A Simple Derivation of Hadamard's Variational Formula

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ここでは、$\mathbb{R}^n$ 内の有界領域上での Poisson 問題と、その問題の Green 関数を考える。約 100 年前、Hadamard は、領域の境界が摂動を受けた際に Green 関数がどのような影響を受けるかという問題を考え、領域の摂動に対する Green 関数の第一変分を求めた。それは、現在 Hadamard の変分公式と呼ばれている。この論文では、Hadamard の変分公式の別証明を与える。さらに、領域の摂動に対する Green 関数の第二変分も計算することができた。我々の公式は、Garabedian-Schiffer の公式の拡張になっている。

1 Introduction

Let $\mathbb{R}^n$ be $n$-dimensional Euclidean space ($n \geq 2$) and $\Omega \subset \mathbb{R}^n$ be a bounded domain. For a given function $f$, we consider Poisson's equation

$$ -\Delta u = f \text{ on } \Omega , \quad u = 0 \text{ on } \partial \Omega . $$

The Green function $G(x,y)$ is a function which provides the solution $u$ of the Poisson equation by

$$ u(x) = \int_{\Omega} G(x,y)f(y)dy. $$

If the domain $\Omega$ is modified, then the Green function $G(x,y)$ would vary. Hadamard considered how $G(x,y)$ would vary and computed the first variation $\delta G(x,y)$ with respect to domain perturbation [3]. His result is now called Hadamard's variational formula. Hadamard showed his formula under the assumption that $\partial \Omega$ and the perturbation are analytic. Later, Garabedian and Schiffer gave a simpler and more rigorous proof of Hadamard's variational formula under the assumption that $\partial \Omega$ and the perturbation are of $C^2$ class (see [1]). Further, they obtained Hadamard's second variational formula [2], [4]. The main aim of this paper is to reconsider Hadamard's variational formula. In particular, we develop a methodology which provides us a much clearer understanding of Hadamard's variational formula. As a result, we obtain a very simple proof of Hadamard's variational formula (see Section 3.1). We also obtain Hadamard's second variational formula which is an extension of Garabedian-Schiffer's formula (Theorem 3.3).

Here, we briefly summarize the notation which we use in this paper. We denote the Euclidean inner product by $x \cdot y$ or $(x,y)_{\mathbb{R}^n}$ for $x, y \in \mathbb{R}^n$. When we do not specify, all vectors in $\mathbb{R}^n$ are regarded as column vectors. Transposing of vectors and matrices are denoted by $(\cdot)^T$. Let $f(x)$ be a smooth function defined in a domain of $\mathbb{R}^n$. The gradient of $f$ is denoted by

$$ \nabla f(x) := \left( \frac{\partial f}{\partial x_1}(x), \cdots, \frac{\partial f}{\partial x_n}(x) \right). $$

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When we need specify the variable of a gradient, we denote such as $\nabla_x f(x)$, $\nabla_x f(x^*)$. We regard gradients of functions as row vectors. Hence, for a vector field $\mathbf{F}(x)$, $\nabla \mathbf{F}(x)$ is the Jacobi matrix $D\mathbf{F}(x)$. Let $\Omega \subset \mathbb{R}^n$ be a domain in $\mathbb{R}^n$. We denote by $L^2(\Omega)$, $H^1(\Omega)$, $H^s(\partial \Omega)$ the usual Lebesgue and Sobolev spaces. The inner product of $L^2(\Omega)$ is denoted by

$$(u,v)_{\Omega} := \int_{\Omega} uv \, dx, \quad u,v \in L^2(\Omega)$$

On a point $x \in \partial \Omega$, we denote the unit outer normal vector of $\partial \Omega$ by $\nu = \nu(x)$. For a subset $\Gamma \subset \partial \Omega$, we denote the duality pair of $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ by $\langle \varphi, v \rangle_{\Gamma}$, $\varphi \in H^{-1/2}(\Gamma)$, $v \in H^{1/2}(\Gamma)$.

2 Basic Definitions

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain and $\tilde{\Omega}$ be a sufficiently larger domain which satisfies $\overline{\Omega} \subset \text{int} \tilde{\Omega}$. For a parameter $t \geq 0$, we define transformation $T_t : \Omega \rightarrow T_t(\Omega) \subset \mathbb{R}^n$ of $\Omega$ with respect $t$ in the following way. Let a $C^{0,1}$-class vector field $S(x)$ be given. We suppose that $\text{supp} S \subset \tilde{\Omega}$. Then, a transformation $T_t(x)$ on $\tilde{\Omega}$ is defined as a solution of the ordinary differential equation

$$\frac{d}{dt} T_t(x) = S(T_t(x)), \quad T_0(x) = x.$$  

That is, for each $x \in \tilde{\Omega}$, $T_t(x)$ is the integral curve generated by (2.1). This $T_t(x)$ satisfies the following properties:

- For any $x \in \Omega$, $T_0(x) = x$.
- For a sufficiently small $t$, $\Omega_t := T_t(\Omega) \subset \tilde{\Omega}$.
- $T_t$ is a diffeomorphism for a sufficiently small $t \geq 0$.
- $T_t$ is smooth with respect to $t$.

From the definition (2.1) we have $S(x) = \frac{\partial}{\partial t} T_t(x) \big|_{t=0}$. Moreover, we define

$$T(x) := \frac{\partial^2}{\partial t^2} T_t(x) \big|_{t=0}.$$

Then, the transformation has the Taylor expansion

$$T_t(x) = x + tS(x) + \frac{1}{2} t^2 T(x) + o(t^2)$$

with respect to $t$. Here, $o(t^2)$ denote a quantity which would be expressed by $t^2 \omega(x,t)$, where $\omega(x,t)$ is a function which converges uniformly (with respect to $x$) to 0 as $t \rightarrow +0$. In the sequel, notations such as $o(t)$, $o(t^2)$ are understood in this way. Let $DS(x)$ be the Jacobi matrix of $S$. From (2.1), we have

$$\frac{d^2}{dt^2} T_t(x) = \frac{d}{dt} S(T_t(x)) = DS(T_t(x)) \frac{d}{dt} T_t(x) = DS(T_t(x)) S(T_t(x)),$$

which implies

$$T(x) = (DS(x))S(x).$$
Let a function φ be defined on \( \tilde{\Omega} \) and \( \varphi \in H^2(\tilde{\Omega}) \). Suppose that a function \( u = u(x, t) \in H^1(\Omega_t) \) is a solution of the boundary value problem
\[
\begin{cases}
\Delta u(\cdot, t) = 0 & \text{in } \Omega_t, \\
u(\cdot, t) = \varphi & \text{on } \partial \Omega_t.
\end{cases}
\]
(2.3)

