On the spaces of equivariant maps between real algebraic varieties

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概要

Recently the author notices that the stability dimension obtained in [1] and [12] can be improved by using the truncated simplicial resolutions invented by J. Mostovoy [15]. The purpose of this note is to announce these improvements.

1 Introduction.

We consider the homotopy types of spaces of algebraic (rational) maps from real projective space $\mathbb{R}P^m$ into the complex projective space $\mathbb{C}P^m$ for $2 \leq m \leq 2n$. It is known in [1] that the inclusion of the space of rational (or regular) maps into the space of all continuous maps is a homotopy equivalence. These results combined with those of [1] can be formulated as a single statement about $\mathbb{Z}/2$-equivariant homotopy equivalence between these spaces, where the $\mathbb{Z}/2$-action is induced by the complex conjugation. This is also one of the generalizations of a theorem of [9], and it is already published in [12]. Recently the author notices that the stability dimensions given in [1] and [12] can be improved by using the truncated simplicial resolutions invented by J. Mostovoy [15]. In this note we shall announce about these improvements (cf. [2]).
1.1 Definitions and notations.

Let $\mathbb{K}$ denote one of the fields $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers and let $d(\mathbb{K}) = \dim_{\mathbb{R}} \mathbb{K} = 1$ if $\mathbb{K} = \mathbb{R}$ and $2$ if $\mathbb{K} = \mathbb{C}$. Let $m$ and $n$ be positive integers such that $1 \leq m < d(\mathbb{K}) \cdot (n + 1) - 1$. We choose $e_{m}^{\mathbb{K}} = [1 : 0 : \cdots : 0] \in \mathbb{K}P^{m}$ as the base point of $\mathbb{K}P^{n}$. For $d(\mathbb{K}) \leq m < d(\mathbb{K}) \cdot (n + 1) - 1$, we denote by $\text{Map}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$ the space consisting of all based maps $f : (\mathbb{R}P^{m}, e_{m}^{\mathbb{R}}) \rightarrow (\mathbb{K}P^{n}, e_{n}^{\mathbb{K}})$, and by $\text{Map}_{\epsilon}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$, where $\epsilon \in \mathbb{Z}/2 = \{0, 1\} = \pi_{0}(\text{Map}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n}))$, the corresponding path component of $\text{Map}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$. Similarly, let $\text{Map}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$ denote the space of all free maps $f : \mathbb{R}P^{m} \rightarrow \mathbb{K}P^{n}$ and $\text{Map}_{\epsilon}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$ the corresponding path component of $\text{Map}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$.

We shall use the symbols $z_{i}$ when we refer to complex valued coordinates or variables or when we refer to complex and real valued ones at the same time while the notation $x_{i}$ will be restricted to the purely real case.

A map $f : \mathbb{R}P^{m} \rightarrow \mathbb{K}P^{n}$ is called a algebraic map of the degree $d$ if it can be represented as a rational map of the form $f = [f_{0} : \cdots : f_{n}]$ such that $f_{0}, \cdots, f_{n} \in \mathbb{K}[z_{0}, \cdots, z_{m}]$ are homogeneous polynomials of the same degree $d$ with no common real roots except $0_{m+1} = (0, \cdots, 0) \in \mathbb{R}^{m+1}$.

We denote by $\text{Alg}_{d}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$ (resp. $\text{Alg}_{d}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$) the space consisting of all (resp. based) algebraic maps $f : \mathbb{R}P^{m} \rightarrow \mathbb{K}P^{n}$ of degree $d$. It is easy to see that there are inclusions $\text{Alg}_{d}(\mathbb{R}P^{m}, \mathbb{K}P^{n}) \subset \text{Map}_{[d]_{2}}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$ and $\text{Alg}_{d}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n}) \subset \text{Map}_{[d]_{2}}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n})$, where $[d]_{2} \in \mathbb{Z}/2 = \{0, 1\}$ denotes the integer $d$ mod 2. Let $A_{d}(m, n)(\mathbb{K})$ denote the space consisting of all $(n + 1)$-tuples $(f_{0}, \cdots, f_{n}) \in \mathbb{K}[z_{0}, \cdots, z_{m}]^{n+1}$ of homogeneous polynomials of degree $d$ with coefficients in $\mathbb{K}$ and without non-trivial common real roots (but possibly with non-trivial common complex ones).

