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A digest: Constant maturity CDS and its rigorous valuation*

The title of the original paper is "Valuation of Constant Maturity Credit Default Swaps."

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1 Introduction

The purpose of this article is to develop a model for valuing constant maturity credit default swap (CMCDS), which is an extension of a vanilla credit default swap (CDS).

A vanilla CDS contract has a fixed premium leg and a contingent default leg. The fixed premium leg corresponds to the periodic payments made by the protection buyer to the seller until either the maturity of the CDS or the occurrence of a credit event, whichever comes first. The default leg corresponds to the net payment made by the protection seller to the buyer in case of default. The fair spread of a CDS is determined by equating the discounted cash flows of these two legs. The premium of a vanilla CDS is fixed throughout the contract, the premium of a CMCDS, however is reset periodically in reference to a prevailing market CDS spread with a specified fixed maturity (see Figure 1).

We derive the CMCDS pricing formula by specifying the stochastic processes followed by both the default intensity and the short rate within a reduced-form framework, different from the previous studies by Brigo [1] and Li [3].

Let me point out that this article digests the original paper [4]. However, we introduce an example other than what is studied in the original paper.

2 The Model and the Main Problem

Let $(\Omega, \mathcal{G}, Q)$ be a complete probability space, where Q is a risk-neutral measure. Denote by τ the default time of an issuer, which is a random time defined on the above space. We implicitly

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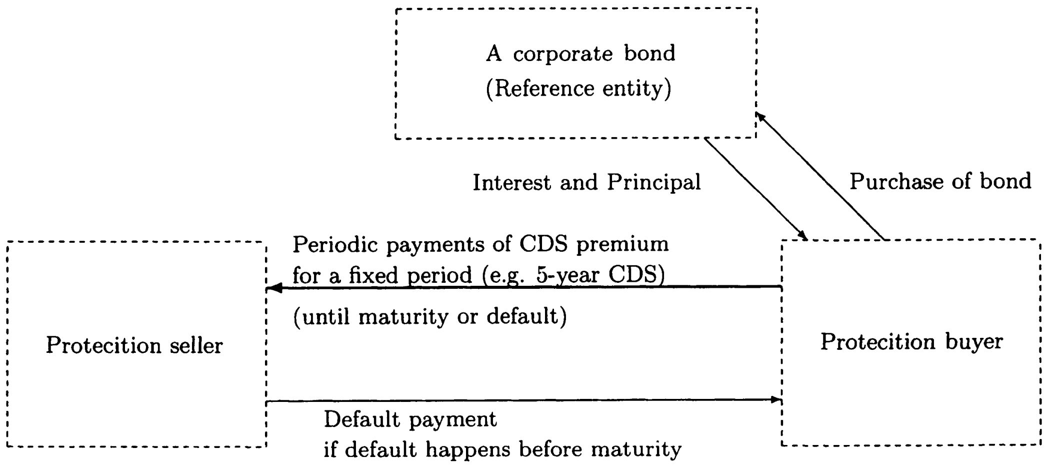

Figure 1: The typical structure of CMCDS. (This figure is not contained in the original paper.)

consider three different filtrations as follows. Let $(\mathcal{H}_t)$ be the filtration generated by only default time τ and $(\mathcal{F}_t)$ be the filtration that includes the market information up to time t except for the default time τ. Thus τ is not an $(\mathcal{F}_t)$-stopping time. Finally, another filtration $(\mathcal{G}_t)$ is defined by the smallest filtration that includes both $(\mathcal{H}_t)$ and $(\mathcal{F}_t)$.

Denote by r_t be the default-free short rate process that is $(\mathcal{F}_t)$-adapted and by $D(t,s)$ for $t \leq s$ the default-free discount factor from s to t:

$$D(t,s) := \exp\left(-\int_t^s r_u du\right).$$

The $(\mathcal{F}_t)$-survival process of τ which is specified by

$$Q(\tau > t|\mathcal{F}_t) = E^Q\left[I_{\{\tau>t\}}|\mathcal{F}_t\right] \qquad t \in \mathbb{R}_+$$

is supposed to satisfy $Q(\tau > t|\mathcal{F}_t) > 0$ for any $t \geq 0$ and is continuous in t.

We assume that the hazard rate process of τ exists. More specifically, there exists a nonnegative $(\mathcal{F}_t)$-progressively measurable process λ_t such that

$$\int_0^t \lambda_s ds = -\ln\left(Q(\tau > t|\mathcal{F}_t)\right) \quad \Leftrightarrow \quad Q(\tau > t|\mathcal{F}_t) = \exp\left(-\int_0^t \lambda_s ds\right).$$

Set $\Lambda_t := \displaystyle\int_0^t \lambda_s ds$

Next, we describe the set-up of CDS and CMCDS markets.

