THE COHOMOLOGY RING OF THE GKM GRAPH
OF A FLAG MANIFOLD

大阪市立大学 大学院理学研究科 福川由貴子 (Yukiko Fukukawa)
Department of Mathematics, Osaka City University

1. Introduction

Let $T$ be a torus of dimension $n$ and $M$ a closed smooth $T$-manifold. The equivariant cohomology of $M$, denoted $H^*_T(M)$, contains a lot of geometric information on $M$. Moreover it is often easier to compute $H^*_T(M)$ than $H^*(M)$ by virtue of the Localization Theorem which implies that the restriction map

\[ \iota^*: H^*_T(M) \to H^*_T(M^T) \]

(1.1)

to the $T$-fixed point set $M^T$ is often injective, in fact, this is the case when $H^{\text{odd}}(M) = 0$. When $M^T$ is isolated, $H^*_T(M^T) = \oplus_{p \in M^T} H^*_T(p)$ and hence $H^*_T(M^T)$ is a direct sum of copies of a polynomial ring in $n$ variables because $H^*_T(p) = H^*(BT)$.

Therefore we are in a nice situation when $H^{\text{odd}}(M) = 0$ and $M^T$ is isolated. Goresky-Kottwitz-MacPherson [2] (see also [3, Chapter 11]) found that under the further condition that the weights at a tangential $T$-module are pairwise linearly independent at each $p \in M^T$, the image of $\iota^*$ in (1.1) above is determined by the fixed point sets of codimension one subtori of $T$ when $\mathbb{Q}$ is tensored in cohomology. Their result motivated Guillemin-Zara [4] to associate a labeled graph $\mathcal{G}_M$ with $M$ and define the “cohomology” ring $\mathcal{H}^*(\mathcal{G}_M)$ of $\mathcal{G}_M$, which is a subring of $\oplus_{p \in M^T} H^*(BT)$. Then the result of Goresky-Kottwitz-MacPherson can be stated that $H^*_T(M) \otimes \mathbb{Q}$ is isomorphic to $\mathcal{H}^*(\mathcal{G}_M) \otimes \mathbb{Q}$ as graded rings when $M$ satisfies the conditions mentioned above.

The result of Goresky-Kottwitz-MacPherson can be applied to many important $T$-manifolds $M$ such as flag manifolds and compact smooth toric varieties etc. When $M$ is such a nice manifold, $H^*_T(M)$ is often known to be isomorphic to $\mathcal{H}^*(\mathcal{G}_M)$ without tensoring with $\mathbb{Q}$ (see [1], [5], [6] for example). We determine the ring structure of $\mathcal{H}^*(\mathcal{G}_M)$ or $\mathcal{H}^*(\mathcal{G}_M) \otimes \mathbb{Z}[rac{1}{2}]$ when $M$ is a flag manifold of classical type directly without using the fact
that $H^*_T(M)$ is isomorphic to $\mathcal{H}^*(G_M)$ ([7]). In my talk, I introduced the result when $M$ is a flag manifold of type A. This is a joint work with Hiroaki Ishida and Mikiya Masuda and the details can be found in [7].

2. Labeled graph and its cohomology for type $A_{n-1}$

Let $\{t_i\}_{i=1}^n$ be a basis of $H^2(BT)$, so that $H^*(BT)$ can be identified with a polynomial ring $\mathbb{Z}[t_1, t_2, \ldots, t_n]$. We take an inner product on $H^2(BT)$ such that the basis $\{t_i\}$ is orthonormal. Then

$$\Phi(A_{n-1}) := \{\pm(t_i - t_j) \mid 1 \leq i < j \leq n\}$$

is a root system of type $A_{n-1}$.

**Definition.** The labeled graph associated with $\Phi(A_{n-1})$, denoted $\mathcal{A}_n$, is a graph with labeling $\ell$ defined as follows.

- The vertex set of $\mathcal{A}_n$ is the permutation group $S_n$ on $\{1, 2, \ldots, n\}$.
- Two vertices $w, w'$ in $\mathcal{A}_n$ are connected by an edge $e_{w,w'}$ if and only if there is a transposition $(i, j) \in S_n$ such that $w' = w(i, j)$, in other words,

$$w'(i) = w(j), \quad w'(j) = w(i) \quad \text{and} \quad w'(r) = w(r) \quad \text{for} \quad r \neq i, j.$$

- The edge $e_{w,w'}$ is labeled by $\ell(e_{w,w'}) := t_{w(i)} - t_{w'(i)}$.

