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An extension of CVaR using non-precise a-priori distribution and its application

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1 Introduction and notation

We consider a stochastic model $X \sim f(\cdot|\theta), \theta \in \Theta$ with continuous parameter space Θ, a-priori density function $\pi(\theta)$, where $X \in L^1(\Omega, \mathcal{A}, P)$ on some probability space $(\Omega, \mathcal{A}, P)$. In classical statistical methods, we usually use only one density function to apply maximum likelihood estimation or Bayesian estimation. However, Viertl and Hareter [9] pointed out that this setting is insufficient because precise a-priori distributions are questionable concerning their justification. This is why they proposed non-precise a-priori densities $\tilde{\pi}(\theta)$ whose precise definition is given by Definition 1.

On the other hand, in recent years some risk measures have been generated and analyzed by an economically motivated optimization problem, for example, value at risk($V@R$), conditional value-at-risk ($CV@R$) [7], convex risk of measure [3] and so on. In particular $CV@R$ is a very useful and important criterion when dealing with real problems, see [4, 5, 8]. In this paper we propose a fuzzy conditional value-at-risk $\widetilde{CV@R}$ using a non-precise density $\tilde{\pi}$ in order to handle risk more flexibly. As far as we know, Yoshida [10] firstly investigated risk measures under fuzzy environment. However prior distributions were not considered in this paper. We believe that introducing prior distributions for risk analysis is crucial in order to analyze more practical problems.

In the reminder of this section we outline fundamental setting of fuzzy theory and fuzzy integral. Let $\mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ be the sets of real numbers, real n-dimensional column vectors and real $m \times n$

matrices, respectively. Let $\mathcal{B}(\mathbb{R})$ be all Borel sets on $\mathbb{R}$. The sets $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ are endowed with the norm $\|\cdot\|$, where for $x = (x(1),\ldots,x(n)) \in \mathbb{R}^n$, $\|x\| = \sum_{j=1}^{n} |x(j)|$ and for $y = (y_{ij}) \in \mathbb{R}^{m\times n}$, $\|y\| = \max_{1\le i\le m} \sum_{j=1}^{n} |y_{ij}|$. For any set X, let $\mathcal{F}(X)$ be the set of all fuzzy sets $X \to [0,1]$. The α-cut of $\tilde{x} \in \mathcal{F}(X)$ is given by $\tilde{x}_\alpha := \{x \in X | \tilde{x}(x) \ge \alpha\} (\alpha \in (0,1])$ and $\tilde{x}_0 := \mathrm{cl}\{x \in X | \tilde{x}(x) > 0\}$, where cl is a closure of a set. Let $\tilde{\mathbb{R}}$ be the set of all fuzzy numbers, i.e., $\tilde{r} \in \tilde{\mathbb{R}}$ means that $\tilde{r} \in \mathcal{F}(\mathbb{R})$ is normal, upper semi-continuous and fuzzy convex and has a compact support. Let $\mathcal{C}$ be the set of all bounded and closed intervals of $\mathbb{R}$. Then, for $\tilde{r} \in \mathcal{F}(\mathbb{R})$, it holds that $\tilde{r} \in \tilde{\mathbb{R}}$ if and only if $\tilde{r}$ normal and $\tilde{r}_\alpha \in \mathcal{C}$ for $\alpha \in [0,1]$. So, for $\tilde{r} \in \tilde{\mathbb{R}}$, we write $\tilde{r}_\alpha = [\tilde{r}_\alpha^-, \tilde{r}_\alpha^+] (\alpha \in [0,1])$. We use the extension principle [2] by Zadeh to define arithmetics with fuzzy numbers and fuzzy functions $\tilde{f}(x) \in \tilde{\mathbb{R}}$ for each $x \in \mathbb{R}$, respectively. Here, we will give a partial order $\preccurlyeq$ on $\mathcal{C}$ by the definition:For $[a,b],[c,d] \in \mathcal{C}$,

$$\begin{aligned} &[a,b] \preccurlyeq [c,d] \quad \text{if} \quad a \le c \quad \text{and} \quad b \le d, \\ &[a,b] \prec [c,d] \quad \text{if} \quad [a,b] \preccurlyeq [c,d] \quad \text{and} \quad [a,b] \ne [c,d]. \end{aligned}$$

