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# A refinement technique to residual evaluation of Computer assisted proofs for Semilinear elliptic boundary value problems

早稲田大学 基幹理工学研究科 高安 亮紀 (Akitoshi Takayasu)[1]
Graduate School of Fundamental Science and Engineering, Waseda University

早稲田大学 大石 進一 (Shin'ichi Oishi)[2]
Department of Applied Mathematics,
Faculty of Science and Engineering, Waseda University,
CREST, JST

## 1 Introduction

Let $\mathbb{R}$ and $\mathbb{N}$ be sets of reals and natural numbers, respectively. Let $\Omega$ be a bounded convex polygonal domain in $\mathbb{R}^m$ with $m=2,3$. This article is concerned with the Dirichlet boundary value problem of the semilinear elliptic equation:

$$\begin{cases} -\nabla\cdot(a\nabla u)=f(u), & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

We have proposed a numerical verification method [1] with Takayuki Kubo at University of Tsukuba for proving the existence of solutions to problem (1). One feature of our method is that verification conditions are based on Newton-Kantorovich theorem. Although this formulation is applicable to higher order finite elements, authors mainly treated the case of piecewise linear elements in the previous paper [1]. Using piecewise linear elements, numerical experiments sometimes require fine meshes to satisfy sufficient conditions of Newton-Kantorovich theorem. Then, our verification method is failed to prove because of computational costs. For example, consider the following nonlinear elliptic equation

$$\begin{cases} -\Delta u=u^2, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{2}$$

The sufficient condition of Newton-Kantorovich theorem is expressed by the condition

$$\alpha\omega\leqq\frac{1}{2},$$

where certain constants $\alpha$ and $\omega$ are explained later. Our numerical experiments are failed to show $\alpha\omega\leqq 1/2$ by using piecewise linear finite elements. In this article, we treat a problem of overcoming such difficulties. Some reformulation are needed to refine the residual estimation by using higher order finite elements. The smoothing technique is modified for our verification method.

In the following section, we briefly explain our computer assisted proof method. The refinement of our method is proposed in Section 3. The smoothing method is introduced. In Section 4, a computational result regarding (2) is presented. Our refined procedure prove the existence and local uniqueness of the exact solution to (2).

## 2 Basic foundations

We would like to explain our computer assisted approach first for the following abstract problem:

$$\text{Find } u\in V \text{ satisfying } \mathcal{F}u=0, \tag{3}$$

[1] takitoshi@suou.waseda.jp
[2] oishi@waseda.jp

with $\langle V, (\cdot,\cdot)_V\rangle$ denoting a Hilbert space with its inner product. We also define the dual space of $V$ as $V^*$. Let $\mathcal{F}: V \to V^*$ denote some Fréchet differentiable mapping. Let $\hat{u} \in V$ be an approximate solution to (3), and Fréchet derivative of $\mathcal{F}$ at $\hat{u}$ denotes $\mathcal{F}'(\hat{u}): V \to V^*$, *i.e.* satisfying

$$\|\mathcal{F}(\hat{u}+\nu) - \mathcal{F}(\hat{u}) - \mathcal{F}'(\hat{u})\nu\|_{V^*} = o(\|\nu\|_V), \ \|\nu\|_V \to 0.$$

Assuming that we can know three constants $C_i$, $(i = 1, 2, 3)$, such that

$$\|\mathcal{F}'(\hat{u})^{-1}\|_{\mathcal{L}(V^*,V)} \leqq C_1, \tag{4}$$

*i.e.*, $C_1$ bounds the inverse operator of $\mathcal{F}'(\hat{u})$. $C_{2,h}$ bounds the residual of approximation:

$$\|\mathcal{F}\hat{u}\|_{V^*} \leqq C_{2,h}. \tag{5}$$

$C_3$ denotes the Lipschitz constant of $\mathcal{F}'$, which is required to be Lipschitz continuous on the certain ball $D$,

$$\|\mathcal{F}'(v) - \mathcal{F}'(w)\|_{\mathcal{L}(V,V^*)} \leqq C_3\|v-w\|_V, \quad \forall v, w \in D. \tag{6}$$

Our main task to computer assisted analysis is the calculation of these constants explicitly. In order to prove the existence and local uniqueness of the exact solution in the neighborhood of $\hat{u}$, the following Newton-Kantorovich theorem is applicable to (3). This form of Newton-Kantorovich theorem is called an affine invariant form [2].

