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Maximizing the Probability of Stopping on Any of the Last m Successes in Bernoulli Trials with Random Horizon

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1 Introduction and summary

Before considering our problems, we review the Sum-the-Multiplicative-Odds Theorem and the secretary problem briefly. Let n be a given positive integer, and suppose that n independent Bernoulli trials are performed one at a time, each of which results in a *success* or a *failure.* That is, if we let X_j equal 1 if the jth trial is a success and 0 if it is a failure, then $X_1, X_2, \ldots, X_n$ are independent Bernoulli random variables that are observed sequentially. When we seek an optimal stopping rule of this sequential observation problem with the objective of maximizing the probability of stopping on any of the last m successes for a predetermined m (we assume $n > m$, because, for $n \leq m$, the optimal rule evidently stops on the first success), the following theorem gives a solution if we let $a_j = P\{X_j = 1\}, b_j = 1 - a_j$ and $r_j = a_j/b_j$, and define, for $j \geq 1$ and $k \geq 0$,

$$R_{k,i,j} = \sum_{k<t_1<t_2<\cdots<t_j\leq i} r_{t_1} r_{t_2} \cdots r_{t_j} \tag{1}$$

for $k + j \leq i \leq n$ and $R_{k,i,j} = 0$ for $k + j > i$.

Sum-the-Multiplicative-Odds Theorem (abbreviated to STMOT). *For the problem of maximizing the probability of stopping on any of the last m successes in n independent Bernoulli trials, the optimal rule stops on the first success $X_k = 1$ with $k \geq s_m(n)$, if any, where*

$$s_m(n) = \min\{k \geq 1 : R_{k,n,m} \leq 1\}. \tag{2}$$

Moreover, the maximal probability of win (i.e. achieving the objective) is

$$u_m(n) = \left(\prod_{j=s_m(n)}^{n} b_j \right) \left(\sum_{j=1}^{m} R_{s_m(n)-1,n,j} \right). \tag{3}$$

See Tamaki (2010) for the STMOT. When $m = 1$, the STMOT is referred to as the Sum-the-Odds Theorem, which was obtained by Bruss (2000) and extended into several directions by Ferguson (2008) later. See also Hill and Krengel (1992), Hsiau and Yang (2002), Bruss and Paindaveine (2000), and Bruss and Louchard (2009) for related works.

An interesting application of the STMOT appears in the *secretary problem* described as follows: A known number n of rankable applicants (1 being the best and n the worst) appear one at a time in random order with all $n!$ permutations equally likely. That is, each of the successive ranks of n applicants constitutes a random permutation. Suppose that all that can be observed are the relative ranks of the applicants as they appear. If Y_j denotes the relative rank of the jth applicant among the first j applicants, the sequentially observed random variables are $Y_1, Y_2, \ldots, Y_n$. It is well known that

(a) $Y_1, Y_2, \ldots, Y_n$ are independent random variables;
(b) $P\{Y_j = i\} = 1/j, 1 \le i \le j, 1 \le j \le n.$

The jth applicant is called a *candidate* if he/she is relatively best, i.e. $Y_j = 1$. If the objective is to stop on any of the last m successes, that is, any of the last m candidates (stopping is identified with selection of an applicant in the secretary problem), independent Bernoulli random variables $X_1, X_2, \ldots, X_n$ are specified by $X_j = I(Y_j = 1)$, where $I(E)$ is the indicator function of an event E, so the STMOT gives the solution $s_m(n)$ and $u_m(n)$ corresponding to $a_j = 1/j$. In particular, as n tends to infinity, we have

$$s_m^* = \lim_{n\to\infty} \frac{s_m(n)}{n} = \exp\left\{-(m!)^{1/m}\right\} \tag{4}$$

and

$$u_m^* = \lim_{n\to\infty} u_m(n) = \exp\left\{-(m!)^{1/m}\right\} \sum_{j=1}^{m} \frac{(m!)^{j/m}}{j!}. \tag{5}$$

See Lemma 3.2 and Table 1 of Tamaki (2010) for (4), (5) and their numerical values. The secretary problem with $m = 1$, referred to as the *best-choice problem* (because the last candidate is best overall), gives the well known result $s_1^* = u_1^* = e^{-1}$. The reader is referred to Ferguson (1989) and Samuels (1991) for reviews of the secretary problem.

In Section 2, the STMOT is extended to allow for a random horizon of length N. That is, N represents the random number of Bernoulli trials to be performed, and is assumed to be a bounded random variable that is also independent of Bernoulli trials. A prior distribution will be given for N. A stopping rule is said to be *threshold* if it only stops on the first success appearing after a given stage. In particular, the optimal rule, as descrbed in the STMOT, is said to be a threshold rule with value $s_m(n)$. It is known that, for a random N, the optimal rule is not necessarily threshold (see, e.g. Section 3 of Petruccelli (1983)). Hence our main concern in Section 2 is to give a simple sufficient condition for the optimal rule to be threshold.