This partial order $\preccurlyeq$ on $\mathcal{C}$ is extended to that of $\tilde{\mathbb{R}}$, called fuzzy max order,

$$\begin{aligned} &\tilde{u} \preccurlyeq \tilde{v} \quad \text{if} \quad \tilde{u}_\alpha \preccurlyeq \tilde{v}_\alpha \quad \text{for all} \quad \alpha \in [0,1], \\ &\tilde{u} \prec \tilde{v} \quad \text{if} \quad \tilde{u} \preccurlyeq \tilde{v} \quad \text{and} \quad \tilde{u} \ne \tilde{v}. \end{aligned}$$

Also, as a further extension, the partial order for fuzzy functions [9] can be defined similarly. The Hausdorff metric on $\mathcal{C}$ is denoted by δ, i.e.,

$$\delta([a,b],[c,d]) = |a-c| \vee |b-d| \quad \text{for} \quad [a,b],[c,d] \in \mathcal{C}.$$

This metric can be extended to $\tilde{\mathbb{R}}$ by

$$\delta(\tilde{u},\tilde{v}) = \sup_{\alpha\in[0,1]} \delta((\tilde{u})_\alpha,(\tilde{v})_\alpha)$$

for $\tilde{u},\tilde{v} \in \tilde{\mathbb{R}}$. Then, it is known that the metric space $(\tilde{\mathbb{R}}^n, \delta)$ is complete [1]. Referring to [9], we define fuzzy integral $\tilde{\mathfrak{F}}$ for the fuzzy functions $\tilde{f}$, non-precise densities $\tilde{\pi}$ and the fuzzy probability $\tilde{P}(A)$, respectively.

Definition 1. *([9]) Assume that $\tilde{f}_\alpha^-(x)$ and $\tilde{f}_\alpha^+(x)$ are integrable functions on $\mathbb{R}$. Then fuzzy integral is defined by*

$$\tilde{\mathfrak{F}} = (F)\int_a^b \tilde{f}(x)dx,$$

where $\tilde{\mathfrak{F}}_\alpha^- := \int_a^b \tilde{f}_\alpha^-(x)dx$ and $\tilde{\mathfrak{F}}_\alpha^+ := \int_a^b \tilde{f}_\alpha^+(x)dx$ for all $\alpha \in (0,1]$ and a condition for $\tilde{\pi}$ is

$$(F)\int_{\mathbb{R}} \tilde{\pi}(x)dx = \tilde{1}.$$

Definition 2. *([9]) Let $S_\alpha = \{f : f \text{ is a probability density s.t.} \tilde{\pi}_\alpha^-(x) \leq f(x) \leq \tilde{\pi}_\alpha^+(x) \text{ for all } x \in \mathbb{R}\}$. The $\alpha-$cut $[\tilde{P}_\alpha^-, \tilde{P}_\alpha^+]$ of the fuzzy probability $\tilde{P}(A)$ is defined by*

$$\begin{aligned}
\tilde{P}_\alpha^+(A) &= \sup_{f\in S_\alpha}\int_A f(x)dx \\
&= \begin{cases} 1 - \int_{A^c}\tilde{\pi}_\alpha^-(x)dx & \text{if } \int_A \tilde{\pi}_\alpha^+(x)dx + \int_{A^c}\tilde{\pi}_\alpha^-(x)dx > 1, \\ \int_A \tilde{\pi}_\alpha^+(x)dx & \text{else}, \end{cases} \\
\tilde{P}_\alpha^-(A) &= \inf_{f\in S_\alpha}\int_A f(x)dx \\
&= \begin{cases} \int_A \tilde{\pi}_\alpha^-(x)dx & \text{if } \int_A \tilde{\pi}_\alpha^-(x)dx + \int_{A^c}\tilde{\pi}_\alpha^+(x)dx > 1, \\ 1 - \int_{A^c}\tilde{\pi}_\alpha^+(x)dx & \text{else} \end{cases}
\end{aligned} \tag{1}$$

for $A \in \mathcal{B}(\mathbb{R})$.