**Theorem 1 (Newton-Kantorovich Theorem).** *Assuming that the Fréchet derivative $\mathcal{F}'(\hat{u})$ is nonsingular and satisfies*

$$\|\mathcal{F}'(\hat{u})^{-1}\mathcal{F}\hat{u}\|_V \leqq \alpha,$$

*for a certain positive $\alpha$. Then, let $D := B(\hat{u}, 2\alpha) = \{v \in V : \|v - \hat{u}\|_V \leqq 2\alpha\} \subset V$ and assume that for a certain positive $\omega$ and any $v, w \in D$, the following holds:*

$$\|\mathcal{F}'(\hat{u})^{-1}(\mathcal{F}'(v) - \mathcal{F}'(w))\|_{\mathcal{L}(V,V)} \leqq \omega\|v-w\|_V.$$

*If $\alpha\omega \leqq 1/2$ holds, then there is a solution $u^* \in V$ of $\mathcal{F}u = 0$ satisfying*

$$\|u^* - \hat{u}\|_V \leqq \rho := \frac{1-\sqrt{1-2\alpha\omega}}{\omega}.$$

*Furthermore, the solution $u^*$ is unique in the ball $B(\hat{u}, \rho)$.*

Since $\alpha \leqq C_1C_{2,h}$ and $\omega \leqq C_1C_3$ form (4)-(6), the concrete computation of $C_1$, $C_{2,h}$ and $C_3$ yields computer assisted proof of the existence and local uniqueness to the problem (3). Therefore, if $\alpha\omega \leqq C_1^2C_{2,h}C_3 < 1/2$ is obtained by verified computation, then the existence and local uniqueness of the solution are proved numerically.

**Remarks**

- The above result does not require elliptic properties of the operator $\mathcal{F}'(\hat{u})$, *i.e.* the existence and local uniqueness can be obtained in the case of the operator is indefinite. This case occurs for several approximate solution. In such a case, the existence and local uniqueness cannot be obtained by the "analytic" way.
- Our computer assisted proof method requires the approximate solution of (3) in a certain finite dimensional subspace, such as the finite element subspace of $V$. It means that we can verify the solution when one have the approximate solution of (1) in the discrete subspace of $V$.
- Another method of proving the existence and inclusion of the exact solution for semilinear elliptic problems has been developed by M.T. Nakao and M. Plum (See, *e.g.* [3] and [4]). Their methods have been demonstrated to be useful for the computer assisted proof. We don't report on their methods in more detail here.

Based on this consideration, we now discuss the detail of our computer assisted proof approach in the below.

## 2.1 Notations

Throughout this article, $L^p(\Omega)$ $(p \in [1, \infty))$ denotes the functional space of Lebesgue-measurable $p$ th power-integrable functions. Especially, let us define $L^2$-inner product $(u, v)$ and $L^2$-norm $\|u\|_{L^2} = \sqrt{(u, u)}$ respectively. Let $H^s(\Omega)$ denote $L^2$-Sobolev space of order $s \in \mathbb{N}$ with the inner product $\langle u, v \rangle_s$. The $H^s$-norm is defined by $\|u\|_{H^s} = \sqrt{\langle u, u \rangle_s}$. Let further define $H_0^1(\Omega)$ by $H_0^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \ (x \in \partial\Omega)\}$ with the inner product $(\nabla u, \nabla v)$ and the norm $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$. Here, $u = 0$ on $\partial\Omega$ is the trace sense. Let $H^{-1}(\Omega)$ be the topological dual space of $H_0^1(\Omega)$, *i.e.*, the space of linear continuous functionals on $H_0^1(\Omega)$. Let $T \in H^{-1}(\Omega)$ and $u \in H_0^1(\Omega)$. We denote $Tu \in \mathbb{R}$ as $< T, u >$. The norm of $T \in H^{-1}(\Omega)$ is defined as

$$\|T\|_{H^{-1}} = \sup_{0 \neq u \in H_0^1(\Omega)} \frac{|< T, u >|}{\|u\|_{H_0^1}}.$$

Let $L^\infty(\Omega)$ denote the essentially bounded functions with the norm $\|u\|_\infty = \operatorname{ess\,sup}_{x \in \Omega} |u(x)|$. For $u(x) \in (L^\infty(\Omega))^m$, let us define

$$|u(x)|_E = \left( \sum_{i=1}^{m} u_i(x)^2 \right)^{\frac{1}{2}}.$$

Assuming that $u^h = (u_1, ..., u_n)$ is $n$-dimensional vector on $\mathbb{R}^n$, let $|u^h|_{l^2}$ be the Euclidean norm:

$$|u^h|_{l^2} = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$$

and the norm $\|\cdot\|_2$ denotes the spectral norm of matrices. Let $X$ and $Y$ be Banach spaces. The set of bounded linear operators is denoted by $\mathcal{L}(X, Y)$ with the operator norm

$$\|T\|_{\mathcal{L}(X,Y)} = \sup_{0 \neq u \in X} \frac{\|Tu\|_Y}{\|u\|_X}, \quad (T \in \mathcal{L}(X, Y)).$$

Here, $\|\cdot\|_X$ is the norm of $X$ and $\|\cdot\|_Y$ is the norm of $Y$. Furthermore, the embedding constant $C_{e,p}$ gives $\|v\|_{L^p} \leqq C_{e,p}\|v\|_{H_0^1}$. Now we choose the spaces $V := H_0^1(\Omega)$ and $V^* := H^{-1}(\Omega)(= \mathcal{L}(H_0^1, \mathbb{R}))$.