Here, \( \Delta := \partial^2/\partial x_1^2 + \cdots + \partial^2/\partial x_n^2 \) is the usual Laplacian with respect to \( x = (x_1, \cdots, x_n)^T \). In this section, we investigate differentiations of quantities which depend on \( u(x, t) \). Such variations of quantities with respect to domain perturbation are called Hadamard’s variation. To compute Hadamard’s variation it is important to know Lagrangian derivative \(^4\) \( \dot{u}_L, \ddot{u}_L \) and Eulerian derivative \(^5\) \( \dot{u}_E, \ddot{u}_E \), defined by, for \( x \in \Omega, \)
\[
\begin{align*}
\dot{u}_L(x) &:= \frac{d}{dt} (u(T_t(x), t)) \bigg|_{t=0}, & \ddot{u}_L(x) &:= \frac{d^2}{dt^2} (u(T_t(x), t)) \bigg|_{t=0}, \\
\dot{u}_E(x) &:= \frac{\partial}{\partial t} u(x, t) \bigg|_{t=0}, & \ddot{u}_E(x) &:= \frac{\partial^2}{\partial t^2} u(x, t) \bigg|_{t=0}.
\end{align*}
\]

For a function \( f(x, t) \), let
\[
\mathcal{H}_x f = \mathcal{H}_x f(x, t) := \left( \frac{\partial^2 f(x, t)}{\partial x_i \partial x_j} \right)_{i,j=1,\cdots,n}
\]
be the Hesse matrix. We use the same notation \( \mathcal{H}_x f \) for the second order tensor \( \mathcal{H}_x f : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R} \) defined by \( \mathcal{H}_x f(X, Y) := ((\mathcal{H}_x f)X, Y)_{\mathbb{R}^n} \) for \( X, Y \in \mathbb{R}^n \). In particular, in the case of \( X = Y \), we denote as \( \mathcal{H}_x f(X, X) = \mathcal{H}_x f \cdot (X)^2 \). A straightforward computation yields
\[
\begin{align*}
\frac{d}{dt} (u(T_t(x), t)) &= \frac{\partial}{\partial t} u(T_t(x), t) + \nabla u(T_t(x), t) \cdot \left( \frac{\partial}{\partial t} T_t(x) \right), \\
\frac{d^2}{dt^2} (u(T_t(x), t)) &= \frac{\partial^2}{\partial t^2} u(T_t(x), t) + 2\nabla \left( \frac{\partial}{\partial t} u(T_t(x), t) \right) \cdot \left( \frac{\partial}{\partial t} T_t(x) \right) \\
&\quad + \nabla u(T_t(x), t) \cdot \left( \frac{\partial^2}{\partial t^2} T_t(x) \right) + \mathcal{H}_x u(T_t(x), t) \cdot \left( \frac{\partial}{\partial t} T_t(x) \right)^2.
\end{align*}
\]
(2.4) (2.5)

2.1 Eulerian Derivatives \( \dot{u}_E, \ddot{u}_E \)

In this subsection, we check properties which Eulerian derivatives \( \dot{u}_E, \ddot{u}_E \) should satisfy. At an inner point \( x \in \Omega \) we have \( \Delta u(\cdot, t) = 0 \) for any \( t \). Hence,
\[
\Delta \dot{u}_E = 0, \quad \dot{\Delta} \dot{u}_E = 0 \quad \text{in } \Omega.
\]

On the boundary \( \partial \Omega \) we have \( u(T_t(x), t) = \varphi(T_t(x)) \). Differentiating the both side and letting \( t \rightarrow +0 \), we see \( \dot{\dot{u}}_E + S \cdot \nabla u = S \cdot \nabla \varphi \). Therefore, we find that the Eulerian derivative \( \dot{u}_E \) is a solution of the following boundary value problem:
\[
\dot{\Delta} \dot{u}_E = 0 \text{ in } \Omega, \quad \dot{u}_E = S \cdot (\nabla \varphi - \nabla u) \text{ on } \partial \Omega.
\]
(2.6)

In the same manner, we conclude that \( \ddot{u}_E \) is a solution of the boundary value problem
\[
\begin{align*}
\Delta \ddot{u}_E &= 0 \quad \text{in } \Omega, \\
\ddot{u}_E &= -2S \cdot \nabla \dot{u}_E + T \cdot (\nabla \dot{\varphi} - \nabla \dot{u}) + (\mathcal{H}_x \dot{\varphi} - \mathcal{H}_x \dot{u}) \cdot (S)^2 \quad \text{on } \partial \Omega.
\end{align*}
\]
(2.7)

\(^1\)It is also called material derivative or covariant derivative.

\(^2\)This is a usual partial derivative with respect to \( t \) which is also called shape derivative.
2.2 Lagrangian Derivatives $\dot{u}_L$, $\ddot{u}_L$

In this subsection, we check properties which Lagrangian derivatives $\dot{u}_L$, $\ddot{u}_L$ should satisfy. Here, variable on $\Omega_t$ is denoted as $x^* = T_t(x)$. A function $\hat{f}(x^*)$ defined on $\Omega_t$ is pulled back by $T_t$ to a function $f(x)$ on $\Omega$ as

$$f(x) := \hat{f}(T_t(x)).$$

Note that we have

$$\nabla_{x^*} \hat{f} = \left( \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n} \right) \left( \begin{array}{ccc} \frac{\partial x_1}{\partial x_1^*} & \cdots & \frac{\partial x_1}{\partial x_n^*} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial x_1^*} & \cdots & \frac{\partial x_n}{\partial x_n^*} \end{array} \right) = (\nabla_x f)(DT_t^{-1}),$$

where $DT_t^{-1}$ is the Jacobi matrix of $T_t^{-1}$.

The weak form of the boundary value problem (2.3) is

$$\begin{aligned}
\{ & (\nabla u(\cdot, t), \nabla \tilde{v})_{\Omega_t} = 0, \forall \tilde{v} \in H^1_0(\Omega_t), \\
& u(\cdot, t) = \varphi \quad \text{on } \partial \Omega_t. 
\end{aligned}
$$

Using the transformation $T_t$, we pull back the problem (2.8) to a problem defined on $\Omega$. Note that

$$\tilde{v} \in H^1_0(\Omega_t) \iff v := \tilde{v} \circ T_t \in H^1_0(\Omega).$$

Then, setting $u_t(x) := u(T_t(x), t)$, we see that

$$\begin{aligned}
(\nabla u(\cdot, t), \nabla \tilde{v})_{\Omega_t} &= \int_{\Omega} (\det DT_t) \left( \nabla u_t \left( DT_t^{-1} \circ T_t \right) \left( DT_t^{-1} \circ T_t \right)^T \right) \cdot \nabla v \, dx \\
&= (A(t) \nabla u_t, \nabla v)_\Omega, \quad \forall v \in H^1_0(\Omega),
\end{aligned}$$

where

$$A(t) := (\det DT_t) \left( DT_t^{-1} \circ T_t \right) \left( DT_t^{-1} \circ T_t \right)^T.$$

That is, the boundary value problem (2.8) on $\Omega_t$ is pulled back to the boundary value problem

$$\begin{aligned}
\{ & (A(t) \nabla u_t, \nabla v)_\Omega = 0, \forall v \in H^1_0(\Omega), \\
u_t = \varphi \circ T_t \quad \text{on } \partial \Omega 
\end{aligned}
$$

on $\Omega$. If $u(x, t)$ is a solution of (2.8), then $u_t(x) = u(T_t(x), t)$ is a solution of (2.9) and vice versa.

