数理解析研究所講究録
第 1737 巻 2011 年 35-39

ASYMMETRIC VARIATION OF CHOI INEQUALITY FOR POSITIVE LINEAR MAP

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Introduction

Let Φ be a unital positive linear map between two matrix algebras $\mathcal{A}$ and $\mathcal{M}$.

Kadison inequality [10] states that for $A \in \mathcal{A}^{sa}$ (the self adjoint elements in $\mathcal{A}$)

$$\Phi(A)^2 \leq \Phi(A^2).$$

It is known in [e,g.,[1]] that $\Phi(A)^r \leq \Phi(A^r)$ holds for $A > 0$ and $r \in [-1, 0]$ and $r \in [1, 2]$, and more genarally

$$f(\Phi(A)) \leq \Phi(f(A))$$

for operator convex function f, and $A \in \mathcal{A}^{sa}$ with spectra of A in the domain of f. We cite nice references [2] and [12] to this subject. Choi [4] shows that for $A \in \mathcal{A}^+$ (the positive cone of $\mathcal{A}$);

(C1) $\Phi(A^p) \leq \Phi(A)^p$ for $0 \leq p \leq 1$. (C2) $\Phi(A)^p \leq \Phi(A^p)$ for $1 \leq p \leq 2$.

The study of positive linear maps is of central importance in several parts of matrix analysis and functional analysis.

J-C.Bourin and E.Ricard show very interesting asymmetric extension of Kadison inequality as follows by using quite ingenious method.

Theorem A (Bourin-Ricard [3]). *Let $A \in \mathcal{A}^+$ and $0 \leq p \leq q$. Then*

$$|\Phi(A^p)\Phi(A^q)| \leq \Phi(A^{p+q}).$$

§1. A result interpolating Theorem A and Choi inequality (C2)

Löwner-Heinz inequality asserts that *If* $A \ge B \ge 0$, *then* $A^\alpha \ge B^\alpha$ *for any* $\alpha \in [0,1]$. As an extension of Löwner-Heinz inequality, we state the following result to give proofs of our results.

Theorem B.
If $A \ge B \ge 0$, *then for each* $r \ge 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \ge (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \ge (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \ge 0$ *and* $q \ge 1$ *with* $(1+r)q \ge p+r$.

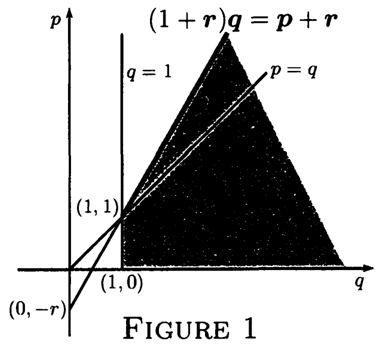

FIGURE 1

The original proof of Theorem B is shown in [6], an elementary one-page proof is in [7] and alternative ones are in [5],[9] and [8]. It is shown in [11] that the conditions p, q and r in FIGURE 1 are best possible.

Theorem 1.1. *Let* $A \in \mathcal{A}^+$, (i) $0 \le p \le q$ *and* (ii) $\frac{q}{q+p} \le r \le \frac{2q}{q+p}$. *Then*

(1.0) $$|\Phi(A^p)^r\Phi(A^q)^r| \le \Phi(A^{(p+q)r}).$$

Proof.

Put $X = \Phi(A^q)^{\frac{p}{q}}$ and $Y = \Phi(A^p)$. Then $X \ge Y \ge 0$ by Choi (C1). Put $\alpha = 2r \ge 0$ and $\beta = \frac{2qr}{p} \ge 0$. Then $(1+\beta)2 \ge \alpha + \beta$ holds by (i) and (ii), so that (ii) of Theorem B ensures

(1.1) $$\Phi(A^q)^{\frac{p}{q}(\frac{\alpha+\beta}{2})} \ge \left(\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}}\Phi(A^p)^\alpha\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}}\right)^{\frac{1}{2}}$$

and (1.1) yields

(1.2) $$\Phi(A^q)^{\frac{(p+q)r}{q}} \ge \left(\Phi(A^q)^r\Phi(A^p)^{2r}\Phi(A^q)^r\right)^{\frac{1}{2}}$$

and

(1.3) $$\Phi(A^{(p+q)r}) \ge \Phi(A^q)^{\frac{(p+q)r}{q}} \quad \text{by Choi (C2) and (ii)}$$

so that we have the desired result (1.0) by (1.2) and (1.3)

$$\Phi(A^{(p+q)r}) \geq \left(\Phi(A^q)^r\Phi(A^p)^{2r}\Phi(A^q)^r\right)^{\frac{1}{2}} = |\Phi(A^p)^r\Phi(A^q)^r|. \square \tag{1.0}$$

Remark 1. Theorem 1.1 implies Theorem A by putting $r = 1$ and also Theorem 1.1 implies Choi inequality (C2) by putting $p = 0$.