**Definition.** The cohomology ring of $\mathcal{A}_n$, denoted $\mathcal{H}^*(\mathcal{A}_n)$, is defined to be the subring of $\text{Map}(V(\mathcal{A}_n), H^*(BT)) = \bigoplus_{v \in V(\mathcal{A}_n)} H^*(BT)$, where $V(\mathcal{A}_n)$ denotes the set of vertices of $\mathcal{A}_n$, i.e. $V(\mathcal{A}_n) = S_n$, satisfying the following condition:

$$f \in \text{Map}(V(\mathcal{A}_n), H^*(BT)) \text{ is an element of } \mathcal{H}^*(\mathcal{A}_n) \text{ if and only if } f(v) - f(v') \text{ is divisible by } \ell(e) \text{ in } H^*(BT) \text{ whenever the vertices } v \text{ and } v' \text{ are connected by an edge } e \text{ in } \mathcal{A}_n.$$

For each $i = 1, \ldots, n$, we define elements $\tau_i, t_i$ of $\text{Map}(V(\mathcal{A}_n), H^*(BT))$ by

$$\tau_i(w) := t_{w(i)}, \quad t_i(w) := t_i \quad \text{for } w \in S_n.$$

In fact, both $\tau_i$ and $t_i$ are elements of $\mathcal{H}^2(\mathcal{A}_n)$.

**Example.** The case $n = 3$. The root system $\Phi(A_2)$ is $\{\pm(t_i - t_j) \mid 1 \leq i < j \leq 3\}$. The labeled graph $\mathcal{A}_3$ and $\tau_i$ for $i = 1, 2, 3$ are as follows.

![Labeled graph for $A_3$ and $\tau_i$](image-url)
Theorem 2.1. Let $\mathcal{A}_n$ be the labeled graph associated with the root system $\Phi(A_{n-1})$ of type $A_{n-1}$ in (2.1). Then

$$H^*(\mathcal{A}_n) = \mathbb{Z}[\tau_1, \ldots, \tau_n, t_1, \ldots, t_n]/(e_i(\tau) - e_i(t) \mid i = 1, \ldots, n),$$

where $e_i(\tau)$ (resp. $e_i(t)$) is the $i^{th}$ elementary symmetric polynomial in $\tau_1, \ldots, \tau_n$ (resp. $t_1, \ldots, t_n$).

To prove this theorem, we need the following two lemmas.

Lemma 2.2. $H^*(\mathcal{A}_n)$ is generated by $\tau_1, \ldots, \tau_n, t_1, \ldots, t_n$ as a ring.

Proof. We shall prove the lemma by induction on $n$. When $n = 1$, $H^*(\mathcal{A}_1)$ is generated by $t_1$ since $\mathcal{A}_1$ is a point; so the lemma holds.

Suppose that the lemma holds for $n - 1$. Then it suffices to show that any homogenous element $f$ of $H^*(\mathcal{A}_n)$, say of degree $2k$, can be expressed as a polynomial in $\tau_i$'s and $t_i$'s. For each $i = 1, \ldots, n$, we set

$$V_i := \{w \in S_n \mid w(i) = n\}$$

and consider the labeled full subgraph $\mathcal{L}_i$ of $\mathcal{A}_n$ with $V_i$ as the vertex set. Note that $\mathcal{L}_i$ can naturally be identified with $\mathcal{A}_{n-1}$ for any $i$.

Let

$$(2.3) \quad 1 \leq q \leq \min\{k + 1, n\}$$

and assume that

$$(2.4) \quad f(\nu) = 0 \quad \text{for any } \nu \in V_i \text{ whenever } i < q.$$ 

A vertex $w$ in $V_q$ is connected by an edge in $\mathcal{A}_n$ to a vertex $\nu$ in $V_i$ if and only if $\nu = w(i, q)$. In this case $f(w) - f(\nu)$ is divisible by $t_{w(i)} - t_{w(q)} = t_{w(i)} - t_n$ and $f(\nu) = 0$ whenever $i < q$ by (2.4), so $f(w)$ is divisible by $t_{w(i)} - t_n$ for $i < q$. Thus, for each $w \in V_q$, there is an element $g^q(w) \in \mathbb{Z}[t_1, \ldots, t_n]$ such that

$$(2.5) \quad f(w) = (t_{w(1)} - t_n)(t_{w(2)} - t_n) \cdots (t_{w(q-1)} - t_n)g^q(w)$$

where $g^q(w)$ is homogeneous and of degree $2(k + 1 - q)$ because $f(w)$ is homogeneous and of degree $2k$.