Let N denotes the notional amount of a contract. T_j denotes the j^{th} discrete premium-payment dates, $j = 1, 2, \cdots, n$ such that

- $T_1, \ldots, T_n$ are deterministic fixed times
- T denotes the maturity date of the CDS/CMCDS. $T = T_n$
- $\triangle_{j-1,j}$: the time increment between payment at the $(j-1)^{th}$ and j^{th} time point in units of years. For simplicity, assume $\triangle_{j-1,j} = \delta$ for all j.

Denote by $R \in [0,1)$ the recovery rate on the CDS/CMCDS upon default of the underlying obligor (assumed to be a constant) and by M the constant maturity defined in the floating premium leg of a CMCDS. Suppose $M = m\delta$.

Let $PV_t^{\mathrm{def}}\left(\mathrm{CDS}^{(T_0,T_n]}\right)$ be the present value at time t of the default (or protection) leg of the CDS. Also, let $PV_t^{\mathrm{prem}}\left(\mathrm{CDS}^{(T_0,T_n]}\right)$ be the present value at time t of the fixed premium leg of a CDS for the period $(T_0, T_n]$, and let $PV_t^{\mathrm{prem}}\left(\mathrm{CMCDS}^{(T_0,T_n]}; M\right)$ the present value at time t of a premium leg of the corresponding CMCDS that pays M-year CDS premium at each payment date for the period $(T_0, T_n]$.

We give the participation rate, which is practically used to express the value of CMCDS.

Definition 1 *The participation rate $\eta(t)$ $(= \eta(t; (T_0, T_n], M))$ applied to the premium leg of a CMCDS with protection against default in the period of $(T_0, T_n]$ is defined by the following equation:*

$$PV_t^{prem}\left(CDS^{(T_0,T_n]}\right) = PV_t^{prem}\left(CMCDS^{(T_0,T_n]}; M\right)\eta(t). \tag{2.1}$$

Now we can mention our main problem. The main problem is to obtain an explicit formula of $\eta(t)$ in terms of the hazard rate λ_t.

For the purpose, we need to model the premium of forward CDS contract and represent it in terms of the hazard rate and default-free interest rate.

For $t \le T_j < T_k$, let $s(t; T_j, T_k)$ be the premium of forward CDS contract, which has first payment at time T_{j+1} and last payment at time T_k, that makes the valuation fair at time t.

Remark that $s(t; T_j, T_k)$ is the $\mathcal{G}_t$-measurable variable and that $s(t; T_j, T_k) \equiv 0$ if $\tau \le t$.

The premium of forward CDS contract can be obtained by equating the present values of both premium leg and default leg of a forward CDS as follows.

Proposition 1 (The premium of forward CDS)

$$\begin{aligned} s(t; T_0, T_n) &= I_{\{\tau > t\}} \frac{(1-R)\, E^Q\left[\int_{T_0}^{T_n} D(t,u) e^{\Lambda_t - \Lambda_u} d\Lambda_u \middle| \mathcal{F}_t\right]}{\delta \sum_{j=1}^n E^Q\left[D(t,T_j) e^{\Lambda_t - \Lambda_{T_j}} | \mathcal{F}_t\right]} \\ &=: I_{\{\tau > t\}} \tilde{s}(t; T_0, T_n), \end{aligned}$$

where $\tilde{s}(t; T_0, T_n)$ is $\mathcal{F}_t$-measurable.

Indeed, the above result immediately follows from

$$\begin{aligned} PV_t^{\mathrm{prem}}\left(\mathrm{CDS}^{(T_0,T_n]}\right) &= E^Q\left[\sum_{j=1}^n s(t; T_0, T_n)\, N\delta D(t,T_j) I_{\{\tau > T_j\}} \middle| \mathcal{G}_t\right] \\ &= s(t; T_0, T_n)\, N\delta \sum_{j=1}^n E^Q\left[D(t,T_j) e^{\Lambda_t - \Lambda_{T_j}} | \mathcal{F}_t\right], \qquad (2.2) \\ PV_t^{\mathrm{def}}\left(\mathrm{CDS}^{(T_0,T_n]}\right) &= E^Q\left[(1-R)\, N D(t,\tau) I_{\{T_0 < \tau \le T_n\}} \middle| \mathcal{G}_t\right] \\ &= (1-R)\, N I_{\{\tau > t\}} E^Q\left[\int_{T_0}^{T_n} D(t,u) e^{\Lambda_t - \Lambda_u} d\Lambda_u \middle| \mathcal{F}_t\right]. \end{aligned}$$

The consequences are achieved via some well-known lemmas in the reduced-form approach of default risk.