We set

$$\begin{aligned}
A' := \frac{d}{dt}A(t) \bigg|_{t=0}, \quad A'' := \frac{d^2}{dt^2}A(t) \bigg|_{t=0}. 
\end{aligned}$$

Suppose that $\varphi \circ T_t$ has the following Taylor expansion:

$$\varphi \circ T_t = \varphi + t \dot{\varphi} + \frac{1}{2} t^2 \ddot{\varphi} + o(t^2).$$

From the definition we find

$$\begin{aligned}
\dot{\varphi} &= S \cdot \nabla \varphi, \\
\ddot{\varphi} &= T \cdot \nabla \varphi + \mathcal{H}_x \varphi \cdot (S)^2.
\end{aligned}$$
Let $u$ be a solution of (2.3). Then, we “differentiate” (2.9) and obtain the equation

$$
(2.11) \quad \begin{cases} 
(\nabla \dot{u}_{\mathcal{L}}, \nabla v)_{\Omega} = -(A'\nabla u, \nabla v)_{\Omega}, & \forall v \in H_{0}^{1}(\Omega), \\
\dot{u}_{\mathcal{L}} = \dot{\varphi} & \text{on } \partial\Omega.
\end{cases}
$$

One more “differentiation” yields the equation

$$
(2.12) \quad \begin{cases} 
(\nabla \ddot{u}_{\mathcal{L}}, \nabla v)_{\Omega} = -2(\mathcal{A}^{f}\nabla \dot{u}_{\mathcal{L}}, \nabla v)_{\Omega} - (\mathcal{A}''\nabla u, \nabla v)_{\Omega}, & \forall v \in H_{0}^{1}(\Omega), \\
\ddot{u}c = \ddot{\varphi} & \text{on } \partial\Omega.
\end{cases}
$$

For solutions of these equations, we have the following lemma.

**Lemma 2.1** Suppose that $u, u_t, \dot{u}_{\mathcal{L}}, \ddot{u}_{\mathcal{L}} \in H^{1}(\Omega)$ are solutions of the equations (2.3), (2.9), (2.11), (2.12), respectively. Then, $u_t$ has a Taylor expansion $u_t = u + t\dot{u}_{\mathcal{L}} + \frac{1}{2}t^{2}\ddot{u}_{\mathcal{L}} + o(t^{2})$ in $H^{1}(\Omega)$. That is, the following is valid:

$$
\lim_{t \to 0+} \frac{\|\chi_{t}\|_{H^{1}(\Omega)}}{t^{2}} = 0, \quad \chi_{t} := v_{t} - (u + t\dot{u}_{\mathcal{L}} + \frac{1}{2}t^{2}\ddot{u}_{\mathcal{L}}).
$$

**Proof** Since $S \in W^{1,\infty}(\tilde{\Omega}; \mathbb{R}^{n})$ and $T \in L^{\infty}(\tilde{\Omega}; \mathbb{R}^{n})$, we see $A(t) \in L^{\infty}(\Omega; \mathbb{R}^{n^{2}})$ and (2.13)

$$
\lim_{t \to 0+} \frac{\|\alpha_{t}\|_{L^{\infty}}}{t^{2}} = 0, \quad \alpha_{t} := A(t) - (1 + tA' + \frac{1}{2}t^{2}A'').
$$

Define $z_{t}, \dot{z}, \ddot{z}$ as solutions of the following boundary value problems:

$$
(\nabla z_{t}, \nabla v)_{\Omega} = 0, \quad \forall v \in H_{0}^{1}(\Omega), \quad z_{t} = \varphi \circ T_{t} \text{ on } \partial\Omega,
$$

$$
(\nabla \dot{z}, \nabla v)_{\Omega} = 0, \quad \forall v \in H_{0}^{1}(\Omega), \quad \dot{z} = \dot{\varphi} \text{ on } \partial\Omega,
$$

$$
(\nabla \ddot{z}, \nabla v)_{\Omega} = 0, \quad \forall v \in H_{0}^{1}(\Omega), \quad \ddot{z} = \ddot{\varphi} \text{ on } \partial\Omega.
$$

Letting

$$
\eta_{t} := z_{t} - (u + t\dot{z} + \frac{1}{2}t^{2}\ddot{z}), \quad \psi_{t} := \varphi \circ T_{t} - (\varphi + t\dot{\varphi} + \frac{1}{2}t^{2}\ddot{\varphi}),
$$

we notice $\eta_{t} - \psi_{t} \in H_{0}^{1}(\Omega)$. Since $0 = (\nabla \eta_{t}, \nabla v)_{\Omega}$ for any $v \in H_{0}^{1}(\Omega)$, we set $v := \eta_{t} - \psi_{t}$ and obtain

$$
\|\nabla \eta_{t}\|_{L^{2}(\Omega)}^{2} = (\nabla \eta_{t}, \nabla \eta_{t})_{\Omega} = (\nabla \eta_{t}, \nabla \psi_{t})_{\Omega} \leq \|\nabla \eta_{t}\|_{L^{2}(\Omega)} \|\nabla \psi_{t}\|_{L^{2}(\Omega)},
$$

$$
\lim_{t \to 0+} \frac{\|\nabla \eta_{t}\|_{L^{2}(\Omega)}}{t^{2}} \leq \lim_{t \to 0+} \frac{\|\nabla \psi_{t}\|_{L^{2}(\Omega)}}{t^{2}} = 0.
$$

Similarly, set

$$
\beta_{t} := u_{t} - (u - z + \frac{1}{2}t^{2}(\ddot{u}_{\mathcal{L}} - \ddot{z})), \quad v_{t} := \varphi \circ T_{t} - (\varphi + t\dot{\varphi} + \frac{1}{2}t^{2}\ddot{\varphi}),
$$

we notice $\beta_{t} - v_{t} \in H_{0}^{1}(\Omega)$. Since $0 = (\nabla \beta_{t}, \nabla v)_{\Omega}$ for any $v \in H_{0}^{1}(\Omega)$, we set $v := \beta_{t} - v_{t}$ and obtain

$$
\|\nabla \beta_{t}\|_{L^{2}(\Omega)}^{2} = (\nabla \beta_{t}, \nabla \beta_{t})_{\Omega} = (\nabla \beta_{t}, \nabla \psi_{t})_{\Omega} \leq \|\nabla \beta_{t}\|_{L^{2}(\Omega)} \|\nabla \psi_{t}\|_{L^{2}(\Omega)},
$$

$$
\lim_{t \to 0+} \frac{\|\nabla \beta_{t}\|_{L^{2}(\Omega)}}{t^{2}} \leq \lim_{t \to 0+} \frac{\|\nabla \psi_{t}\|_{L^{2}(\Omega)}}{t^{2}} = 0.
$$

Then, from (2.11), (2.12), we find that for any $v \in H_{0}^{1}(\Omega),

$$
(\mathcal{A}(t) \nabla \beta_{t}, \nabla v)_{\Omega} = (\mathcal{A}(t) \nabla z_{t}, \nabla v)_{\Omega} - t(\mathcal{A}(t) \nabla (u_{L} - \dot{z}), \nabla v)_{\Omega} - \frac{1}{2}t^{2}(\mathcal{A}(t) \nabla (u_{L} - \ddot{z}), \nabla v)_{\Omega}
$$

$$
= (\mathcal{A}(t) \nabla \eta_{t} + \frac{1}{2}t^{2}\mathcal{A}'u_{L}), \nabla v)_{\Omega} + t((I + tA' - \mathcal{A}(t)) \nabla u_{L}, \nabla v)_{\Omega}
$$

$$
+ ((I + tA' + \frac{1}{2}t^{2}A'' - \mathcal{A}(t)) \nabla u, \nabla v)_{\Omega}.
$$
By (2.13) there exists a positive constant $\lambda$ such that, for any sufficiently small $t > 0$,

$$\lambda \| \nabla v \|_{L^2(\Omega)}^2 \leq (A(t) \nabla v, \nabla v)_\Omega, \quad \forall v \in H^1_0(\Omega).$$