## 2.2 Weak formulation

Let $\Omega$ be a bounded convex polygonal domain in $\mathbb{R}^m$ with $m = 2, 3$. Present authors have presented with T. Kubo a method of a computer assisted proof for the Dirichlet boundary value problem of the semilinear elliptic equation [1] of the form:

$$\begin{cases} -\nabla \cdot (a \nabla u) = f(u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where $a(x)$ is a smooth function on $\Omega \cup \partial\Omega$ with $a(x) \geqq a_0 > 0$ for some $a_0 \in \mathbb{R}$. Here, $f : V \to L^2(\Omega)$ is assumed to be Fréchet differentiable. For example, the following function

$$f(u) = -b \cdot \nabla u - cu + c_2 u^2 + c_3 u^3 + g$$

with $b(x) \in (L^\infty(\Omega))^m$, $c, c_2, c_3 \in L^\infty(\Omega)$ and $g \in L^2(\Omega)$ satisfies this condition. Here, we will briefly review our proposed method. For $u, v \in V$, we define a continuous bilinear form $A(u, v)$ as $A(u, v) = (a\nabla u, \nabla v)$. Note that the bilinear form $A(u, v)$ is an inner product on $V$ and there exist positive constants $C_a$ and $c_a$ satisfying

$$c_a \|u\|_V \leqq \|u\|_a \leqq C_a \|u\|_V \quad \text{for} \quad u \in V, \tag{8}$$

where $\|u\|_a = \sqrt{A(u, u)}$. In fact, we can choose $c_a = \sqrt{a_0}$ and $C_a = \sqrt{\|a\|_\infty}$.

If we fix $u \in V$, then $A(u, \cdot) \in V^*$ is a linear functional. Thus, we can define an operator $\mathcal{A} : V \to V^*$ by $< \mathcal{A}u, v >= A(u, v)$. Note that the bilinear form $A$ is coercive, *i.e.*

$$A(u, u) \geqq a_0 \|u\|_V^2. \tag{9}$$

Then, for $v \in V$, Lax-Milgram's theorem states the existence of the inverse of $\mathcal{A} : V \to V^*$ as $\mathcal{A}^{-1} : V^* \to V$. Similarly, for $u, v \in V$ we can define a nonlinear operator $\mathcal{N} : V \to V^*$ by $< \mathcal{N}u, v >= (f(u), v)$. A weak form of Eq.(7) can be transformed into

$$\mathcal{A}u = \mathcal{N}u. \tag{10}$$

We define the operator $\mathcal{F} : V \to V^*$ by $\mathcal{F}u = (\mathcal{A} - \mathcal{N})u$. Then, Eq.(10) can be written as

$$\mathcal{F}u = 0. \tag{11}$$

This is nothing but the abstract problem (3).

In order to apply Newton-Kantorovich theorem, the Fréchet derivative of $\mathcal{F}$ is needed. The Fréchet differentiability of $\mathcal{F}$ is derived by that of $f$. We define the Fréchet derivative of $\mathcal{N}$ at $\hat{u} \in V$, *i.e.*, $\mathcal{N}'(\hat{u}) : V \to V^*$ is given by $< \mathcal{N}'(\hat{u})u, v >= (f'(\hat{u})u, v)$. Here, $f'(\hat{u}) : V \to L^2(0,1)$ is the Fréchet derivative of $f : V \to L^2(0,1)$ at $\hat{u}$. Thus, for a given $u \in V$ the Fréchet derivative $\mathcal{F}'(u) : V \to V^*$ is defined as

$$\mathcal{F}'(u) = \mathcal{A} - \mathcal{N}'(u). \tag{12}$$

## 2.3 Finite element approximation

Next we define the finite element approximation. Let $V_h$ denote a finite-dimensional space spanned by linearly independent $V$-conforming finite element basis functions depending on the mesh size $h$, $(0 < h < 1)$. For the piecewise linear base functions $\phi_i^l$, we define $V_h^l = \text{span}\{\phi_1^l, \phi_2^l, ..., \phi_{N_l}^l\} \subset V$ where $N_l$ denotes the number of node points in $\Omega \setminus \partial\Omega$. On the other hand, for piecewise quadratic base functions $\phi_i^q$, we define