An application of this condition again appears in the secretary problem (i.e. $a_j = 1/j$) with a random number N of applicants. In particular, for the problem with N uniform on $\{1, 2, \ldots, n\}$, the optimal rule will be shown to be threshold with value $t_m(n)$ having the limiting property

$$t_m^* \quad = \quad \lim_{n\to\infty} \frac{t_m(n)}{n} = \exp\left\{-[(m+1)!]^{1/m}\right\}. \tag{6}$$

The corresponding probability of win $v_m(n)$ also has the limit

$$v_m^* \quad = \quad \lim_{n\to\infty} v_m(n) = \exp\left\{-[(m+1)!]^{1/m}\right\} \sum_{j=1}^{m} \frac{[(m+1)!]^{(j+1)/m}}{(j+1)!}. \tag{7}$$

See Lemma 3.1 and Table 1 in Section 3 for (6), (7) and their numerical values. We see $t_1^* = e^{-2}$ and $v_1^* = 2e^{-2}$, which coincides with the result derived by Presman and Sonin (1972) who is the first to study the best-choice problem with a random number of applicants. See also Irle (1980) and Petruccelli (1983). It may be interestng to compare (t_m^*, v_m^*) to (s_m^*, u_m^*) as two extremes. A generalized uniform prior that can be a bridge between (s_m^*, u_m^*) and (t_m^*, v_m^*) will be also discussed. In addition to the uniform prior, a curtailed geometric prior is also examined in detail. See also Samuel-Cahn (1995) for a best-choice problem with random freeze.

2 Bernoulli trials with random horizon

For ease of description, let, for a given prior $\{p_k, 1 \leq k \leq n\}$,

$$\pi_k = P\{N \geq k\} = p_k + p_{k+1} + \cdots + p_n, \qquad 1 \leq k \leq n$$

with $\pi_1 = 1$ and $\pi_n > 0$ (π_0 is interpreted as 1 if it appears). Write $b_i = 1 - a_i$ and $r_i = a_i/b_i$ as before, and let

$$B_{k,i} = b_{k+1}b_{k+2}\cdots b_i, \qquad 0 \leq k < i \leq n$$

with $B_{k,k} = 1$ for convenience. The proofs will be omitted due to space restriction.

Lemma 2.2. *Whatever the distribution of N might be, the first time the optimal rule will stop on a success occurs no later than the $s_m(n)$th trial, where $s_m(n)$ is as defined by (2) in the STMOT. Moreover, the optimal rule stops on the first success among trials $s_m(n), s_m(n)+1, \ldots, n$ if stopping has not occurred previously.*

Theorem 2.3. *Let*

$$Q_j(k) = \sum_{i=k+j}^{n} (B_{k,i}R_{k,i,j})\,\frac{p_i}{\pi_k}, \qquad k + j \leq n$$

with $Q_j(k) = 0, k + j > n$, and

$$t_m(n) \quad = \quad \min\{j : Q_0(k) - Q_m(k) \geq 0 \text{ for all } j \leq k \leq n - m\}$$

with $\min\{\phi\} = n - m + 1$. *Then a necessary and sufficient condition for the optimal rule to be threshold with value $t_m(n)$ is that, for all $1 \leq k < t_m(n) - 1$ (if such k exists),*

$$\pi_k \sum_{j=0}^{m-1} Q_j(k) < \pi_{t_m(n)-1} \sum_{j=1}^{m} Q_j(t_m(n) - 1).$$

Let

$$G(k) = p_k - r_{k+1} \sum_{i=k+m}^{n} B_{k,i}R_{k+1,i,m-1}p_i.$$

Then

Theorem 2.6. *A sufficient condition for the optimal rule to be threshold is that $G(k)$ changes its sign from $-$ to $+$ at most once before $s_m(n)$, that is,*

$$\text{once } G(k) \geq 0 \text{ for some } k, \text{ then } G(j) \geq 0 \text{ for all } k \leq j < s_m(n).$$

For the purposes of most applications to the secretary problem, the following corollary is useful.

Corollary 2.7. *For the secretary problem with $p_k > 0$ for all $1 \leq k \leq n$, a sufficient condition for the optimal rule to be threshold is that*

$$p_{k+j}/p_k \text{ is non} - \text{increasing in } k(< s_m(n))$$

for each possible value of j.

The optimal rules for the following examples are threshold.

Example 1. Let N take only on the value greater than $s_m(n) - 1$, i.e. $p_1 = p_2 = \cdots = p_{s_m(n)-1} = 0$.

Example 2 (secretary problem with fixed population size).

Example 3 (uniform prior). Let N be a uniform random variable on $\{1, 2, \ldots, n\}$, i.e. $p_k = 1/n, 1 \leq k \leq n$. Then $p_{k+j}/p_k = 1$.

Example 4 (geometric prior). Let N be a curtailed geometric random variable, i.e. $p_k = (1-q)q^{k-1}/(1-q^n), 1 \leq k \leq n$ for a given parameter $0 < q < 1$.