One expresses

$$(2.6) \quad g^q(w) = \sum_{r=0}^{k+1-q} g_r^q(w)t_n^r$$

with homogenous polynomials $g^q_r(w)$ of degree $2(k+1-q-r)$ in $\mathbb{Z}[t_1, \ldots, t_{n-1}]$. Then there is a polynomial $G^q_r$ in $\tau_i$'s (except $\tau_q$) and $t_i$'s (except $t_n$) such that $G^q_r(w) = g^q_r(w)$ for any $w \in V_q$, because $g_r^q$ restricted to $\mathcal{L}_q$ is an element of $H^*(\mathcal{L}_q) = H^*(\mathcal{A}_{n-1})$. 

Since \( \tau_i(w) = t_{w(i)} \) and \( w(i) = n \) for \( w \in V_i \), we have

\[
(2.7) \prod_{i=1}^{q-1} (\tau_i - t_n)(w) = 0 \quad \text{for any } w \in V_i \text{ whenever } i < q.
\]

Therefore, it follows from (2.5), (2.6), the Claim above and (2.7) that putting \( G^q = \sum_{r=0}^{k+1-q} G_{r}^{q} t_{n}^{r} \), we have

\[
(f - G^q \prod_{i=1}^{q-1} (\tau_i - t_n))(w) = f(w) - g^{q}(w) \prod_{i=1}^{q-1} (t_{w(i)} - t_n)
\]

\[
= 0 \quad \text{for any } w \in V_i \text{ whenever } i \leq q.
\]

Therefore, subtracting the polynomial \( G^q \prod_{i=1}^{q-1} (\tau_i - t_n) \) from \( f \), we may assume that

\[
f(v) = 0 \quad \text{for any } v \in V_i \text{ whenever } i < q + 1.
\]

The above argument implies that \( f \) finally takes zero on all vertices of \( \mathcal{A}_n \) (which means \( f = 0 \)) by subtracting a polynomial in \( \tau_i \)'s and \( t_i \)'s, and this completes the induction step. \( \square \)

We abbreviate the polynomial ring \( \mathbb{Z}[\tau_1, \ldots, \tau_n, t_1, \ldots, t_n] \) as \( \mathbb{Z}[\tau, t] \). The canonical map \( \mathbb{Z}[\tau, t] \to \mathcal{H}^\ast(\mathcal{A}_n) \) is a grade preserving homomorphism which is surjective by Lemma 2.2. Let \( e_i(\tau) \) (resp. \( e_i(t) \)) denote the \( i \)th elementary symmetric polynomial in \( \tau_1, \ldots, \tau_n \) (resp. \( t_1, \ldots, t_n \)). It easily follows from (2.2) that \( e_i(\tau) = e_i(t) \) for \( i = 1, \ldots, n \). Therefore the canonical map above induces a grade preserving epimorphism

\[
(2.8) \quad \mathbb{Z}[\tau, t]/(e_1(\tau) - e_1(t), \ldots, e_n(\tau) - e_n(t)) \to \mathcal{H}^\ast(\mathcal{A}_n).
\]

Remember that the Hilbert series of a graded ring \( A^\ast = \bigoplus_{j=0}^{\infty} A^j \), where \( A^j \) is the degree \( j \) part of \( A^\ast \) and of finite rank over \( \mathbb{Z} \), is a formal power series defined by

\[
F(A^\ast, s) := \sum_{j=0}^{\infty} \text{rank}_\mathbb{Z} A^j s^j.
\]

In order to prove that the epimorphism in (2.8) is an isomorphism, it suffices to verify the following lemma because the modules in (2.8) are both torsion free.

Lemma 2.3. The Hilbert series of the both sides at (2.8) coincide, in fact, they are given by \( \frac{1}{(1-s^2)^n} \prod_{i=1}^{n} (1 - s^{2i}) \).

Proof. (1) Calculation of LHS at (2.8). Let \( e_i := e_i(\tau) - e_i(t) \). It follows from the exact sequence

\[
0 \to (e_1, \ldots, e_n) \to \mathbb{Z}[\tau, t] \to \mathbb{Z}[\tau, t]/(e_1, \ldots, e_n) \to 0
\]
that we have

\[(2.9) \quad F(\mathbb{Z}[\tau, t]/(e_1, \ldots, e_n), s) = F(\mathbb{Z}[\tau, t], s) - F((e_1, \ldots, e_n), s).\]

Here, since \(\deg \tau_i = \deg t_i = 2\), we have

\[(2.10) \quad F(\mathbb{Z}[\tau, t], s) = \frac{1}{(1 - s^2)^{2n}}\]

as easily checked; so it suffices to calculate \(F((e_1, \ldots, e_n), s)\).

