfaces,* math.AG/0307164, Duke Math. J. **141** (2008), 241–291

[BrMa] Bridgeland, T., Maciocia, A., *Fourier-Mukai transforms for $K3$ and elliptic fibrations,* J. Algebraic Geom. **11** (2002), no. 4, 629–657

[Br] Brînzănescu, V., *Holomorphic vector bundles over compact complex surfaces,* Lecture Notes in Mathematics, 1624. Springer-Verlag, Berlin, 1996. x+170 pp.

[BM1] Brînzănescu, V., Moraru, R., *Stable bundles on non-Kähler elliptic surfaces,* Comm. Math. Phys. **254** (2005), no. 3, 565–580

[BM2] Brînzănescu, V., Moraru, R., *Holomorphic rank-2 vector bundles on non-Kähler elliptic surfaces,* Ann. Inst. Fourier (Grenoble) **55** (2005), no. 5, 1659–1683

[BM3] Brînzănescu, V., Moraru, R., *Twisted Fourier-Mukai transforms and bundles on non-Kähler elliptic surfaces,* Math. Res. Lett. **13** (2006), no. 4, 501–514

[GNY] Göttsche, L., Nakajima, H., Yoshioka, K., *K-theoretic Donaldson invariants via instanton counting*, Pure and Applied Math. Quarterly, **5** (2009), 1029–1110; `arXiv:math/0611945`.

[H] Huybrechts, D., *Chow groups of K3 surfaces and spherical objects,* to appear in Journal EMS arXiv:0809.2606v1

[H-Sc] Huybrechts, D., Schröer, S., *The Brauer group of analytic $K3$ surfaces,* Int. Math. Res. Not. (2003), no. 50, 2687–2698

[H-St] Huybrechts, D., Stellari, P., *Equivalences of twisted $K3$ surfaces,* math.AG/0409030

[In] Inaba, M., *Moduli of stable objects in a triangulated category,* arXiv:math/0612078

[KY] Kurihara, K., and Yoshioka, K., *Holomorphic vector bundles on non-algebraic tori of dimension 2,* manuscripta mathematica, **126** (2008), 143–166

[Li1] Li, J., *Kodaira dimension of moduli space of vector bundles on surfaces,* Invent. Math. **115** (1994), no. 1, 1–40.

[Li2] Li, J., *The first two Betti numbers of the moduli spaces of vector bundles on surfaces,* Comm. Anal. Geom. **5** (1997), no. 4, 625–684

[Ma] Maruyama, M., *Moduli of stable sheaves II,* J. Math. Kyoto Univ. **18** (1978), pp. 557–614

[Mo] Mochizuki, T., *Donaldson type invariants for algebraic surfaces. Transition of moduli stacks*, Lecture Notes in Mathematics, **1972**. Springer-Verlag, Berlin, 2009. xxiv+383 pp.

[Mu1] Mukai, S., *Semi-homogeneous vector bundles on an Abelian variety,* J. Math. Kyoto Univ. **18** (1978), 239–272

[Mu2] Mukai, S., *Duality between $D(X)$ and $D(\hat{X})$ with its application to Picard sheaves,* Nagoya Math. J., **81** (1981), 153–175

[Mu3] Mukai, S., *Symplectic structure of the moduli space of sheaves on an abelian or K3 surface,* Invent. math. **77** (1984), 101–116

[Mu4] Mukai, S., *On the moduli space of bundles on $K3$ surfaces. I,* in "Vector bundles on algebraic varieties (Bombay, 1984)", 341–413, Tata Inst. Fund. Res., Bombay, 1987.

[Sch] Schuster, H.-W., *Locally free resolutions of coherent sheaves on surfaces,* J. Reine Angew. Math **337** (1982), 159–165

[SV] Schröer, S., Vezzosi, G., *Existence of vector bundles and global resolutions for singular surfaces,* Compos. Math. **140** (2004), no. 3, 717–728

[Sw] Schwarzenberger, R. L. E., *Vector bundles on algebraic surfaces,* Proc. London Math. Soc. (3) **11** (1961) 601–622

[T1] Toma, M., *Stable bundles on non-algebraic surfaces giving rise to compact moduli spaces,* C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), no. 5, 501–505

[T2] Toma, M., *Stable bundles with small c_2 over 2-dimensional complex tori,* Math. Z. **232** (1999), 511–525

[T-T] Teleman, A., Toma, M., *Holomorphic vector bundles on non-algebraic surfaces,* C. R. Math. Acad. Sci. Paris **334** (2002), no. 5, 383–388

[Y1] Yoshioka, K., *Moduli of twisted sheaves on a projective variety,* math.AG/0411538, Adv. Stud. Pure Math. **45** (2006), 1–30

[YY] Yanagida, S., Yoshioka, K., *Semi-homogeneous sheaves, Fourier-Mukai transforms and moduli of stable sheaves on abelian surfaces,* preprint

[Y2] K. Yoshioka: *Fourier-Mukai transform on abelian surfaces,* Math. Ann. **345** (2009), 493–524

[V] Voisin, C., *A counterexample to the Hodge conjecture extended to Kähler varieties,* Int. Math. Res. Not. 2002, no. 20, 1057–1075

Department of Mathematics, Faculty of Science, Kobe University, Kobe, 657, Japan

E-mail address: yoshioka@math.kobe-u.ac.jp