Inserting $v = \beta_t$ into the above equation, we obtain

$$\frac{\lambda}{t^2} \| \nabla \beta_t \|_{L^2(\Omega)} \leq \| 1 - A(t) \|_{L^\infty(\Omega)} \left( \frac{\| \nabla \eta_t \|_{L^2(\Omega)}}{t^2} + \| \nabla \beta_t \|_{L^2(\Omega)} \right) + \frac{1}{t^2} \| \alpha_t \|_{L^\infty(\Omega)} \| \nabla u \|_{L^2(\Omega)}.$$

Therefore, we conclude $\lim_{t \to 0^+} \| \nabla \beta_t \|_{L^2(\Omega)}/t^2 = 0$ and complete the proof since $\chi_t = \beta_t + \eta_t$. □

2.3 The Relationship between Eulerian and Lagrangian derivatives

In this subsection we consider the relationship between Eulerian and Lagrangian derivatives. From (2.4) we immediately notice

$$\dot{u}_\mathcal{L} = \dot{u}_\mathcal{E} + S \cdot \nabla u$$

in $\Omega$.

Since $(\nabla \dot{u}_\mathcal{L}, \nabla v) = -(A' \nabla u, \nabla v)$ and $(\nabla \dot{u}_\mathcal{E}, \nabla v) = 0$ for any $v \in H^1_0(\Omega)$, we have

$$(\nabla (S \cdot \nabla u), \nabla v)_\Omega = -(A' \nabla u, \nabla v)_\Omega, \quad \forall v \in H^1_0(\Omega).$$

Similarly, from (2.5) we obtain

$$\ddot{u}_\mathcal{L} = \ddot{u}_\mathcal{E} + 2S \cdot \nabla \dot{u}_\mathcal{E} + T \cdot \nabla u + \mathcal{H}_x u \cdot (S)^2$$

in $\Omega$. Since

$$\langle (\nabla \ddot{u}_\mathcal{L}, \nabla v) \rangle_{\Omega} = -(2A' \nabla \dot{u}_\mathcal{L} + A'' \nabla u, \nabla v)_{\Omega}, \quad \forall v \in H^1_0(\Omega),$$

we have, for any $v \in H^1_0(\Omega)$,

$$\langle (2\nabla (S \cdot \nabla u) + \nabla (T \cdot \nabla u) + \nabla (\mathcal{H}_x u \cdot (S)^2), \nabla v) \rangle_{\Omega} = -(2A' \nabla \dot{u}_\mathcal{L} + A'' \nabla u, \nabla v)_{\Omega}.$$  

2.4 Liouville’s Theorem

In this section we prepare Liouville’s theorem which plays an important role in calculus of Hadamard’s variation. Following Garabedian [1] and (2.2), we denote normal components of $S$ and $T$ by $\delta \rho$ and $\delta^2 \rho$, respectively:

$$\delta \rho := S \cdot \nu, \quad \delta^2 \rho := T \cdot \nu = \nu^t DS(x) S(x).$$

Theorem 2.2 (Liouville’s Theorem) Let a sufficiently smooth function $c(x, t)$ be defined on the domain $\Omega_t := \mathcal{T}_t(\Omega)$ for each $t \geq 0$. Suppose also that $c(x, t)$, $c_t(x, t) := \frac{\partial c}{\partial t}(x, t)$ are measurable on $\Omega_t$. Then, the following holds:

$$\frac{d}{dt} \left( \int_{\Omega_t} c(x, t) dx \right)|_{t=0} = \int_{\Omega} (c_t(x, 0) + \nabla \cdot (c(x, 0) S(x))) dx = \int_{\Omega} c_t(x, 0) dx + \langle c(\cdot, 0), \delta \rho \rangle_{\partial \Omega}.$$
Proof. We may suppose without loss of generality that \( \partial \Omega, c, c_t \) are all sufficiently smooth. The proof for general cases follows from the density property of \( C^\infty(\tilde{\Omega}) \) in \( H^1(\tilde{\Omega}) \). Let \( J\mathcal{T}_t(x) \) be the Jacobi matrix of \( \mathcal{T}_t(x) \). Differentiating the both sides of

\[
\int_{\tilde{\Omega}} c(x, t) \, dx = \int_{\tilde{\Omega}} c(\mathcal{T}_t(x), t) \det(J\mathcal{T}_t(x)) \, dx
\]

(2.16) with respect to \( t \), we have

\[
\frac{d}{dt} \left( \int_{\tilde{\Omega}} c(x, t) \, dx \right) = \int_{\tilde{\Omega}} \left( c_t(\mathcal{T}_t(x), t) + \nabla c(\mathcal{T}_t(x), t) \cdot \frac{\partial}{\partial t} \mathcal{T}_t(x) \right) \det(J\mathcal{T}_t(x)) \, dx
\]

(2.17)\]

Then, letting \( t \to 0^+ \), we obtain the first equality of (2.15). Here, we use

\[
\frac{\partial}{\partial t} \det(J\mathcal{T}_t(x)) \Big|_{t=0} = \nabla \cdot S(x).
\]

The second equality immediately follows from the divergence theorem. \( \Box \)

**Corollary 2.3** Suppose that a function \( f(x, t) \) is in \( H^1(\Omega_t) \) for each \( t \geq 0 \) and harmonic on \( \Omega_t \) with respect to \( x \in \mathbb{R}^n \). Then, we have

\[
\frac{d}{dt} \int_{\Omega} \left| \nabla_x f(x, t) \right|^2 \, dx \bigg|_{t=0} = 2 \left( \frac{\partial f}{\partial \nu} \cdot \dot{f}_E \right)_{\partial \Omega} + \langle |\nabla f|^2, \delta \rho \rangle_{\partial \Omega}.
\]

Proof: Set \( c(x, t) := \left| \nabla_x f(x, t) \right|^2 \) and apply Theorem 2.2. \( \Box \)

We now try to obtain a second order Liouville's theorem. Assume that \( \partial \Omega, c, S \) are sufficiently smooth. We have obtained (2.17) by differentiating the both side of (2.16) with respect to \( t \). One more differentiation of the both side of (2.17) and letting \( t \to 0 \) yield

\[
\int_{\Omega} \left[ c_{tt}(x, 0) + 2 \nabla_x c_t(x, 0) \cdot S + \nabla_x c(x, 0) \cdot T + \mathcal{H}_x c(x, 0) \cdot (S)^2 \right] \, dx
\]

\[
+ 2 \int_{\Omega} \left[ c_t(x, 0) + \nabla_x c(x, 0) \cdot S \right] (\nabla \cdot S) \, dx
\]

\[
+ \int_{\Omega} c(x, 0) \left[ \nabla \cdot T + 2 \sum_{i<j}(S_{ix}, S_{jx}, -S_{ix}S_{jx}) \right] \, dx.
\]