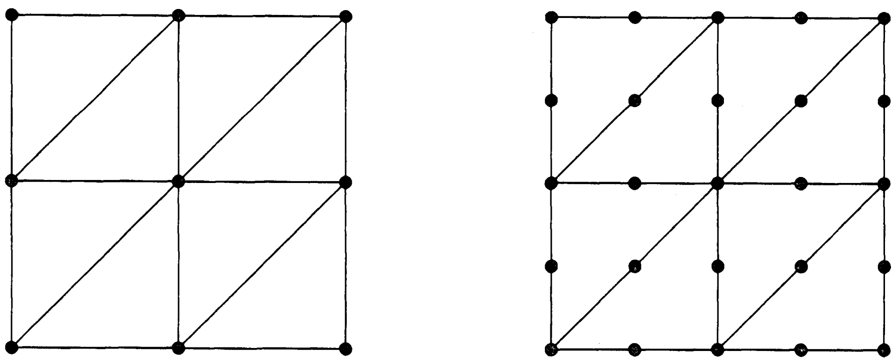

Figure 1: Piecewise linear ($N_l = 1$) & quadratic elements ($N_q = 9$)

$V_h^q = \text{span}\{\phi_1^q, \phi_2^q, ..., \phi_{N_q}^q\} \subset V$ where $N_q$ denotes the number of node points in $\Omega \setminus \partial\Omega$ (See Figure 1). If we use the piecewise linear or quadratic base functions, $V_h = V_h^l$ or $V_h = V_h^q$, respectively. In the following, by $\phi_i$ we designate $\phi_i^l$ or $\phi_i^q$ according to the base function being linear or quadratic.

The Ritz-projection $\mathcal{P}_h : V \to V_h$ is defined by $(a(x)(\nabla u - \nabla(\mathcal{P}_h u)), \nabla v_h) = 0$, $\forall v_h \in V_h$. For $u \in V \cap H^2(\Omega)$ and its approximation $\mathcal{P}_h u \in V_h$, a priori error estimate is given as

$$\|u - \mathcal{P}_h u\|_V \leqq C_0(h) \|f(u)\|_{L^2}.$$

In case of $a(x) = 1$, for the rectangular mesh, Nakao, Yamamoto and Kimura [5] have shown that one can take $C_0(h) = h/\pi$ and $h/2\pi$ for bilinear and biquadratic element, respectively. Kikuchi and Liu [6] have proved that for $a(x) = 1$ and for the linear and equilateral triangle mesh of the convex polygonal domain, $C_0(h)$ can be taken as $0.493h$. Now, we show how to calculate $C_0(h)$ for the case of $a(x) \neq 1$. Let $\Pi_h : V \to V_h$ be the orthogonal projection defined by $(\nabla u - \nabla(\Pi_h u), \nabla v_h) = 0$, $\forall v_h \in V_h$. For convex polygonal domain, it is known that the following a priori error estimate holds:

$$\|u - \Pi_h u\|_V \leqq C(h) \|\Delta u\|_{L^2}.$$

Assuming that we know the explicit formula for $C(h)$, *e.g.* in case of the linear and equilateral triangle mesh, one can take $C(h) = 0.493h$ as mentioned above. From (9), $\mathcal{P}_h u$ and $\Pi_h u \in V_h$, it follows

$$\begin{aligned} c_a^2 \|u - \mathcal{P}_h u\|_V^2 &\leqq A(u - \mathcal{P}_h u, u - \Pi_h u) \\ &\leqq C_a^2 \|u - \mathcal{P}_h u\|_V \|u - \Pi_h u\|_V \\ &\leqq C_a^2 \|u - \mathcal{P}_h u\|_V C(h) \|\Delta u\|_{L^2}. \end{aligned}$$

Thus, we have

$$\|u - \mathcal{P}_h u\|_V \leqq \left(\frac{C_a}{c_a}\right)^2 C(h) \|\Delta u\|_{L^2}. \tag{13}$$

Put $-\nabla \cdot (a \nabla u) = g_d$. Then, we have

$$\begin{aligned} \|\Delta u\|_{L^2} &= \left\| \frac{\nabla a \cdot \nabla u + g_d}{a} \right\|_{L^2} \\ &\leqq \frac{1}{a_0} \left( \|\nabla a \cdot \nabla u\|_{L^2} + \|g_d\|_{L^2} \right) \\ &\leqq \frac{1}{a_0} \left( \| |\nabla a|_E \|_\infty \|\nabla u\|_{L^2} + \|g_d\|_{L^2} \right). \end{aligned}$$