Let $A^{\mathbb{K}}_{d}(m, n) \subset A_{d}(m, n)(\mathbb{K})$ be the subspace consisting of $(n + 1)$-tuples $(f_{0}, \cdots, f_{n}) \in A_{d}(m, n)(\mathbb{K})$ such that the coefficient of $z_{0}^{d}$ in $f_{0}$ is 1 and 0 in the other $f_{k}$'s ($k \neq 0$). Then there is a natural surjective projection map

$$\Psi^{\mathbb{K}}_{d} : A^{\mathbb{K}}_{d}(m, n) \rightarrow \text{Alg}_{d}^{*}(\mathbb{R}P^{m}, \mathbb{K}P^{n}).$$

For $m \geq 2$ and $g \in \text{Alg}_{d}^{*}(\mathbb{R}P^{m-1}, \mathbb{K}P^{n})$ a fixed algebraic map, we denote
by $\text{Alg}_d^K(m, n; g)$ and $F(m, n; g)$ the spaces defined by
\[
\begin{align*}
\text{Alg}_d^K(m, n; g) &= \{ f \in \text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n) : f|\mathbb{R}P^{m-1} = g \}, \\
F^K(m, n; g) &= \{ f \in \text{Map}_{[d_2]}^*(\mathbb{R}P^m, \mathbb{K}P^n) : f|\mathbb{R}P^{m-1} = g \}.
\end{align*}
\]
Note that there is a homotopy equivalence $F^K(m, n; g) \simeq \Omega^m \mathbb{K}P^n$. Let $A_d^K(m, n; g) \subset A_d^K(m, n)$ denote the subspace given by
\[
A_d^K(m, n; g) = (\Psi_d^K)^{-1}(\text{Alg}_d^K(m, n; g)).
\]
Observe that if an algebraic map $f \in \text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n)$ can be represented as $f = [f_0 : \cdots : f_n]$ for some $(f_0, \cdots, f_n) \in A_d^K(m, n)$ then the same map can also be represented as $f = [\tilde{g}_m f_0 : \cdots : \tilde{g}_m f_n]$, where $\tilde{g}_m = \sum_{k=0}^{m} z_k^2$. So there is an inclusion
\[
\text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n) \subset \text{Alg}_{d+2}^*(\mathbb{R}P^m, \mathbb{K}P^n)
\]
and we can define the stabilization map $s_d : A_d^K(m, n) \to A_{d+2}^K(m, n)$ by $s_d(f_0, \cdots, f_n) = (\tilde{g}_m f_0, \cdots, \tilde{g}_m f_n)$. It is easy to see that there is a commutative diagram
\[
\begin{array}{ccc}
A_d^K(m, n) & \xrightarrow{s_d} & A_{d+2}^K(m, n) \\
\downarrow\Psi^K_d & & \downarrow\Psi^K_{d+2} \\
\text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n) & \subset & \text{Alg}_{d+2}^*(\mathbb{R}P^m, \mathbb{K}P^n)
\end{array}
\]
A map $f \in \text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n)$ is called an algebraic map of minimal degree $d$ if $f \in \text{Alg}_d^*(\mathbb{R}P^m, \mathbb{K}P^n) \setminus \text{Alg}_{d-2}^*(\mathbb{R}P^m, \mathbb{K}P^n)$. It is easy to see that if $g \in \text{Alg}_d^*(\mathbb{R}P^{m-1}, \mathbb{K}P^n)$ is an algebraic map of minimal degree $d$, then the restriction
\[
\Psi^K_d|A_d^K(m, n; g) : A_d^K(m, n; g) \xrightarrow{\cong} \text{Alg}_d^K(m, n; g)
\]
is a homeomorphism. Let
\[
\begin{align*}
i_{d, K} : \text{Alg}_{d}^*(\mathbb{R}P^m, \mathbb{K}P^n) &\hookrightarrow \text{Map}_{[d_2]}^*(\mathbb{R}P^m, \mathbb{K}P^n), \\
i_{d, K}' : \text{Alg}_d^K(m, n; g) &\hookrightarrow F(m, n; g) \simeq \Omega^m \mathbb{K}P^n
\end{align*}
\]
denote the inclusions and let
\[
i_d^K = i_{d, K} \circ \Psi_d^K : A_d^K(m, n) \to \text{Map}_{[d_2]}^*(\mathbb{R}P^m, \mathbb{K}P^n).
\]
be the natural projection.
1.2 The case $m = 1$.