Here, we used (2.18) and, for \( S = (S_1, \cdots, S_n)^T \),

\[
\frac{\partial^2}{\partial t^2} \det(D\mathcal{T}_t(x)) \bigg|_{t=0} = \nabla \cdot T(x) + 2 \sum_{i<j}(S_{ix}, S_{jx}, -S_{ix}S_{jx}), \quad S_{kx_i} := \frac{\partial S_k}{\partial x_i}.
\]

Hence, we obtain

\[
\frac{d^2}{dt^2} \left( \int_{\tilde{\Omega}} c(x, t) \, dx \right) \bigg|_{t=0} = \int_{\Omega} c_{tt}(x, 0) \, dx + \int_{\Omega} \mathcal{H}_x c(x, 0) \cdot (S)^2 \, dx
\]

\[
+ 2 \int_{\partial \Omega} c_t(x, 0) \delta \rho \, ds + \int_{\partial \Omega} c(x, 0) \delta^2 \rho \, ds
\]

\[
+ 2 \int_{\Omega} (\nabla_x c(x, 0) \cdot S)(\nabla \cdot S) \, dx + 2 \int_{\Omega} c(x, 0) \sum_{i<j}(S_{ix}, S_{jx}, -S_{ix}S_{jx}) \, dx.
\]
We try to simplify this formula. Recall that $DS$ is the Jacobi matrix of the vector field $S$. Since it follows from the divergence theorem that

$$2 \int_{\Omega} (\nabla x c(x, 0) \cdot S)(\nabla \cdot S)dx = 2 \int_{\partial \Omega} (\nabla x c(x, 0) \cdot S)\rho ds$$

$$- 2 \int_{\Omega} \mathcal{H}_x c(x, 0) \cdot (S)^2 dx - 2 \int_{\Omega} ((DS) \cdot \nabla x c(x, 0)dx,$$

we have

$$\frac{d^2}{dt^2} \left( \int_{\Omega} c(x, t) dx \right)_{t=0} = \int_{\Omega} c_{tt}(x, 0)dx + 2 \int_{\partial \Omega} c_{t}(x, 0)\rho ds$$

$$+ \int_{\partial \Omega} c(x, 0)\delta^2 ds + 2 \int_{\partial \Omega} (\nabla x c(x, 0) \cdot S)\rho ds$$

$$- \int_{\Omega} \mathcal{H}_x c(x, 0) \cdot (S)^2 dx - 2 \int_{\Omega} ((DS) \cdot \nabla x c(x, 0)dx$$

$$+ 2 \int_{\Omega} c(x, 0) \sum_{i<j}(S_{ix}S_{jx} - S_{ix}S_{jx})dx.$$
Therefore, we have

\[
X + Y + Z = - \int_{\partial\Omega} (\nabla_{x} c \cdot S) \delta \rho + \sum_{i<j} \int_{\partial\Omega} c S_{ix} S_{jx} \nu_{i} - \sum_{i<j} \int_{\partial\Omega} c S_{ix} S_{j}\nu_{i}
+ \sum_{i<j} \int_{\partial\Omega} c S_{ix} S_{jx} \nu_{j} - \sum_{i<j} \int_{\partial\Omega} c S_{ix} S_{jx} \nu_{i}
= - \int_{\partial\Omega} (\nabla_{x} c \cdot S) \delta \rho + \int_{\partial\Omega} c (\nabla \cdot S) \delta \rho - \int_{\partial\Omega} c \delta^{2} \rho.
\]

Here, we used (2.14). Hence, \( W + X + Y + Z \) becomes

\[
W + X + Y + Z = \int_{\partial\Omega} (\nabla_{x} c(x, 0) \cdot S) \delta \rho \, ds
+ \int_{\partial\Omega} c(x, 0) (\nabla \cdot S) \delta \rho \, ds - \int_{\partial\Omega} c(x, 0) \delta^{2} \rho \, ds
\]

Gathering all terms, we finally obtain

\[
\frac{d^{2}}{dt^{2}} \left( \int_{\Omega_{t}} c(x, t) \, dx \right) |_{t=0} = \int_{\Omega} c_{tt}(x, 0) \, dx + 2 \int_{\partial\Omega} c_{t}(x, 0) \delta \rho \, ds
+ \int_{\partial\Omega} \nabla \cdot (c(x, 0) S) \delta \rho \, ds.
\]

We have done these computation under the assumption that \( c, \partial\Omega \) are sufficiently smooth. With a usual density argument we obtain the following theorem for general cases.

**Theorem 2.4 (Extended Liouville’s Theorem)** Suppose that a \( C^{2} \)-class function \( c(x, t) \) is given on \( \tilde{\Omega} \) and, for each \( t \geq 0 \), \( c(x, t), c_{t}(x, t) := \frac{\partial c}{\partial t}(x, t), c_{tt}(x, t) := \frac{\partial^{2}}{\partial t^{2}} c(x, t) \) are integrable on \( \Omega_{t} := \mathcal{T}_{t}(\Omega) \). Then, we have the following equality:

\[
\frac{d^{2}}{dt^{2}} \left( \int_{\Omega_{t}} c(x, t) \, dx \right) |_{t=0} = \int_{\Omega} c_{tt}(x, 0) \, dx + 2 \int_{\partial\Omega} c_{t}(x, 0) \delta \rho \, ds
+ \int_{\partial\Omega} \nabla \cdot (c(x, 0) S) \delta \rho \, ds.
\]

Here, \( \nu \) is unit outer normal and \( \delta \rho := S \cdot \nu \).

### 3 Hadamard’s Variational Formula

Let \( \Omega \subset \mathbb{R}^{n} \) be a domain and \( G(x, y) \) be the Green function of the Laplacian \( \Delta \) on \( \Omega \). If \( \Omega \) is perturbed, \( G(x, y) \) also varies. Hadamard presented the first variation \( \delta G(x, y) \) of \( G(x, y) \) with respect to domain perturbation. His formula is called Hadamard’s variational formula. Later, Garabedian and Schiffer gave an rigorous alternative proof of Hadamard’s variational formula. They also computed the second variation of the Green function. In this section, using the result obtained in the previous section, we give an alternative and much simpler proof of Hadamard’s variation formula, the both first and second variations. In the sequel, the boundary \( \partial\Omega \) is assumed to be sufficiently smooth. The fundamental solution \( \Gamma(x) \) of the Laplacian \( \Delta \) is defined by

\[
\Gamma(x) := \begin{cases}
-\frac{1}{2\pi} \log |x|, & n = 2, \\
\frac{1}{(n-2)\omega_{n}} |x|^{2-n}, & n \geq 3.
\end{cases}
\]
Here, $\omega_n$ is the measure of $(n - 1)$-dimensional sphere $S^{n-1}$. Then, for sufficiently smooth function $f$ we have Green's formula

\[ (3.1) \quad - \int_{\Omega} \Gamma(x-y)\Delta f(x)dx + \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x)\Gamma(x-y)ds_x = \int_{\partial\Omega} f(x) \frac{\partial}{\partial \nu} \Gamma(x-y)ds_x + f(y). \]

For the fundamental solution $\Gamma(x-y)$, define $u$ as a solution of the following boundary value problem:

\[ \Delta u = 0 \quad \text{in} \: \Omega, \quad u(x) = -\Gamma(x-y), \quad x \in \partial\Omega. \]