On the other hand, from (9), we have the following inequality

$$c_a^2 \|\nabla u\|_{L^2}^2 \leqq A(u, u) = (g_d, u) \leqq \|g_d\|_{L^2} \|u\|_{L^2} \leqq C_{e,2} \|g_d\|_{L^2} \|\nabla u\|_{L^2}.$$

Therefore, it turns out that

$$\|\Delta u\|_{L^2} \leqq \frac{1}{a_0} \left( \frac{C_{e,2}}{c_a^2} \| |\nabla a|_E \|_\infty + 1 \right) \|g_d\|_{L^2} = C' \|g_d\|_{L^2}. \tag{14}$$

Finally, from (13) and (14), we can derive the formula for $C_0(h)$ in the case of $a(x) \neq 1$ as

$$C_0(h) = \left(\frac{C_a}{c_a}\right)^2 C(h) C'.$$

## 2.4 Each constants

By the notation of Fréchet derivative (12), condition (4) turns out to be the inverse norm estimation:

$$\|(\mathcal{A} - \mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(V^*, V)} \leqq C_1.$$

In our method, this is estimated by the following theorem given by S. Oishi [7]. This theorem is based on perturbation lemma of linear operators [8].

**Theorem 2** (Oishi 1995). *Let $\hat{u} \in V$ and $\mathcal{N}'(\hat{u}) : V \to V^*$ be a linear compact operator. Let $V_h$ be a finite dimensional subspace of $V$. Let $\mathcal{P}_h : V \to V_h$ be the Ritz-projection. Assuming that $\mathcal{P}_h \mathcal{A}^{-1} \mathcal{N}'(\hat{u}) : V \to V$ is bounded and satisfies*

$$\|\mathcal{P}_h \mathcal{A}^{-1} \mathcal{N}'(\hat{u})\|_{\mathcal{L}(V,V)} \leqq K,$$

*the difference between $\mathcal{A}^{-1} \mathcal{N}'(\hat{u})$ and $\mathcal{P}_h \mathcal{A}^{-1} \mathcal{N}'(\hat{u})$ is bounded and enjoys*

$$\|(\mathcal{A}^{-1} - \mathcal{P}_n \mathcal{A}^{-1}) \mathcal{N}'(\hat{u})\|_{\mathcal{L}(V,V)} \leqq L,$$

*and the finite dimensional operator $\mathcal{P}_h(\mathcal{I} - \mathcal{A}^{-1} \mathcal{N}'(\hat{u}))|_{V_h} : V_h \to V_h$ is invertible with*

$$\|(\mathcal{P}_h(\mathcal{I} - \mathcal{A}^{-1} \mathcal{N}'(\hat{u}))|_{V_h})^{-1}\|_{\mathcal{L}(V,V)} \leqq M.$$

*Here, $\mathcal{P}_h(\mathcal{I} - \mathcal{A}^{-1} \mathcal{N}'(\hat{u}))|_{V_h} : V_h \to V_h$ is the restriction of the operator $\mathcal{P}_h(\mathcal{I} - \mathcal{A}^{-1} \mathcal{N}'(\hat{u})) : V \to V_h$ on $V_h$. If $(1 + MK)L < 1$, then the operator $\mathcal{A} - \mathcal{N}'(\hat{u})$ is also invertible and*

$$\|\mathcal{F}'(\hat{u})^{-1}\|_{\mathcal{L}(V^*, V)} = \|(\mathcal{A} - \mathcal{N}'(\hat{u}))^{-1}\|_{\mathcal{L}(V^*, V)} \leqq \frac{1 + MK}{a_0(1 - (1 + MK)L)} =: C_1.$$

□

In the previous paper [1], we have shown that the residual of the operator equation (11) can be bounded by

$$\|\mathcal{F}\hat{u}\|_{V^*} = \|\mathcal{A}\hat{u} - \mathcal{N}\hat{u}\|_{V^*} \leqq C_a^2 \left(\|\hat{u} - \mathcal{P}_h\mathcal{A}^{-1}\mathcal{N}(\hat{u})\|_V + C_0(h)\|f(\hat{u})\|_{L^2}\right) =: C_{2,h}. \tag{15}$$

It is noted that the term $C_0(h)\|f(\hat{u})\|_{L^2}$ is included in this expression of $C_{2,h}$. Usually, since $C_0(h)$ is proportional to $h$, $C_{2,h}$ decreases only proportional to $h$ even if we use smaller $h$. Namely, if $\|f(\hat{u})\|_{L^2}$ term becomes large, the condition of Newton-Kantorovich theorem ($\alpha\omega \leqq 1/2$) might not be satisfied unless fine mesh is used. To overcome this, this article will present a refined method of evaluating the residual of the operator equation (11). In the previous paper, we have also shown how to calculate the Lipschitz constant $C_3$ defined through $C_3 := C_{e,2}C_L$ where $C_L$ is the Lipschitz constant of $f'$.