Theorem 1.1

$r = 1 \swarrow$ $\searrow p = 0$

Theorem A Choi inequality (C2)

Theorem 1.1 can be extended to the class of positive , sub-unital linear maps. The result also holds in the general setting of positive linear maps between unital C^*-albebra.

Corollary 1.2. *Let $A \in \mathcal{A}^+$ and $0 \leq p \leq q$. Then*

$$|\Phi(A^p)^{\frac{q}{q+p}}\Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^q) \tag{1.4}$$

and

$$|\Phi(A^p)^{\frac{2q}{q+p}}\Phi(A^q)^{\frac{2q}{q+p}}| \leq \Phi(A^{2q}). \tag{1.5}$$

Proof. Put $r = \dfrac{q}{q+p}$ and $r = \dfrac{2q}{q+p}$ in Theorem 1.1 respectively.

§2. Asymmetric variations of $\Phi(A)^{-1} \leq \Phi(A^{-1})$ paralleled to Theorem 1.1

Let $A \in \mathcal{A}^{++}$ be defined by $A \in \mathcal{A}^+$ and A is invertible and let Φ be strictly positive and unital. By the almost similar way to Theorem 1.1, we show the following result.

Theorem 2.1. *Let $A \in \mathcal{A}^{++}$,* (i) $0 \leq p \leq q$ *and* (ii) $\dfrac{q}{q+p} \leq r \leq \dfrac{2q}{q+p}$. *Then*

$$|\Phi(A^{-p})^{-r}\Phi(A^q)^r| \leq \Phi(A^{(p+q)r}). \tag{2.0}$$

Proof. Since $f(t) = t^s$ is operator convex for $s \in [-1, 0]$, $\Phi(A)^s \leq \Phi(A^s)$ holds for $A > 0$ and $s \in [-1, 0]$ as stated in Introduction. Put $X = \Phi(A^q)^{\frac{p}{q}}$ and $Y = \Phi(A^{-p})^{-1}$. Then $X \geq Y > 0$. Put $\alpha = 2r \geq 0$ and $\beta = \dfrac{2qr}{p} \geq 0$. Then $(1 + \beta)2 \geq \alpha + \beta$ by (i) and (ii). so that (ii) of Theorem B ensures

$$\Phi(A^q)^{\frac{p}{q}(\frac{\alpha+\beta}{2})} \geq \left(\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}}\Phi(A^{-p})^{-\alpha}\Phi(A^q)^{\frac{p}{q}\frac{\beta}{2}}\right)^{\frac{1}{2}} \tag{2.1}$$

and (2.1) yields

$$\Phi(A^q)^{\frac{(p+q)r}{q}} \geq \left(\Phi(A^q)^r\Phi(A^{-p})^{-2r}\Phi(A^q)^r\right)^{\frac{1}{2}} \tag{2.2}$$

and

$$\Phi(A^{(p+q)r}) \geq \Phi(A^q)^{\frac{(p+q)r}{q}} \quad \text{by Choi (C2) and (ii)} \tag{2.3}$$

so that we have the desired (2.0) by (2.2) and (2.3)

$$\Phi(A^{(p+q)r}) \geq \left(\Phi(A^q)^r\Phi(A^{-p})^{-2r}\Phi(A^q)^r\right)^{\frac{1}{2}} = |\Phi(A^{-p})^{-r}\Phi(A^q)^r| \quad \square \tag{2.0}$$

Corollary 2.2. *Let $A \in \mathcal{A}^{++}$ and $0 \leq p \leq q$. Then*

$$|\Phi(A^{-p})^{\frac{-q}{q+p}}\Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^q). \tag{2.4}$$

$$|\Phi(A^{-p})^{\frac{-2q}{q+p}}\Phi(A^q)^{\frac{q}{q+p}}| \leq \Phi(A^{2q}). \tag{2.5}$$

Proof. Put $r = \dfrac{q}{q+p}$ and $r = \dfrac{2q}{q+p}$ in Theorem 2.1 respectively.□

Remark 2. Theorem 2.1 interpolating Choi inequality (C2) by potting $p = 0$ and $|\Phi(A^{-p})^{-1}\Phi(A^q)| \leq \Phi(A^{p+q})$ for $0 \leq p \leq q$ by putting $r = 1$.

Theorem 2.1

$r = 1 \swarrow$ $\searrow p = 0$

$|\Phi(A^{-p})^{-1}\Phi(A^q)| \leq \Phi(A^{p+q})$. Choi inequality (C2)

The complete form of this talk has been published in the following paper:

T.Furuta, Around Choi inequalities for positive linear maps, Linear Algebra Appl., **434**(2011), 14-17.

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