Then, $G(x,y) := \Gamma(x-y) + u(x)$ is the Green function of $\Delta$ on $\Omega$. It follows from the definition that $G(x,y) = 0$ for $x \in \partial\Omega$ and $y \in \Omega$. Adding the following Green's formula with respect to $f$ and $u$

\[ - \int_{\Omega} u(x)\Delta f(x)dx + \int_{\partial\Omega} \frac{\partial f}{\partial \nu}(x)u(x)ds = \int_{\partial\Omega} f(x) \frac{\partial u}{\partial \nu}(x)ds \]

to (3.1), we obtain Green's second formula

\[ (3.2) \quad f(y) = - \int_{\Omega} G(x,y)\Delta f(x)dx - \int_{\partial\Omega} f(x) \frac{\partial}{\partial \nu} G(x,y)ds_x. \]

3.1 First Variation

Now, we consider domain perturbation $\Omega_t = T_t(\Omega)$ defined in the previous section. The Green function $G(x,y,t)$ on $\Omega_t$ is written as

\[ G(x,y,t) = \Gamma(x-y) + u(x,t), \]

where $u(x,t)$ is the harmonic function which satisfies

\[ (3.3) \quad \Delta_x u(x,t) = 0 \quad \text{in} \: \Omega_t, \quad u(x,t) = -\Gamma(x-y), \quad x \in \partial\Omega_t. \]

Obviously, we have $G(x,y,0) = G(x,y)$ and $u(x,0) = u(x)$. For two inner points $x, y \in \Omega$ and sufficiently small $t > 0$, we have $x, y \in \Omega_t$. The first variation $\delta G(x,y)$ with respect to domain perturbation is defined by

\[ \delta G(x,y) := \lim_{t \rightarrow 0^+} \frac{G(x,y,t) - G(x,y,0)}{t} = \frac{u(x,t) - u(x,0)}{t} = \dot{u}_\mathcal{E}(x), \]

and is equal to the Eulerian derivative $\dot{u}_\mathcal{E}$ of $u$.

By (2.6), we confirm that $\dot{u}_\mathcal{E}$ is a solution of the boundary value problem

\[ \Delta \dot{u}_\mathcal{E} = 0 \quad \text{in} \: \Omega, \quad \dot{u}_\mathcal{E} = S \cdot (-\nabla x \Gamma(x-y) - \nabla u) = -S \cdot \nabla x G(x,y) = -\delta \rho \frac{\partial}{\partial \nu} G(x,y) \quad \text{on} \: \partial\Omega. \]

Here, we use the fact that $S \cdot \nabla x G(x,y) = (S \cdot \nu) \frac{\partial}{\partial \nu_x} G(x,y)$ on $\partial\Omega$. Applying the formula (3.2) to $\dot{u}_\mathcal{E}$, we obtain Hadamard's variational formula.

**Theorem 3.1 (Hadamard's variational formula)** The first variation $\delta G(w,y)$ of the Green function $G(w,y)$ of $\Delta$ with respect to domain perturbation is given by

\[ \delta G(w,y) = \int_{\partial\Omega} \frac{\partial}{\partial \nu_x} G(x,y) \frac{\partial}{\partial \nu_x} G(x,w) \delta \rho ds_x, \quad \delta \rho := S \cdot \nu. \]
3.2 Second Variation

In this subsection, we compute the second variation of the Green function with respect to domain perturbation. We prepare a lemma. Let a harmonic function $u(x, t)$ be a solution of the Dirichlet problem (3.3). Since $\delta G(x, y) = \dot{u}_\mathcal{E}(x)$, we recall that

\begin{equation}
\delta G(x, y) = -\delta \rho \frac{\partial}{\partial \nu_x} G(x, y), \quad x \in \partial \Omega.
\end{equation}

Hence, for a harmonic function $g(x)$, we find

\begin{align*}
0 &= \int_{\partial \Omega} \left( \delta G(x, y) + \delta \rho \frac{\partial}{\partial \nu_x} G(x, y) \right) \frac{\partial g}{\partial \nu}(x) ds_x \\
&= \int_{\Omega} \nabla_x \delta G(x, y) \cdot \nabla g(x) dx + \int_{\partial \Omega} \delta \rho \frac{\partial}{\partial \nu_x} G(x, y) \frac{\partial g}{\partial \nu}(x) ds_x,
\end{align*}

and obtain the following lemma.

**Lemma 3.2** For a harmonic function $g$ on $\Omega$, the following equality holds:

\begin{equation}
\int_{\Omega} \nabla_x \delta G(x, y) \cdot \nabla g(x) dx = -\int_{\partial \Omega} \frac{\partial}{\partial \nu_x} G(x, y) \frac{\partial g}{\partial \nu}(x) \delta \rho ds_x.
\end{equation}

The second variation $\delta^2 G(x, y)$ of the Green function $G(x, y)$ is defined by

\begin{equation}
\delta^2 G(x, y) := \left. \frac{\partial^2}{\partial t^2} G(x, y, t) \right|_{t=0} = \ddot{u}_\mathcal{E}(x),
\end{equation}

and, therefore, we only need to compute $\ddot{u}_\mathcal{E}$. Recall that the harmonic function is a solution of the Dirichlet problem

\begin{equation*}
\Delta u = 0 \text{ in } \Omega, \quad u = -\Gamma(\cdot - y) \text{ on } \partial \Omega.
\end{equation*}

By (2.7), the boundary value of $\ddot{u}_\mathcal{E}$ on $\partial \Omega$ is

\begin{equation*}
\ddot{u}_\mathcal{E} = -2S \cdot \nabla_x G(x, y) - T \cdot \nabla_x G(x, y) - \mathcal{H}_x G(x, y) \cdot (S)^2.
\end{equation*}

Here, we use $G(x, y) = \Gamma(x - y) + u(x)$ and $\dot{u}_\mathcal{E}(x) = \delta G(x, y)$. From (3.2), we find

\begin{equation}
\delta^2 G(x, y) = \ddot{u}_\mathcal{E}(x) = 2 \int_{\partial \Omega} S \cdot \nabla_w \delta G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w
\end{equation}

\begin{align*}
&+ \int_{\partial \Omega} T \cdot \nabla_w G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w \\
&+ \int_{\partial \Omega} \mathcal{H}_w G(w, y) \cdot (S)^2 \frac{\partial}{\partial \nu_x} G(x, w) ds_w.
\end{align*}

We denote the first, second and third terms of the right-hand side of (3.5) by $X$, $Y$, $Z$, respectively. As before, the term $Y$ can be written as

\begin{equation}
Y = \int_{\partial \Omega} \frac{\partial}{\partial \nu_w} G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) \delta^2 \rho ds_w.
\end{equation}

To understand the terms $X$ and $Z$, we consider the $(n - 1)$-dimensional tangent space $T_x \partial \Omega$ of $\partial \Omega$ at $x \in \partial \Omega$. Let $\{s_1, \cdots, s_{n-1}\}$ be the orthonormal basis of $T_x \partial \Omega$. Then, $\{s_1, \cdots, s_{n-1}, \nu\}$
is an orthonormal basis of the tangent space $T_x \mathbb{R}^n$ at $x \in \mathbb{R}^n$. For a generic function $f$, directional derivatives are defined by

$$\frac{\partial f}{\partial \nu} = \nabla f \cdot \nu, \quad \frac{\partial f}{\partial s_i} = \nabla f \cdot s_i, \quad i = 1, \ldots, n - 1.$$  