Hereafter we will write $PV_t^{\text{prem}}\left(\text{CMCDS}^{(T_0,T_n]}\right)$ for $PV_t^{\text{prem}}\left(\text{CMCDS}^{(T_0,T_n]}; M\right)$.

Then, we also have

$$
\begin{aligned}
&PV_t^{\text{prem}}\left(\text{CMCDS}^{(T_0,T_n]}\right)\\
&= E^Q\left[\sum_{j=1}^{n} s\left(T_{j-1};T_{j-1},T_{j-1}+M\right) N\delta D(t,T_j) I_{\{\tau>T_j\}}\Big|\mathcal{G}_t\right]\\
&= N\delta I_{\{\tau>t\}}\sum_{j=1}^{n} E^Q\left[\tilde{s}\left(T_{j-1};T_{j-1},T_{j-1}+M\right) D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}|\mathcal{F}_t\right],
\end{aligned}
\tag{2.3}
$$

Substituting (2.2) and (2.3) into (2.1), we obtain the participation rate $\eta(t)$ as:

$$
\eta(t) = \frac{s\left(t;T_0,T_n\right)\displaystyle\sum_{j=1}^{n} E^Q\left[D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}|\mathcal{F}_t\right]}{\displaystyle\sum_{j=1}^{n} E^Q\left[\tilde{s}\left(T_{j-1};T_{j-1},T_{j-1}+M\right) D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}|\mathcal{F}_t\right]}. \tag{2.4}
$$

The denominator seems still complicated, but anyway the determination of $\eta(t)$ is reduced to the calculation of $E^Q[\tilde{s}\left(T_{j-1};T_{j-1},T_{j-1}+M\right) D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}|\mathcal{F}_t]$.

At last, we just mention the following lemma without proof.

Lemma 1 *Let $\tilde{s}\left(T_{j-1};T_{j-1},T_{j-1}+M\right)$ be the future time T_{j-1} credit spread that makes the M-year CDS contract fair at future time T_{j-1}. Then for any j, $j=1,\cdots,n$, we have*

$$
\begin{aligned}
&E^Q\left[\tilde{s}\left(T_{j-1};T_{j-1},T_{j-1}+M\right) D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}|\mathcal{F}_t\right]\\
&= \tilde{s}\left(t;T_{j-1},T_{j-1}+M\right)\frac{\sum_{k=j}^{j+m-1} E^Q\left[D(t,T_k)e^{\Lambda_t-\Lambda_{T_k}}|\mathcal{F}_t\right]}{E^Q\left[\int_{T_{j-1}}^{T_{j-1}+M} D(t,u)e^{\Lambda_t-\Lambda_u}d\Lambda_u\Big|\mathcal{F}_t\right]}\\
&\times E^Q\left[\frac{E^Q\left[\int_{T_{j-1}}^{T_{j-1}+M} D(T_{j-1},u)e^{\Lambda_{T_{j-1}}-\Lambda_u}d\Lambda_u\Big|\mathcal{F}_{T_{j-1}}\right]}{\sum_{k=j}^{j+m-1} E^Q\left[D(T_{j-1},T_k)e^{\Lambda_{T_{j-1}}-\Lambda_{T_k}}|\mathcal{F}_{T_{j-1}}\right]} D(t,T_j)e^{\Lambda_t-\Lambda_{T_j}}\Bigg|\mathcal{F}_t\right].
\end{aligned}
\tag{2.5}
$$

Further discussion about analysis based on two-factor Gaussian model, convexity adjustments and numerical works are skipped. Please see the original paper [4] for the details.

3 An Example

In this section, we give an example that is not given in the original paper [4].

We suppose that both the default-free instantaneous spot rate r_t and the instantaneous hazard rate λ_t follow so-called CIR model as follows.

$$
\begin{aligned}
dr_t &= \kappa^r(\theta^r - r_t)dt + \sigma^r\sqrt{r_t}dW_t^{r,Q},\\
d\lambda_t &= \kappa^\lambda(\theta^\lambda - \lambda_t)dt + \sigma^\lambda\sqrt{\lambda_t}dW_t^{\lambda,Q}.
\end{aligned}
$$

Here we suppose all the parameters like κ^r, θ^r and so on are positive. Also assume that $2\kappa^r\theta^r > (\sigma^r)^2$ and $2\kappa^\lambda\theta^\lambda > (\sigma^\lambda)^2$, so that r_t and λ_t can remain positive. Furthermore, let $W_t^{r,Q}$ and $W_t^{\lambda,Q}$ are $(\mathcal{F}_t)$-conditionally independent Brownian motions under the measure Q.

As one can see, this model belongs to the affine term-structure class. This implies all the components appeared in (2.4) and (2.5) can be reduced to solving some versions of Ricatti equations, at least numerically.