# 3 Refinement for residual evaluation

Since the expression (15) of $C_{2,h}$ includes the term $C_0(h)\|f(\hat{u})\|_{L^2}$, $C_{2,h}$ is difficult to decrease less than 1 when the maximam value of $\hat{u}$ becomes large. If we use the piecewise linear finite element, $C_0(h)$ is usually decreasing $O(h)$. Thus, in order to satisfy the condition of Newton-Kantorovich theorem, $C_1^2C_{2,h}C_3 \leqq 1/2$, the mesh size $h$ should be taken sufficiently small such that $C_0(h)\|f(\hat{u})\|_{L^2} \ll 1$ holds. This means that $h$ should be taken very small. It causes a problem of increasing computational costs. In fact, we cannot success the verification of the problem (2). In order to overcome such difficulties for verifying the solution, we use the smoothing technique. It is a method of improving the accuracy of the residual norm estimation.

Here, elements of the finite dimensional subspace $V_h$ are assumed to be piecewise linear or quadratic finite elements ($V_h^l$ or $V_h^q$). We define $N = \dim V_h$. Let $N_b$ be the number of grid points on the boundary $\partial\Omega$. Let $g_i$ ($i = 1, 2, ..., N_b$) be grid points on $\partial\Omega$. Let further $\phi_1^*, ..., \phi_{N_b}^*$ be piecewise linear or quadratic finite element basses defined by

$$\begin{cases} \phi_i^*(g_i) = 1, & i = 1, ..., N_b, \\ \phi_i^*(g_j) = 0, & j \neq i. \end{cases}$$

Thus, $V_h^* \subset H^1(\Omega)$ is a finite element subspace defined by

$$V_h^* = \operatorname{span}\{\phi_1^*, ..., \phi_{N_b}^*, \phi_1, ..., \phi_N\}.$$

Let $\bar{\nabla}\hat{u} \in V_h^* \times V_h^*$ be the vector function defined by

$$(\bar{\nabla}\hat{u}, v^*) = (\nabla\hat{u}, v^*), \quad \forall v^* \in V_h^* \times V_h^*.$$

Namely it is an $L^2$-projection of $\nabla\hat{u} \in L^2(\Omega) \times L^2(\Omega)$ to $V_h^* \times V_h^*$. Further, $\bar{\Delta}\hat{u} \in L^2(\Omega)$ is defined by

$$\bar{\Delta}\hat{u} = \nabla \cdot (a\bar{\nabla}\hat{u}).$$

Then the following Green's formula holds between $\bar{\nabla}\hat{u}$ and $\bar{\Delta}\hat{u}$:

$$(a\bar{\nabla}\hat{u}, \nabla v) + (\bar{\Delta}\hat{u}, v) = 0, \quad \forall v \in V. \tag{16}$$

Hence, $\bar{\nabla}\hat{u}$ can be seen as an approximation of $\nabla u$. This statement is argued in [9]. Using this fact, we present a refined estimation. Let $v_h \in V_h$ be the Ritz-projection of $v \in V$, satisfying

$$A(v - v_h, \phi_h) = 0, \quad \phi_h \in V_h.$$

From this, we have

$$\|v - v_h\|_{L^2} \leqq C_a^2 C_0(h)\|v - v_h\|_V. \tag{17}$$

This is nothing but Aubin-Nitsche's trick. The orthogonality of the Ritz-projection and (8) yield

$$\|v - v_h\|_V \leqq \frac{C_a}{c_a}\|v\|_V \tag{18}$$

and

$$\|v_h\|_V \leqq \frac{C_a}{c_a}\|v\|_V. \tag{19}$$

Using inequalities (18) and (19), we have

$$
\begin{aligned}
\|\mathcal{F}\hat{u}\|_{V^*} &= \sup_{0\neq v\in V} \frac{|<\mathcal{A}\hat{u}-\mathcal{N}\hat{u},v>|}{\|v\|_V} \\
&= \sup_{0\neq v\in V} \frac{|A(\hat{u},v)-(f(\hat{u}),v)|}{\|v\|_V} \\
&= \sup_{0\neq v\in V} \frac{|A(\hat{u},v-v_h)-(f(\hat{u}),v-v_h)+A(\hat{u},v_h)-(f(\hat{u}),v_h)|}{\|v\|_V} \\
&\leqq \sup_{0\neq v\in V} \frac{|A(\hat{u},v-v_h)-(f(\hat{u}),v-v_h)|}{\|v\|_V} + \left(\frac{C_a}{c_a}\right) \sup_{0\neq v_h\in V_h} \frac{|A(\hat{u},v_h)-(f(\hat{u}),v_h)|}{\|v_h\|_V}. \qquad (20)
\end{aligned}
$$