We denote the calculation of fuzzy probability $\tilde{P}(A)$ by

$$\tilde{P}(A) = (FP)\int_A \tilde{\pi}(x)dx \quad \text{for } A \in \mathcal{B}(\mathbb{R}). \tag{2}$$

Recall that $X \sim f(\cdot|\theta), \theta \in \Theta$ with continuous parameter space Θ, a-priori density function $\pi(\theta)$. Then, a distribution function of X for a prior density $\pi(\theta)$ is given by

$$F_X(x|\pi) = \int_{-\infty}^x \int_\Theta \pi(\theta) f(y|\theta)d\theta dy. \tag{3}$$

In case of non-precise a-prior density $\tilde{\pi}(\theta)$,

$$\tilde{F}_X(x|\tilde{\pi}) = (FP)\int_{-\infty}^{x} \tilde{f}(y|\tilde{\pi})dy, \tag{4}$$

where $\tilde{f}(y|\tilde{\pi}) = (F)\int_{\Theta} \tilde{\pi}(\theta)f(y|\theta)d\theta$.

Theorem 1. *([6]) For any $\tilde{\pi}$, we have the following.*
(i)$\lim_{x\to\infty} \tilde{F}_X(x|\tilde{\pi}) = 1$, $\lim_{x\to-\infty} \tilde{F}_X(x|\tilde{\pi}) = 0$.
(ii)$\tilde{F}_X(x|\tilde{\pi}) \succeq \tilde{F}_X(y|\tilde{\pi})$ *if* $x \geq y$.
(iii) $\lim_{x\to y+0} \tilde{F}_X(x|\tilde{\pi}) = \tilde{F}_X(y|\tilde{\pi})$.

According to Zadeh's extension principle [2], we define the fuzzy value at risk $\widetilde{V@R}_\gamma(X|\tilde{\pi})(\gamma \in (0,1))$ and conditional value at risk $\widetilde{CV@R}_\gamma(X|\tilde{\pi})(\gamma \in (0,1))$. Let $V@R_\gamma(F) = \inf\{y|F(y) \geq \gamma\}(\gamma \in (0,1)), CV@R_\gamma(F) = \frac{1}{1-\gamma}\int_\gamma^1 V@R_p(F)dp(\gamma \in (0,1))$, respectively, where F are distribution functions.

Definition 3. *([6]) For a give $\tilde{\pi}$ and a density function $f(x|\theta)$ we define the fuzzy value at risk $\widetilde{V@R}_\gamma(X|\tilde{\pi})$ and conditional value at risk $\widetilde{CV@R}_\gamma(X|\tilde{\pi})(\gamma \in (0,1))$ as follows:*

$$\begin{aligned} \widetilde{V@R}_\gamma(X|\tilde{\pi})(x) &= \sup_{V@R_\gamma(F)=x} \inf_y \tilde{F}_X(y|\tilde{\pi})(F(y)), \\ \widetilde{CV@R}_\gamma(X|\tilde{\pi})(x) &= \sup_{CV@R_\gamma(F)=x} \inf_y \tilde{F}_X(y|\tilde{\pi})(F(y)). \end{aligned} \tag{5}$$