Indeed, for the purpose of doing with the conditional expectation with respect to $\mathcal{F}_{T_{j-1}}$ ($t < T_{j-1}$) in (2.5), it is useful to have the joint conditional distribution of r_t and λ_t under another equivalent probability measure $\hat{Q}_j$ specified as follows.

$$E^Q\left[\frac{d\hat{Q}_j}{dQ}\Big|\mathcal{F}_t\right] = E^Q\left[\frac{D(0,T_j)e^{-\Lambda_{T_j}}}{E^Q\left[D(0,T_j)e^{-\Lambda_{T_j}}\right]}\Big|\mathcal{F}_t\right] = \frac{D(0,t)P(t,T_j)}{P(0,T_j)} \times \frac{E^Q\left[e^{-\Lambda_{T_j}}|\mathcal{F}_t\right]}{E^Q\left[e^{-\Lambda_{T_j}}\right]}$$

From the results in subsection 3.2.3 of Brigo and Mercurio [2], it follows that the Brownian motions under the new measure $\hat{Q}_j$ are given by

$$\begin{aligned}
\hat{W}_t^{r,j} &:= W_t^{r,Q} + \sigma^r \int_0^t B^r(u,T_j)\sqrt{r_u}du, \\
\hat{W}_t^{\lambda,j} &:= W_t^{\lambda,Q} + \sigma^\lambda \int_0^t B^\lambda(u,T_j)\sqrt{\lambda_u}du,,
\end{aligned}$$

where for $* = r, \lambda$,

$$\begin{aligned}
B^*(t,T_j) &:= \frac{2\left\{\exp\left(h^*\left(T_j - t\right)\right) - 1\right\}}{2h^* + (\kappa^* + h^*)\left\{\exp\left(h^*\left(T_j - t\right)\right) - 1\right\}}, \\
h^* &:= \sqrt{(\kappa^*)^2 + 2(\sigma^*)^2}.
\end{aligned}$$

Therefore, the dynamics of r_t and λ_t under the "T_j-forward" measure $\hat{Q}_j$ is given by

$$\begin{aligned}
dr_t &= \left\{\kappa^r\theta^r - \left(\kappa^r + (\sigma^r)^2 B^r(t,T_j)\right) r_t\right\} dt + \sigma^r\sqrt{r_t}d\hat{W}_t^{r,j}, \\
d\lambda_t &= \left\{\kappa^\lambda\theta^\lambda - \left(\kappa^\lambda + (\sigma^\lambda)^2 B^\lambda(t,T_j)\right) r_t\right\} dt + \sigma^\lambda\sqrt{\lambda_t}d\hat{W}_t^{\lambda,j}.
\end{aligned}$$

Since conditional independence between r_t and λ_t is invariant by this measure change, we have

$$\hat{Q}_j(\lambda_{T_j} \in d\lambda, r_{T_j} \in dr|r_t, \lambda_t) = \hat{Q}_j(\lambda_{T_j} \in d\lambda|\lambda_t) \times \hat{Q}_j(r_{T_j} \in dr|r_t)$$

Now, for $t < s (\leq T_j)$, we obtain the distribution of r_s conditional on r_t under $\hat{Q}_j$ as below

$$\begin{aligned}
\hat{Q}_j(r_s \in dy|r_t) &= \xi(t,s)\hat{q}_{\chi^2(4\kappa^r\theta^r/(\sigma^r)^2, \eta(t,s))}\left(\xi(t,s)y\right) dy, \\
\xi(t,s) &:= 2\left[B^r(t,T_j) + \frac{\kappa^r + h^r}{(\sigma^r)^2} + \frac{2h^r(s-t)}{(\sigma^r)^2\left\{\exp\left(h^r\left(T_j - t\right)\right) - 1\right\}}\right], \\
\eta(t,s) &:= \frac{4}{\xi(t,s)}\left[\frac{2h^r(s-t)^2 r_t \exp\left(h^r(s-t)\right)}{(\sigma^r)^2\left\{\exp\left(h^r\left(T_j - t\right)\right) - 1\right\}}\right],
\end{aligned}$$

where $\hat{q}_{\chi^2(v,\gamma)}(z)$ is the density function of the non-central χ^2-distribution with v degrees of freedom and non-centrality parameter γ.

The distribution of λ_s conditional on λ_t $(t < s)$ under $\hat{Q}_j$ can be achieved similarly. Concretely,

$$\hat{q}_{\chi^2(v,\gamma)}(z) := \sum_{k=0}^{\infty} \frac{e^{-\gamma/2}(\gamma/2)^k}{k!} \times \frac{(1/2)^{k+v/2}}{\Gamma(k+v/2)} z^{k+v/2-1} e^{-z/2},$$
$$\Gamma(z) := \int_0^{\infty} x^{z-1} e^{-x} dx \qquad \text{(Gamma function)}$$

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