First, recall the following old result for the case $m = 1$.

**Theorem 1.1** ([10], [20] (cf. [13])). Let $n \geq 2$ and $d \geq 1$ be integers.

(i) If $\mathbb{K} = \mathbb{R}$ and $m = 1$, the map $i_d^\mathbb{R} : A_d^\mathbb{R}(1, n) \to \text{Map}_{d+2}^\mathbb{R}(\mathbb{R}P^1, \mathbb{R}P^n) \simeq \Omega S^n$ is a homotopy equivalence up to dimension $D_1(d, n)$, where $D_1(d, n)$ denotes the integer given by $D_1(d, n) = (d + 1)(n - 1) - 1$.

Moreover, if $n \geq 3$ or $n = 2$ with $d \equiv 1 \pmod{2}$, there is a homotopy equivalence $A_d^\mathbb{R} \simeq J_d(\Omega S^n)$, where $J_d(\Omega S^n)$ denotes the d-th stage James filtration of $\Omega S^n$ given by

$$J_d(\Omega S^n) = S^{n-1} \cup e^{2(n-1)} \cup e^{3(n-1)} \cup \cdots \cup e^{d(n-1)} \subset \Omega S^n.$$

(ii) If $\mathbb{K} = \mathbb{C}$ and $m = 1$, the map $i_d^\mathbb{C} : A_d^\mathbb{C}(1, n) \to \Omega S^{2n+1}$ is a homotopy equivalence up to dimension $D_1(d, 2n+1) = 2n(d + 1) - 1$ and there is a homotopy equivalence $A_d^\mathbb{C}(1, n) \simeq J_d(\Omega S^{2n+1})$.

**Remark.** (i) A map $f : X \to Y$ is called a homotopy (resp. a homology) equivalence up to dimension $D$ if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$. Similarly, it is called a homotopy (resp. a homology) equivalence through dimension $D$ if $f_* : \pi_k(X) \to \pi_k(Y)$ (resp. $f_* : H_k(X, \mathbb{Z}) \to H_k(Y, \mathbb{Z})$) is an isomorphism for any $k \leq D$.

(ii) Let $G$ be a finite group and let $f : X \to Y$ be a $G$-equivariant map. Then a map $f : X \to Y$ is called a $G$-equivariant homotopy (resp. homology) equivalence up to dimension $D$ if for each subgroup $H \subset G$ the induced homomorphism $f_*^H : \pi_k(X^H) \to \pi_k(Y^H)$ (resp. $f_*^H : H_k(X^H, \mathbb{Z}) \to H_k(Y^H, \mathbb{Z})$) is an isomorphism for any $k < D$ and an epimorphism for $k = D$.

Similarly, it is called a $G$-equivariant homotopy (resp. homology) equivalence through dimension $D$ if for each subgroup $H \subset G$ the induced homomorphism $f_*^H : \pi_k(X^H) \to \pi_k(Y^H)$ (resp. $f_*^H : H_k(X^H, \mathbb{Z}) \to H_k(Y^H, \mathbb{Z})$) is an isomorphism for any $k \leq D$. 

The complex conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on $A_d^\mathbb{C}(m, n)$ and $S^{2n+1}$, where we identify $S^{2n+1}$ with the space

$$S^{2n+1} = \{(w_0, \ldots, w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^n |w_k|^2 = 1\}.$$ 

It is easy to see that $A_d^\mathbb{C}(m, n)^{\mathbb{Z}/2} = A_d^\mathbb{R}(m, n)$ and $(i_d^\mathbb{C})^{\mathbb{Z}/2} = i_d^\mathbb{R}$. Hence, we also have:

**Corollary 1.2 ([10]).** If $n \geq 2$ and $d \geq 1$ are integers, the map $i_d^\mathbb{C} : A_d^\mathbb{C}(1, n) \to \Omega S^{2n+1}$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence up to dimension $D_1(d, n)$.