Thus, defining the orthogonal matrix $P$ by $P := (s_1, \ldots, s_{n-1}, \nu)$, we may write

$$\left( \frac{\partial f}{\partial s_1}, \ldots, \frac{\partial f}{\partial s_{n-1}}, \frac{\partial f}{\partial \nu} \right) = (\nabla f) P, \quad \text{or}$$

$$\begin{equation}
(\nabla f)^T = \sum_{i=1}^{n-1}s_i \frac{\partial f}{\partial s_i} + \nu \frac{\partial f}{\partial \nu} \quad \text{and} \quad \nabla = \sum_{i=1}^{n-1}s_i^T \frac{\partial}{\partial s_i} + \nu^T \frac{\partial}{\partial \nu}.
\end{equation}
$$

If we write $S$ as

$$S = \sum_{i=1}^{n-1} \mu_i s_i + \delta \rho \nu, \quad \delta \rho = S \cdot \nu, \quad \mu_i = S \cdot s_i, \quad i = 1, \ldots, n - 1$$

on $\partial \Omega$, we obtain

$$S \cdot \nabla_w \delta G(w, y) = \sum_{i=1}^{n-1} \mu_i \frac{\partial}{\partial s_i} \delta G(w, y) + \delta \rho \frac{\partial}{\partial \nu_w} \delta G(w, y).$$

Using Lemma 3.2 with $g := \delta G$, the term $X$ (the first term of the right-hand side of (3.5)) is written as

$$X = 2 \sum_{i=1}^{n-1} \int_{\partial \Omega} \mu_i \frac{\partial}{\partial s_i} \delta G(w, y) \frac{\partial}{\partial \nu} G(x, w) ds_w + 2 \int_{\partial \Omega} \delta \rho \frac{\partial}{\partial \nu_w} \delta G(w, y) \frac{\partial}{\partial \nu} G(x, w) ds_w$$

$$= 2 \sum_{i=1}^{n-1} \int_{\partial \Omega} \mu_i \frac{\partial}{\partial s_i} \delta G(w, y) \frac{\partial}{\partial \nu} G(x, w) ds_w - 2 \int_{\Omega} \nabla_w \delta G(w, y) \cdot \nabla_x \delta G(x, w) ds_w.$$

Next, we try to rewrite the third term $Z$ of the right-hand side of (3.5). To this end, we consider the curved coordinate defined by $\{s_1, \ldots, s_{n-1}, \nu\}$ in the neighborhood of $x \in \partial \Omega$ and second order differentiation on the coordinate. For a $C^2$ class generic function $f$, the Hesse matrix $\mathcal{H}f$ is written by $\mathcal{H}f = \nabla (\nabla f)^T$ and

$$\begin{equation}
\mathcal{H}f = \nabla \left( \sum_{i=1}^{n-1} s_i \frac{\partial f}{\partial s_i} + \nu \frac{\partial f}{\partial \nu} \right)
= \sum_{i=1}^{n-1} \frac{\partial f}{\partial s_i} Ds_i + \frac{\partial f}{\partial \nu} D\nu + \sum_{i=1}^{n-1} s_i \nabla \left( \frac{\partial f}{\partial s_i} \right) + \nu \nabla \left( \frac{\partial f}{\partial \nu} \right)
= \sum_{i=1}^{n-1} \frac{\partial f}{\partial s_i} Ds_i + \frac{\partial f}{\partial \nu} D\nu + \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial s_i s_j} s_i s_j + \sum_{i=1}^{n-1} \frac{\partial^2 f}{\partial s_i \nu} (s_i \nu^T + \nu s_i^T) + \frac{\partial^2 f}{\partial \nu^2} \nu \nu^T.
\end{equation}$$
Similarly, $\Delta f$ is computed as

$$\Delta f = \nabla \cdot \nabla f = \nabla \cdot \left( \sum_{i=1}^{n-1} s_i \frac{\partial f}{\partial s_i} + \nu \frac{\partial f}{\partial \nu} \right)$$

$$= (\nabla \cdot \nu) \frac{\partial f}{\partial \nu} + \nu \cdot \nabla \frac{\partial f}{\partial \nu} + \sum_{i=1}^{n-1} \left( (\nabla \cdot s_i) \frac{\partial f}{\partial s_i} + s_i \cdot \nabla \frac{\partial f}{\partial s_i} \right)$$

(3.10)

$$= (\nabla \cdot \nu) \frac{\partial f}{\partial \nu} + \frac{\partial^2 f}{\partial \nu^2} + \sum_{i=1}^{n-1} \left( (\nabla \cdot s_i) \frac{\partial f}{\partial s_i} + \frac{\partial^2 f}{\partial s_i^2} \right).$$

Since the Green function $G$ satisfies $G(x, y) = 0$ on $\partial \Omega$, we have

$$\frac{\partial}{\partial s_i} G(x, y) = \frac{\partial^2}{\partial s_i \partial s_j} G(x, y) = 0,$$

$i, j = 1, \ldots, n - 1.$

and, thus,

$$0 = \Delta_w G(w, y) = (\nabla \cdot \nu) \frac{\partial}{\partial \nu_w} G(w, y) + \frac{\partial^2}{\partial \nu_w^2} G(w, y), \quad w \in \partial \Omega.$$

Applying these results to computation of $\mathcal{H}_w G(w, y) \cdot S^2$ with $S = \sum_{i=1}^{n-1} \mu_i s_i + \delta \rho \nu$, we obtain

$$\mathcal{H}_w G(w, y) \cdot (S)^2 = 2 \delta \rho \sum_{i=1}^{n-1} \mu_i (D\nu)^{s_i} G(w, y) - (\nabla \cdot \nu)(\delta \rho)^2 \frac{\partial}{\partial \nu_w} G(w, y)$$

$$+ \left( \sum_{i=1}^{n-1} \mu_i s_i^T (D\nu) s_j + 2 \sum_{i=1}^{n-1} \mu_i \delta \rho s_i^T (D\nu) \nu + (\delta \rho)^2 \nu^T (D\nu) \nu \right) \frac{\partial}{\partial \nu_w} G(w, y)$$

$$= 2 \delta \rho \sum_{i=1}^{n-1} \mu_i \frac{\partial^2}{\partial s_i \partial \nu_w} G(w, y) + \left( \sum_{i=1}^{n-1} \kappa_i \left( \mu_i^2 - (\delta \rho)^2 \right) \right) \frac{\partial}{\partial \nu_w} G(w, y).$$

Here, we use the fact $\nabla \nu = \sum_{i=1}^{n-1} \kappa_i s_i s_i^T$, $\nabla \cdot \nu = \text{tr} (\nabla \nu) = \sum_{i=1}^{n-1} \kappa_i$, where $\kappa_i$ is the curvature of the cross-section of $(n-1)$-dimensional surface $\partial \Omega$ by a two-dimensional plane defined by $s_i$ and $\nu$. Therefore, the third term $Z$ of the right-hand side of (3.5) is written by

$$Z = \int_{\partial \Omega} \sum_{i=1}^{n-1} \kappa_i \left( \mu_i^2 - (\delta \rho)^2 \right) \frac{\partial}{\partial \nu_w} G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w$$

$$+ 2 \int_{\partial \Omega} \sum_{i=1}^{n-1} \mu_i \delta \rho \frac{\partial^2}{\partial s_i \partial \nu_w} G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w.$$