Lemma 1. *([6]) The $\alpha-$cut of the fuzzy value at risk $\widetilde{V@R}_\gamma(X|\tilde{\pi})$ and conditional value at risk $\widetilde{CV@R}_\gamma(X|\tilde{\pi})(\gamma \in (0,1))$ are given by*

$$\begin{aligned} \widetilde{V@R}^{+}_{\gamma,\alpha}(X|\tilde{\pi}) &= \inf\{x|\tilde{F}^{-}_{X,\alpha}(x|\tilde{\pi}) \geq \gamma\}, \\ \widetilde{V@R}^{-}_{\gamma,\alpha}(X|\tilde{\pi}) &= \inf\{x|\tilde{F}^{+}_{X,\alpha}(x|\tilde{\pi}) \geq \gamma\}, \\ \widetilde{CV@R}^{+}_{\gamma,\alpha}(X|\tilde{\pi}) &= \frac{1}{1-\gamma}\int_\gamma^1 \widetilde{V@R}^{+}_{p,\alpha}(X|\tilde{\pi})dp, \\ \widetilde{CV@R}^{-}_{\gamma,\alpha}(X|\tilde{\pi}) &= \frac{1}{1-\gamma}\int_\gamma^1 \widetilde{V@R}^{-}_{p,\alpha}(X|\tilde{\pi})dp. \end{aligned} \tag{6}$$

2 Properties of $\widetilde{CV@R}$

Proposition 1. *([6]) For any random variables X, Y and $\tilde{\pi}$, $\widetilde{CV@R}_\gamma$ has the following* (i)-(iv)*:*
(i) *(Monotonicity) If $X \le Y$, $\widetilde{CV@R}_\gamma(X|\tilde{\pi}) \preceq \widetilde{CV@R}_\gamma(Y|\tilde{\pi})$.*
(ii) *(Translation invariance) For X and $c \in \mathbb{R}$, $\widetilde{CV@R}_\gamma(X+c|\tilde{\pi}) = \widetilde{CV@R}_\gamma(X|\tilde{\pi}) + c$.*
(iii) *(Homogeneity) For X and $\lambda > 0$, $\widetilde{CV@R}_\gamma(\lambda X|\tilde{\pi}) = \lambda \widetilde{CV@R}_\gamma(X|\tilde{\pi})$.*
(iv) *(Convexity) For X, Y and $0 \le \lambda \le 1$, $\widetilde{CV@R}_\gamma(\lambda X + (1-\lambda)Y|\tilde{\pi}) \preceq \lambda\widetilde{CV@R}_\gamma(X|\tilde{\pi}) + (1-\lambda)\widetilde{CV@R}_\gamma(Y|\tilde{\pi})$.*

Definition 4. *([6]) For any $\tilde{\pi}$, the acceptance set of $\widetilde{CV@R}_\gamma$ is defined by*

$$\mathfrak{A}_{\widetilde{CV@R}_\gamma} := \{X|\widetilde{CV@R}_\gamma(X|\tilde{\pi}) \preceq \tilde{0}\}. \tag{7}$$

Proposition 2. *([6]) Let $\mathfrak{A} := \mathfrak{A}_{\widetilde{CV@R}_\gamma}$. Then,*
(i) *For $X \in \mathfrak{A}$, if it holds that $Y \le X$, $Y \in \mathfrak{A}$.*
(ii) *$\mathfrak{A}$ is convex cone.*

3 A Numerical Example

Recall the example by Viertl [9]. Let $\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta)$ be the density of a classical gamma distribution $\gamma(2, \frac{1}{4})$, i.e.

$$\tilde{\pi}_1^-(\theta) = \tilde{\pi}_1^+(\theta) = 4^2 \times \theta \times e^{-4\theta} \times I_{[0,\infty)}(\theta)$$

and we consider triangle fuzzy numbers for each $\theta \in \Theta$, that is,

$$\tilde{\pi}_\alpha(\theta) = \left[\frac{(\alpha+1)\tilde{\pi}_1^-(\theta)}{2}, \frac{(3-\alpha)\tilde{\pi}_1^+(\theta)}{2}\right] \text{ for each } \theta \in \Theta.$$