In the following, we show how to bound the second term of (20). Let $\varepsilon_i$ be

$$\varepsilon_i := A(\hat{u},\phi_i)-(f(\hat{u}),\phi_i), \quad (i=1,...,N).$$

Since $v_h \in V_h$, we can express $v_h$ as

$$v_h = \sum_{i=1}^{N} c_i\phi_i.$$

Let us put $c=(c_1,...,c_N)^t$ and $\varepsilon=(\varepsilon_1,...,\varepsilon_N)^t$. Let further $D$ be $n\times n$ matrix whose $(i,j)$-elements are given by $(a\nabla\phi_j,\nabla\phi_i)$. Then, we have

$$\left(\frac{C_a}{c_a}\right) \sup_{0\neq v_h\in V_h} \frac{|A(\hat{u},v_h)-(f(\hat{u}),v_h)|}{\|v_h\|_V} \leqq \frac{C_a\sum_{i=1}^N c_i\varepsilon_i}{c_a^2\sqrt{c^tDc}} \leqq \frac{C_a|c|_{l^2}|\varepsilon|_{l^2}}{c_a^2\sqrt{c^tDc}} \leqq \left(\frac{C_a}{c_a^2}\right)\|D^{-1}\|_2|\varepsilon|_{l^2} =: C_r. \qquad (21)$$

Finally, using a smoothing element $\bar{\nabla}\hat{u}$ and inequalities (16)-(21), we have

$$
\begin{aligned}
\|\mathcal{F}\hat{u}\|_{V^*} &\leqq \sup_{0\neq v\in V} \frac{|(a(\nabla\hat{u}-\bar{\nabla}\hat{u}),\nabla(v-v_h))+(a\bar{\nabla}\hat{u},\nabla(v-v_h))-(f(\hat{u}),v-v_h)|}{\|v\|_V} + C_r \\
&\leqq \frac{C_a^3}{c_a}\left(\|\nabla\hat{u}-\bar{\nabla}\hat{u}\|_{L^2}+C_0(h)\|\bar{\Delta}\hat{u}+f(\hat{u})\|_{L^2}\right)+C_r =: C_{R,h}.
\end{aligned}
$$

One can replace $C_{2,h}$ by $C_{R,h}$. For a "certain" good approximation, $\|\bar{\Delta}\hat{u}+f(\hat{u})\|_{L^2}$ becomes relatively smaller than $\|f(\hat{u})\|_{L^2}$. Then, the condition $C_1^2C_{R,h}C_3 \leqq 1/2$ is easier to be fulfilled. Table 1 shows quantities $C_{2,h}$ and $C_{R,h}$ in the case of the problem (2). We use piecewise linear and quadratic basses on an uniform triangular mesh. In fact, smoothing technique doesn't work drastically by piecewise linear finite elements. On the other hand, in case of piecewise quadratic elements, $C_{R,h}$ becomes much less than $C_{2,h}$ of piecewise linear elements.

Table 1: Comparing $C_{2,h}$ with $C_{R,h}$ by piecewise linear & quadratic elemets

| Mesh size: $\frac{1}{2^r}$ | $C_{2,h}$ (Linear) | $C_{R,h}$ (Linear) | $C_{R,h}$ (Quadratic) |
|---|---|---|---|
| 4 | 12.115 | 7.3393 | 0.6186 |
| 5 | 5.9337 | 3.4043 | 0.1585 |
| 6 | 2.9516 | 1.6335 | 0.0399 |
| 7 | 1.4740 | 0.7968 | 0.0108 |
| 8 | 0.7368 | 0.3921 | 0.0072 |

## 4 Computational result

For an application of our verification method, we consider the following semilinear Dirichet boundary value problem on $\Omega=(0,1)\times(0,1)$:

$$\begin{cases} -\Delta u = u^2, & \text{in } \Omega, \\ u(x)=0, & \text{on } \partial\Omega. \end{cases}$$

An approximate solution $\hat{u}$ is computed by the finite element method with piecewise linear and quadratic base functions. Figure 2 shows the shape of the approximate solution with piecewise linear elements. The proposed computer assisted proof method is applied to this approximate solution. All computations are