Noticing (3.4), we see the equality

$$\frac{\partial}{\partial s_i} \delta G(w, y) + \delta \rho \frac{\partial^2}{\partial s_i \partial \nu_w} G(w, y) = - \frac{\partial (\delta \rho)}{\partial s_i} \frac{\partial}{\partial \nu_w} G(w, y).$$

Computing $X + Z$ using this equality, we find

$$X + Z = -2 \int_{\Omega} \nabla_w \delta G(w, y) \nabla_w \delta G(x, w) ds_w$$

$$+ \int_{\partial \Omega} \sum_{i=1}^{n-1} \left[ \kappa_i \left( \mu_i^2 - (\delta \rho)^2 \right) - 2 \mu_i \frac{\partial (\delta \rho)}{\partial s_i} \right] \frac{\partial}{\partial \nu_w} G(w, y) \frac{\partial}{\partial \nu_x} G(x, w) ds_w.$$

(3.11)
Gathering (3.5), (3.6), and (3.11) we obtain
\[
\delta^2 G(x, y) = -2 \int_{\Omega} \nabla_{w} \delta G(w, y) \nabla_{w} \delta G(x, w) ds_{w} + \int_{\partial \Omega} \frac{\partial}{\partial \nu_{w}} G(w, y) \frac{\partial}{\partial \nu_{x}} G(x, w) ds_{w},
\]
\[\chi := \delta^2 \rho + \sum_{i=1}^{n-1} \left[ \kappa_{i} \left( \mu_{i}^2 - (\delta \rho)^2 \right) - 2 \mu_{i} \frac{\partial (\delta \rho)}{\partial s_{i}} \right].\]

We further try to simplify \( \chi \). At first, since \( \frac{\partial^2}{\partial \nu_{w}} = \kappa_{i} s_{i} \), we notice
\[
\sum_{i=1}^{n-1} \kappa_{i} \mu_{i}^2 = \sum_{i=1}^{n-1} \kappa_{i} (S \cdot s_{i})^2 = \sum_{i=1}^{n-1} S \cdot s_{i} \left( \frac{\partial (\delta \rho)}{\partial s_{i}} - \left( \frac{\partial S}{\partial s_{i}} \cdot \nu \right) \right).
\]
Thus, recalling from (3.7) that \( DS = \sum_{n=1}^{n-1} \frac{\partial S}{\partial s_{i}} s_{i}^T + \frac{\partial S}{\partial \nu} \nu^T \), we see
\[
\sum_{i=1}^{n-1} S \cdot s_{i} \frac{\partial (\delta \rho)}{\partial s_{i}} = (S \cdot \nabla) \delta \rho - \delta \rho \frac{\partial (\delta \rho)}{\partial \nu} = (S \cdot \nabla) \delta \rho - \frac{1}{2} \frac{\partial (\delta \rho)^2}{\partial \nu}.
\]
Here, we use the facts \( \delta^2 \rho = \nu^T (DS) S \) by (2.14) and \( \frac{\partial S}{\partial \nu} = 0 \). Similarly, since \( S \cdot \nabla = \sum_{n=1}^{n-1} (S \cdot s_{i})_{T} s_{i}^T + (S \cdot \nu)_{T} \nu^T \) by (3.7), we have
\[
\sum_{i=1}^{n-1} S \cdot s_{i} \frac{\partial (\delta \rho)}{\partial s_{i}} = (S \cdot \nabla) \delta \rho - \delta \rho \frac{\partial (\delta \rho)}{\partial \nu} = (S \cdot \nabla) \delta \rho - \frac{1}{2} \frac{\partial (\delta \rho)^2}{\partial \nu}.
\]
Letting \( \bar{\kappa} := \sum_{i=1}^{n-1} \kappa_{i} \), \( \chi \) is rewritten as
\[
\chi = -\bar{\kappa} (\delta \rho)^2 - (S \cdot \nabla) \delta \rho + \frac{\partial (\delta \rho)^2}{\partial \nu}.
\]
If the domain perturbation \( T_{t}(x) \) satisfies \( S \cdot s_{i} = 0 \), \( i = 1, \cdots, n - 1 \), it follows from (3.12), (3.13) that
\[
\delta^2 \rho = (S \cdot \nabla) \delta \rho = \frac{1}{2} \frac{\partial (\delta \rho)^2}{\partial \nu}.
\]
Therefore, in this case, we find
\[
\chi = -\bar{\kappa} (\delta \rho)^2 + \delta^2 \rho.
\]

So far, computation has been done under smoothness assumptions. A usual density argument yields the following theorem:

**Theorem 3.3 (Hadamard's second variational formula)** Let \( \Omega \subset \mathbb{R}^{n} \) be a Lipschitz domain and \( T_{t}(x) \) be a \( W^{1,\infty} \) class domain perturbation on \( \Omega \). Then, the second variation \( \delta^2 G(w, y) \) of the Green function \( G(w, y) \) of Laplacian on \( \Omega \) is written by
\[
\delta^2 G(x, y) = \left\langle \chi \frac{\partial}{\partial \nu} G(x, \cdot), \frac{\partial}{\partial \nu} G(\cdot, y) \right\rangle_{\partial \Omega} - 2 \langle \nabla \delta G(x, \cdot), \nabla \delta G(\cdot, y) \rangle_{\partial \Omega},
\]
\[
\chi = -\bar{\kappa} (\delta \rho)^2 - (S \cdot \nabla) \delta \rho + \frac{\partial (\delta \rho)^2}{\partial \nu}, \quad \delta \rho := S \cdot \nu, \quad \bar{\kappa} := \sum_{i=1}^{n-1} \kappa_{i}.
\]
where \( \{s_i\}_{i=1}^{n-1} \) is an orthonormal basis of the tangent space of \( \partial\Omega \) and \( \kappa_i \) is the curvature of \( \partial\Omega \) along \( s_i \). In particular, if the perturbation satisfies \( S \cdot s_i = 0, i = 1, \cdots, n - 1 \), we have

\[
\delta^2 G(x, y) = \left( (\delta^2 \rho - \kappa(\delta \rho)^2) \frac{\partial}{\partial \nu} G(x, \cdot), \frac{\partial}{\partial \nu} G(\cdot, y) \right)_{\partial \Omega} - 2 (\nabla \delta G(x, \cdot), \nabla \delta G(\cdot, y))_\Omega.
\]

Remark: Garabedian and Schiffer [2] dealt with domain perturbation such as

\[ T_t(x) = x + t h(x) \nu(x), \quad t \geq 0, \]

where \( h(x) \) is a scalar function defined on \( \partial\Omega \). In this case, Hadamard’s second variational formula is (3.15) with \( \delta \rho = h \) and \( \delta^2 \rho(x) = 0 \) which is exactly same to Garabedian-Schiffer’s formula [2]. Therefore, Theorem 3.3 is an extension of Garabedian-Schiffer’s formula.

References