Also, $f(x|\theta)$ is the density of the exponential distribution, i.e.

$$f(x|\theta) = \theta e^{-\theta x} \qquad x \ge 0, \quad x \in \mathbb{R}.$$

From (1), we can get the following for each α.

$$\begin{aligned}\tilde{F}^{+}_{X,\alpha}(y|\tilde{\pi}) &= \begin{cases} 1-\frac{8(1+\alpha)}{(4+y)^2} & \text{if } y > 4\sqrt{2}-4, \\ (3-\alpha)(\frac{1}{2}-\frac{8}{(4+y)^2}) & \text{else,} \end{cases} \\ \tilde{F}^{-}_{X,\alpha}(y|\tilde{\pi}) &= \begin{cases} (1+\alpha)(\frac{1}{2}-\frac{8}{(4+y)^2}) & \text{if } 0 \le y < 4\sqrt{2}-4, \\ 1-\frac{8(3-\alpha)}{(4+y)^2} & \text{else.} \end{cases}\end{aligned} \tag{8}$$

Here, let $\gamma = 0.99$, then we have $\widetilde{V@R}^{+}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{\sqrt{8(3-\alpha)}}{\sqrt{1-0.99}} - 4$, $\widetilde{V@R}^{-}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{\sqrt{8(\alpha+1)}}{\sqrt{1-0.99}} - 4$, $\widetilde{CV@R}^{+}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{4\sqrt{2(3-\alpha)(1-0.99)}}{1-0.99} - 4$ and $\widetilde{CV@R}^{-}_{0.99,\alpha}(X|\tilde{\pi}) = \frac{4\sqrt{2(1+\alpha)(1-0.99)}}{1-0.99} - 4$ from Lemma 1. Fig.1 and 2 show $\widetilde{V@R}_{0.99}$ and $\widetilde{CV@R}_{0.99}$, respectively.

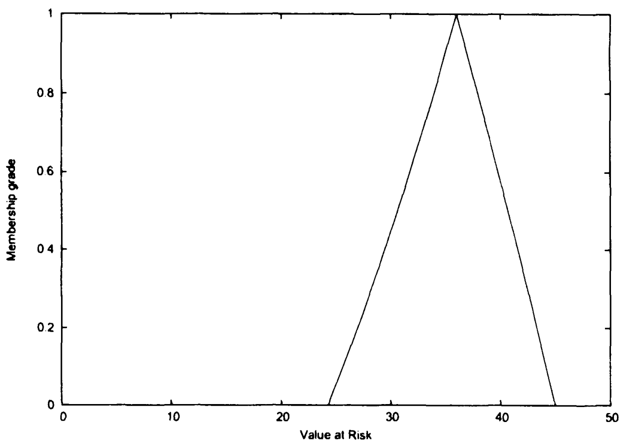

Fig.1. Representation of $\widetilde{V@R}_{0.99}$

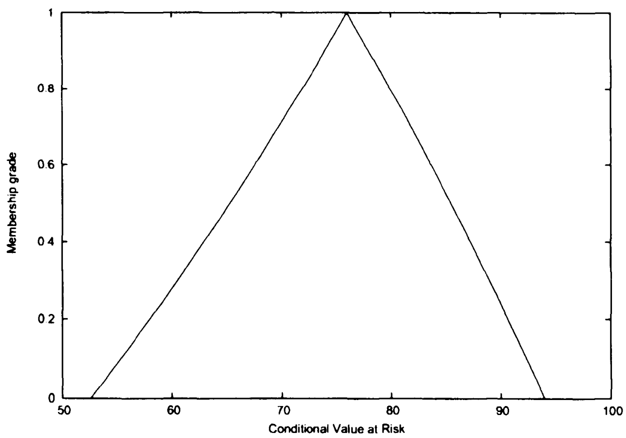

Fig.2. Representation of $\widetilde{CV@R}_{0.99}$

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