## 2 The case $m \geq 2$.

### 2.1 The improvements of the stability dimensions.

For a space $X$, let $F(X, r)$ denote the configuration space of distinct $r$ points in $X$ given by $F(X, r) = \{(x_1, \ldots, x_r) \in X^r : x_i \neq x_j$ if $i \neq j\}$. The symmetric group $S_r$ of $r$ letters acts on $F(X, r)$ freely by permuting coordinates. Let $C_r(X)$ be the configuration space of unordered $r$-distinct points in $X$ given by the orbit space $C_r(X) = F(X, r)/S_r$.

It is known ([8], [18]) that there are the stable homotopy equivalence and the isomorphism of abelian groups

$$\begin{cases} 
\Omega^m S^{m+l} \simeq_s \bigvee_{r=1}^{\infty} D_r(\mathbb{R}^m; S^l) \quad (\text{stable homotopy equivalence}) \\
H_k(D_r(\mathbb{R}^m, S^l), \mathbb{Z}) \cong H_{k-rl}(C_r(\mathbb{R}^m), (\pm \mathbb{Z})^{\otimes r}) \quad (k, l \geq 1),
\end{cases}$$

where we set $\wedge^r X = X \wedge \cdots \wedge X$ ($r$ times), $X_+ = X \cup \{*\}$ ($*$ is the disjoint base point), and $D_r(\mathbb{R}^m, S^l) = F(\mathbb{R}^m, r)_+ \wedge_{S_r} (\wedge^r S^l)$.

Let $G^M_{m,N;k}$ and $D_K(d; m, n)$ be the abelian group and the positive in-
teger defined by

$$G_{m,N;k}^{M} = \bigoplus_{r=1}^{M} H_{k-(N-m)r}(C_{r}(\mathbb{R}^{m}), (\pm \mathbb{Z})^{\otimes(N-m)}),$$

where $\lfloor x \rfloor$ denotes the integer part of a real number $x$. Note that there is an isomorphism $H_{k}(\Omega^{m}S^{m+l}, \mathbb{Z}) \cong G_{m,m+l;k}^{\infty}$ for any $k \geq 1$.

Then we have the following results.

**Theorem 2.1** (cf. [1]). Let $2 \leq m < n$ and let $g \in \text{Alg}_{d}(\mathbb{R}P^{m-1}, \mathbb{R}P^{n})$ be an algebraic map of minimal degree $d$.

(i) The inclusion $i_{d,\mathbb{R}}^{'} : \text{Alg}_{d}^{\mathbb{R}}(m, n; g) \rightarrow F^{\mathbb{R}}(m, n; g) \simeq \Omega^{m}S^{n}$ is a homotopy equivalence through dimension $D_{\mathbb{R}}(d; m, n)$ if $m+2 \leq n$ and a homology equivalence through dimension $D_{\mathbb{R}}(d; m, n)$ if $m+1 = n$.

(ii) For any $k \geq 1$, $H_{k}(\text{Alg}_{d}^{\mathbb{R}}(m, n; g), \mathbb{Z})$ contains the subgroup $G_{m,n;k}^{d}$ as a direct summand. Moreover, the induced homomorphism $i_{d,\mathbb{R}}^{'}: H_{k}(\text{Alg}_{d}^{\mathbb{R}}(m, n; g), \mathbb{Z}) \rightarrow H_{k}(\Omega^{m}S^{n}, \mathbb{Z})$ is an epimorphism for any $k \leq (n-m)(d+1)-1$.

**Theorem 2.2** (cf. [1]). If $2 \leq m < n$ are positive integers,

$$i_{d}^{\mathbb{R}} : A_{d}^{\mathbb{R}}(m, n) \rightarrow \text{Map}_{[d]_{2}}^{*}(\mathbb{R}P^{m}, \mathbb{R}P^{n})$$

is a homotopy equivalence through dimension $D_{\mathbb{R}}(d; m, n)$ if $m+2 \leq n$ and a homology equivalence through dimension $D_{\mathbb{R}}(d; m, n)$ if $m+1 = n$.

**Theorem 2.3** (cf. [12]). Let $2 \leq m \leq 2n$, and let $g \in \text{Alg}_{d}(\mathbb{R}P^{m-1}, \mathbb{C}P^{n})$ be an algebraic map of minimal degree $d$.