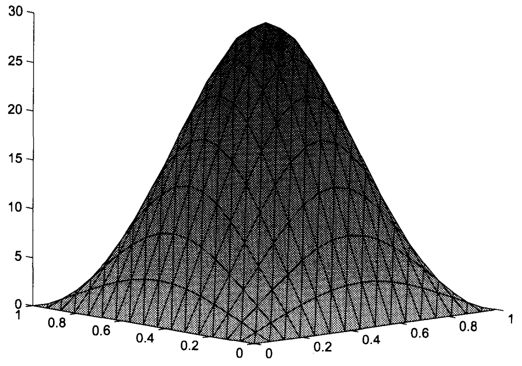


Figure 2: Approximate solution $\hat{u}$, Mesh size $\frac{1}{16}$.

carried out on Windows Server 2008 Enterprise, Quad-Core AMD Opteron(tm) Processor 8384, 2.70 GHz with 128 GByte Memory by using MATLAB 2010a with a toolbox for verified computations, INTLAB [10].

Obviously, the Fréchet derivative of $f(u) = u^2$ is given by $f'(u) = 2u$. The calculated approximate solution $\hat{u}$ is bounded on $\Omega$ so that $\hat{u} \in L^\infty(\Omega)$ in this case. Therefore, for $\hat{u}, v, w \in V$ it follows

$$K := \frac{C_{e,2}}{a_0}\|f'(\hat{u})\|_{\mathcal{L}(V,L^2)} \leqq 2C_{e,2}^2\|\hat{u}\|_\infty,$$
$$L := C_0(h)\|f'(\hat{u})\|_{\mathcal{L}(V,L^2)} \leqq 2C_{e,2}C_0(h)\|\hat{u}\|_\infty$$

and let $D$ and $G$ be $n \times n$ matrices whose $(i,j)$-elements are given by

$$(\nabla\phi_j, \nabla\phi_i) \text{ and } (\nabla\phi_j, \nabla\phi_i) - (2\hat{u}\phi_j, \phi_i),$$

respectively. Let a lower triangular matrix $\hat{L}$ be the Cholesky decomposition of $D$, *i.e.*, $D = \hat{L}\hat{L}^t$.

$$M := \frac{C_a}{c_a}\|\hat{L}^t G^{-1}\hat{L}\|_2.$$

Furthermore,

$$\|f'(v) - f'(w)\|_{\mathcal{L}(V,L^2)} \leqq 2C_{e,4}^2\|v - w\|_V.$$

Thus, we put $C_L := 2C_{e,4}^2$.

Using piecewise linear finite elements, our verification is fail to prove the existence of the exact solution. Table 2 shows the failure in case of piecewise linear elements. We cannot obtain an improved result even if we use smoothing technique. On the other hand, the drastic refinement is occurred by piecewise quadratic elements. In case of $1/128$, our computer assisted proof method yields

$$C_1 = 12.1493,\ C_{R,h} = 0.0108,\ C_3 = 0.2252.$$

Thus, we have

$$C_1^2 C_{R,h} C_3 < 0.3549.$$

Therefore, our method succeeded the verification of the approximate solution. It turns out that there exists an exact solution in the ball $B = B(\hat{u}, \rho)$ with the radius

$$\rho = 1.687 \times 10^{-1}.$$

By increasing grid points, guaranteed error bounds are improved. The improvement of the guaranteed error is presented in Table 3. We use piecewise quadratic basses on an uniform triangular mesh to compute the verified result.

Table 2: Verification Results by piecewise linear elements

| Mesh size: $\frac{1}{2^r}$ | $C_1$ | $C_1^2C_{2,h}C_3$ | $C_1^2C_{R,h}C_3$ | Verification |
|---|---|---|---|---|
| 5 | Failed | - | - | Failed |
| 6 | 130.1 | 17535 | 9704 | Failed |
| 7 | 17.09 | 152.07 | 82.20 | Failed |
| 8 | 11.93 | 37.007 | 19.70 | Failed |
| 9 | 10.42 | 14.137 | 7.455 | Failed |

Table 3: Verification Results by piecewise quadratic elements.

| Mesh size: $\frac{1}{2^r}$ | $C_1$ | $C_{R,h}$ | $C_1^2C_{R,h}C_3$ | Error: $\rho$ |
|---|---|---|---|---|
| 5 | 440.1612 | 0.1585 | 6910 | Failed |
| 6 | 17.9291 | 0.0399 | 2.8885 | Failed |
| 7 | 12.1491 | 0.0108 | 0.3548 | $1.687\times10^{-1}$ |
| 8 | 10.5144 | 0.0072 | 0.1770 | $8.286\times10^{-2}$ |

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