For \(I \subset [n]\) we set \(e_I := \prod_{i \in I} e_i\). Then it follows from the Inclusion-Exclusion principle that

\[(2.11) \quad F((e_1, \ldots, e_n), s) = \sum_{\emptyset \neq l \subset [n]} (-1)^{|l|-1} F((e_l), s)\]

and since \(F((e_l), s) = s^{\deg e_l}/(1 - s^2)^{2n}\) and \(\deg e_l = \sum_{i \in I} 2i\), it follows from (2.11) that

\[(2.12) \quad F((e_1, \ldots, e_n), s) = \sum_{\emptyset \neq I \subset [n]} (-1)^{|I|-1} \frac{s^{\sum_{j \in I} 2i}}{(1 - s^2)^{2n}}.\]

Therefore it follows from (2.9), (2.10) and (2.12) that

\[(2.13) \quad F(\mathbb{Z}[\tau, t]/(e_1, \ldots, e_n), s) = \frac{1}{(1 - s^2)^{2n}} - \sum_{\emptyset \neq l \subset [n]} (-1)^{|l|-1} \frac{s^{\sum_{j \in l} 2i}}{(1 - s^2)^{2n}}\]

(2) Calculation of RHS at (2.8). Let \(d_n(k) := \text{rank}_\mathbb{Z} \mathcal{H}^{2k}(\mathcal{A}_n)\). Then

\[(2.14) \quad F(\mathcal{H}^*(\mathcal{A}_n), s) = \sum_{k=0}^{\infty} d_n(k)s^{2k}.\]

Recall the argument in the proof of Lemma 2.2. Since \(g_r^q\) in (2.6) belongs to \(\mathcal{H}^{2(k+1-q-r)}(\mathcal{L}_q) = \mathcal{H}^{2(k+1-q-r)}(\mathcal{A}_{n-1})\) as shown in the Claim there, the rank of the module consisting of those \(g_r^q\) in (2.5) and (2.6) is given by

\[\sum_{r=0}^{k+1-q} d_{n-1}(k + 1 - q - r) = \sum_{r=0}^{k+1-q} d_{n-1}(r).\]

Therefore, noting (2.3), we see that the argument in the proof of Lemma 2.2 implies

\[d_n(k) = \sum_{q=1}^{\min\{k+1,n\}} \sum_{r=0}^{k+1-q} d_{n-1}(r),\]
in other words, if we set \( d_{n-1}(j) = 0 \) for \( j < 0 \), then
\[
(2.15) \quad d_n(k) = \begin{cases} 
\sum_{i=1}^{n} \frac{i}{d_{n-1}(k+1-i)} & \text{if } k \leq n - 1, \\
\sum_{i=1}^{n} \frac{i}{d_{n-1}(k+1-i)} + n \sum_{i=n+1}^{k+1} d_{n-1}(k+1-i) & \text{if } k \geq n.
\end{cases}
\]

We shall abbreviate \( F(\mathcal{H}^*(\mathcal{A}_n), s) \) as \( F_n(s) \). Then, plugging \((2.15)\) in \((2.14)\), we obtain
\[
F_n(s) = \sum_{k=0}^{\infty} \left( d_{n-1}(k) + 2d_{n-1}(k-1) + \cdots + nd_{n-1}(k+1-n) \right) s^{2k}
\]
\[
+ n \sum_{k=n}^{\infty} \left( d_{n-1}(k-n) + \cdots + d_{n-1}(1) + d_{n-1}(0) \right) s^{2k}
\]
\[
= F_{n-1}(s) + 2s^2 F_{n-1}(s) + \cdots + ns^{2n-2} F_{n-1}(s)
\]
\[
+ n \left( d_{n-1}(0)s^{2n} \frac{1}{1-s^2} + d_{n-1}(1)s^{2n+2} \frac{1}{1-s^2} + \cdots \right)
\]
\[
= F_{n-1}(s) \left( 1 + 2s^2 + \cdots + ns^{2n-2} \right) + n \frac{s^{2n}}{1-s^2} F_{n-1}(s)
\]
\[
= \frac{1-s^{2n}}{1-s^2} F_{n-1}(s).
\]

On the other hand, \( F_1(s) = 1/(1-s^2) \) since \( \mathcal{H}^*(\mathcal{A}_1) = \mathbb{Z}[t_1] \). It follows that
\[
F_n(s) = \frac{1}{(1-s^2)^{2n}} \prod_{i=1}^{n} (1-s^{2i}).
\]

This together with \((2.13)\) proves the lemma. \( \square \)

**REFERENCES**