(i) The inclusion $i_{d,\mathbb{C}}^{'} : \text{Alg}_{d}^{\mathbb{C}}(m, n; g) \rightarrow F^{\mathbb{C}}(m, n; g) \simeq \Omega^{m}S^{2n+1}$ is a homotopy equivalence through dimension $D_{\mathbb{C}}(d; m, n)$ if $m < 2n$ and a homology equivalence through dimension $D_{\mathbb{C}}(d; m, n)$ if $m = 2n$. 
(ii) For any $k \geq 1$, $H_k(Alg^C_d(m, n; g), \mathbb{Z})$ contains the subgroup $G^d_{m,2n+1;k}$ as a direct summand. Moreover, the induced homomorphism $i'_{d,C_*} : H_k(Alg^C_d(m, n; g), \mathbb{Z}) \to H_k(\Omega^m S^{2n+1}, \mathbb{Z})$ is an epimorphism for any $k \leq (2n - m + 1)(d + 1) - 1$.

**Theorem 2.4** (cf. [12]). If $2 \leq m \leq 2n$ are positive integers,

$$i^C_d : A^C_d(m, n) \to \text{Map}^{*}_{[d]_2} (\mathbb{R}P^m, \mathbb{C}P^n)$$

is a homotopy equivalence through dimension $D_{\mathbb{C}}(d;m,n)$ if $m < 2n$ and a homology equivalence through dimension $D_{\mathbb{C}}(d;m,n)$ if $m = 2n$.

Note that the complex conjugation on $\mathbb{C}$ naturally induces $\mathbb{Z}/2$-actions on the spaces $Alg^C_d(m, n; g)$ and $A^C_d(m, n)$ as before. In the same way it also induces a $\mathbb{Z}/2$-action on $\mathbb{C}P^n$ and this action extends to actions on the spaces $\text{Map}^*(\mathbb{R}P^m, S^{2n+1})$ and $\text{Map}^*(\mathbb{R}P^m, \mathbb{C}P^n)$, where we identify $S^{2n+1} = \{(w_0, \ldots, w_n) \in \mathbb{C}^{n+1} : \sum_{k=0}^{n} |w_k|^2 = 1\}$ and regard $\mathbb{R}P^m$ as a $\mathbb{Z}/2$-space with the trivial $\mathbb{Z}/2$-action.

**Corollary 2.5** (cf. [12]). Let $2 \leq m \leq 2n$, $d \geq 1$ be positive integers and $g \in Alg^C_d(\mathbb{R}P^{m-1}, \mathbb{C}P^n)$ be a fixed algebraic map of the minimal degree $d$.

(i) If $m < 2n$, the inclusion map $i'_{d,C} : Alg^C_d(m, n; g) \to F^C(m, n; g) \simeq \Omega^m S^{2n+1}$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D_{\mathbb{R}}(d;m,n)$.

(ii) If $m = 2n$, the above inclusion map $i'_{d,C}$ is and a $\mathbb{Z}/2$-equivariant homology equivalence through dimension $D_{\mathbb{R}}(d;m,n)$.

(iii) The map $i^C_d : A^C_d(m, n) \to \text{Map}^{*}_{[d]_2} (\mathbb{R}P^m, \mathbb{C}P^n)$ is a $\mathbb{Z}/2$-equivariant homotopy equivalence through dimension $D_{\mathbb{R}}(d;m,n)$ if $m < 2n$ and a $\mathbb{Z}/2$-equivariant homology equivalence through the same dimension $D_{\mathbb{R}}(d;m,n)$ if $m = 2n$.

### 2.2 Conjectures.

Finally we report several related questions.

**Conjecture 2.6.** Is the projection $\Psi^K_d : A^K_d(m, n) \to \text{Alg}^*_{d}(\mathbb{R}P^m, \mathbb{K}P^n)$ a homotopy equivalence?
Let $\hat{D}_K(d;m,n)$ denote the integer given by

$$
\hat{D}_K(d;m,n) = \begin{cases} 
(n-m)(d+1) - 1 & \text{if } K = \mathbb{R}, \\
(2n-m+1)(d+1) - 1 & \text{if } K = \mathbb{C}.
\end{cases}
$$

**Conjecture 2.7.** Is the map $i_d^K : A_d^K(m,n) \to \text{Map}_{[d]}^*(\mathbb{R}P^m, \mathbb{K}P^n)$ a homotopy (or homology) equivalence up to dimension $\hat{D}_K(d;m,n)$?

**Remark.** The above conjectures are correct if $m = 1$.

参考文献