Example 5 (Poisson prior). Let N be a curtailed Poisson random variable, i.e. $p_k = e^{-\lambda}\frac{\lambda^k}{k!}/\sum_{j=1}^n e^{-\lambda}\frac{\lambda^j}{j!}, 1 \leq k \leq n$ for a given parameter $0 < \lambda$.

Example 6 (binomial prior). Let N be a curtailed binomial random variable, i.e. $p_k = \binom{n}{k}p^k(1-p)^{n-k}/(1-(1-p)^n), 1 \leq k \leq n$ for a given parameter $0 < p < 1$.

Example 7 (generalized uniform prior). Let N be a uniform random variable on $\{T, T+1, \ldots, n\}$ for a given parameter $T = 1, 2, \ldots, n$, i.e.

$$p_k = \begin{cases} 0, & \text{if } 1 \leq k \leq T-1 \\ \frac{1}{n-T+1}, & \text{if } T \leq k \leq n. \end{cases}$$

3 Asymptotic results for the secretary problem

Lemma 3.1.(Uniform distribution.) *Let n tend to ∞ for a uniform prior given in Example 3. Then, we have (6) and (7) asymptotically.*

Lemma 3.2.(Geometric distribution.) *Let q depend on n through $q = 1-c/n$ for a fixed positive value $c(< n)$ and define $t_{m,c} = \lim_{n\to\infty} t_m(n)/n$. Then $t_{m,c}$ is a unique root z of the equation*

$$\int_1^{1/z} \frac{e^{-czx}}{x}\left[1 - \frac{(\log x)^m}{m!}\right] dx = 0.$$

Moreover, as $n \to \infty$, the optimal probability tends to

$$v_{m,c} = \frac{ct}{1-e^{-c}} \int_1^{1/t} \frac{e^{-ctx}}{x}\left[\sum_{j=1}^{m} \frac{(\log x)^j}{j!}\right] dx,$$

where $t = t_{m,c}$. See Table 2 for some values of $t_{m,c}$ and $v_{m,c}$.

Lemma 3.3.(Generalized uniform distribution.) *Let $t_{m,\alpha}$ denote the optimal threshold value and $v_{m,\alpha}$ the optimal probability ($t_{m,0}$ and $v_{m,0}$ are already given as t_m^* in (6) and v_m^* in (7) respectively). Two cases are distinguished according to $\alpha \leq t_m^*$ or $\alpha > t_m^*$.*

Case (i): $0 \leq \alpha \leq t_m^*$

$$\begin{aligned} t_{m,\alpha} &= t_m^*, \\ v_{m,\alpha} &= \frac{v_m^*}{1-\alpha}. \end{aligned}$$

Case (ii): $t_m^* < \alpha < 1$

$t_{m,\alpha}$ is a unique root $z \in (0, \alpha)$ of the equation

$$\{\log(1/z)\}^{m+1} - \{\log(\alpha/z)\}^{m+1} = (m+1)! \log(1/\alpha)$$

or, equivalently,

$$\sum_{j=0}^{m} \frac{(\log \alpha)^j (-\log z)^{m-j}}{(j+1)!(m-j)!} = 1.$$

Moreover,

$$v_{m,\alpha} = \frac{t_{m,\alpha}}{1-\alpha} \sum_{j=1}^{m} \frac{\{\log(1/t_{m,\alpha})\}^j - \{\log(\alpha/t_{m,\alpha})\}^j}{j!}.$$

In particular,

$$\lim_{\alpha \to 1} t_{m,\alpha} = s_m^*,$$
$$\lim_{\alpha \to 1} v_{m,\alpha} = u_m^*,$$

where s_m^* and u_m^* are as given in (4) and (5) respectively.

For $m = 1$ and $m = 2$, we can give explicit expressions for $t_{m,\alpha}$ and $v_{m,\alpha}$.

Table 1

Values of t_m^* and v_m^* for several m

m	1	2	3	4	5	10
t_m^*	0.1353	0.0863	0.0559	0.0365	0.0240	0.0032
v_m^*	0.2707	0.4705	0.6172	0.7243	0.8020	0.9635

Table 2

Values of $t_{m,c}$ (upper) and $v_{m,c}$ (lower) for several pairs of (m, c)

	m				
c	1	2	3	4	5
0	0.1353	0.0863	0.0559	0.0365	0.0240
	0.2707	0.4705	0.6172	0.7243	0.8020
0.1	0.1317	0.0840	0.0543	0.0355	0.0234
	0.2689	0.4681	0.6149	0.7222	0.8004
1	0.1008	0.0643	0.0416	0.0272	0.0179
	0.2546	0.4494	0.5964	0.7059	0.7867
5	0.0346	0.0225	0.0148	0.0098	0.0065
	0.2337	0.4209	0.5671	0.6792	0.7641
10	0.0174	0.0113	0.0075	0.0049	0.0033
	0.2329	0.4196	0.5656	0.6778	0.7628
50	0.0035	0.0023	0.0015	0.0010	0.0007
	0.2329	0.4196	0.5656	0.6778	0.7628

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