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ON SUBNORMAL AND COMPLETELY HYPEREXPANSIVE COMPLETION PROBLEMS

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ABSTRACT. In this note we discuss the completely hyperexpansive completion problem about finite sequences of positive numbers in terms of positivity of attached matrices. In particular, we obtain formulas to solve the completely hyperexpansive completion problem about low numbers such as two, three, four, five and six numbers. As an application, we also discuss an explicit solution of the subnormal completion problem for five numbers.

1. Introduction and definitions

This was presented at the 2010 RIMS workshop: research and its application of noncommutative structure in operator theory, which was held at Kyoto University in Japan on October 27-29, 2010. And this is the joint work with Z. Jabłoński, J. A. Kwak, and J. Stochel, which will be appeared in some other journal as a full context.

The completion problem for completely hyperexpansive weighted shift operators with applications to subnormal completion problems will be discussed in this note. In particular, we give a general solution of the completely hyperexpansive completion problem using a different approach than that in [**4**]. Our method is based on a characterization of truncations of completely alternating sequences. The aforesaid characterization relies on the solution of the truncated Hausdorff moment problem due to Kreĭn and Nudel'man (cf. [**8**, Theorems III.2.3 and III.2.4]).

The following notation is made for convenience and ease of presentation. We write δ_t for the Borel probability measure on $[0,1]$ concentrated at $t \in [0,1]$. Given $m, n \in \{0,1,2,\ldots\} \cup \{\infty\}$, we define $\lfloor m,n \rfloor = \{i\colon i \text{ is an integer}, m \leqslant i \leqslant n\}$. Let $\gamma = \{\gamma_i\}_{i=0}^{m}$ and $\widehat{\gamma} = \{\hat{\gamma}_i\}_{i=0}^{n}$ be sequences of real numbers with m,n as above. If $m \leqslant n$ and $\gamma_i = \hat{\gamma}_i$ for $i \in \lfloor 0,m \rfloor$, then we write $\gamma \subseteq \widehat{\gamma}$. Given a finite number of real numbers $\zeta_0, \ldots, \zeta_k$, we denote by $[\zeta_j]_{j=0}^{k}$ the column matrix and regard it as a vector in the vector space $\mathbb{R}^{k+1}$, where $\mathbb{R}$ stands for the field of real numbers.

Throughout this note we assume that X is a real vector space and k is a nonnegative integer. Let $\boldsymbol{x} = \{x_i\}_{i=1}^{k}$ be a sequence of vectors in X with $k \geqslant 1$ and $x_1 \neq 0$. The largest integer $j \in \lfloor 1,k \rfloor$ for which the vectors $x_1, \ldots, x_j$ are

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linearly independent is called the *rank* of $\{x_i\}_{i=1}^k$. For $x_{k+1} \in X$, we assume that x_{k+1} belongs to the linear span of $\boldsymbol{x}$ whenever the rank r of $\boldsymbol{x}$ is equal to k. Then there exists a unique r-tuple $(\varphi_1, \ldots, \varphi_r) \in \mathbb{R}^r$ such that $x_{r+1} = \varphi_1 \cdot x_1 + \ldots + \varphi_r \cdot x_r$. The *generating function* $\mathfrak{g}_{\tilde{\boldsymbol{x}}}$ of $\tilde{\boldsymbol{x}} = \{x_i\}_{i=1}^{k+1}$ is given by

$$\mathfrak{g}_{\tilde{\boldsymbol{x}}}(t) = -(\varphi_1 t^0 + \ldots + \varphi_r t^{r-1}) + t^r, \quad t \in \mathbb{R}.$$

Recall that a sequence $\{y_i\}_{i=0}^k \subseteq X$ $(k \geqslant 1)$ is said to be *affinely independent* if for every sequence $\{\lambda_i\}_{i=0}^k \subseteq \mathbb{R}$, if $\sum_{i=0}^k \lambda_i y_i = 0$ and $\sum_{i=0}^k \lambda_i = 0$, then $\lambda_i = 0$ for $i \in \lfloor 0, k \rfloor$. A vector $y_{k+1} \in X$ is an *affine combination* of a sequence $\{y_i\}_{i=0}^k \subseteq X$ $(k \geqslant 0)$ with *coefficients* $(\psi_0, \ldots, \psi_k) \in \mathbb{R}^{k+1}$ if $y_{k+1} = \sum_{i=0}^k \psi_i y_i$ and $\sum_{i=0}^k \psi_i = 1$. Let $\boldsymbol{y} = \{y_i\}_{i=0}^k$ be a sequence of vectors in X with $k \geqslant 1$ and $y_0 \neq y_1$. The largest integer $j \in \lfloor 1, k \rfloor$ for which the sequence $\{y_i\}_{i=0}^j$ is affinely independent is called the *affine rank* of $\{y_i\}_{i=0}^k$. For $y_{k+1} \in X$. We assume that y_{k+1} belongs to the affine span of $\boldsymbol{y}$ whenever the affine rank r of $\boldsymbol{y}$ is equal to k. Then there exists a unique $(r+1)$-tuple $(\psi_0, \ldots, \psi_r) \in \mathbb{R}^{r+1}$ such that $y_{r+1} = \psi_0 y_0 + \ldots + \psi_r y_r$ and $\sum_{i=0}^r \psi_i = 1$. The *affine generating function* $\mathfrak{G}_{\tilde{\boldsymbol{y}}}$ of $\tilde{\boldsymbol{y}} = \{y_i\}_{i=0}^{k+1}$ is given by

$$\mathfrak{G}_{\tilde{\boldsymbol{y}}}(t) = \psi_0 t^0 + (\psi_0 + \psi_1) t^1 + \ldots + (\psi_0 + \ldots + \psi_{r-1}) t^{r-1} + t^r, \quad t \in \mathbb{R}.$$

2. Truncations of monotone and alternating sequences

A sequence $\{\gamma_n\}_{n=0}^\infty$ of real numbers is said to be a Hausdorff moment sequence if there exists a positive Borel measure μ on $[0,1]$ such that for all $n \geqslant 0$

$$\gamma_n = \int_{[0,1]} s^n \mathrm{d}\mu(s), \tag{2.1}$$

where $0^0 = 1$. The measure μ is unique and finite. Call it an $\mathfrak{H}$*-representing measure* for $\{\gamma_n\}_{n=0}^\infty$. By the Hausdorff theorem (cf. [**5**] and [**2**, Proposition 4.6.11]), a sequence $\gamma = \{\gamma_n\}_{n=0}^\infty$ of real numbers is a Hausdorff moment sequence if and only if it is *completely monotone*, i.e., $(\nabla^m \gamma)_k \geqslant 0$ for all integers $k, m \geqslant 0$, where ∇^m is the mth power of the difference operator ∇ which acts on γ via

$$(\nabla \gamma)_n = \gamma_n - \gamma_{n+1}, \quad n = 0, 1, 2, \ldots \quad (\nabla^0 \gamma = \gamma). \tag{2.2}$$

To consider truncated Hausdorff moment problem, we give an integer $m \geqslant 0$. Then we say that a positive Borel measure μ on $[0,1]$ is an $\mathfrak{H}$*-representing measure* for a sequence $\{\gamma_n\}_{n=0}^m$ of real numbers if (2.1) holds for $n \in \lfloor 0, m \rfloor$. If $m = 2k$ for some integer $k \geqslant 0$, and $\gamma_0 > 0$, then the rank of the sequence $\{[\gamma_{i+j-1}]_{i=0}^k\}_{j=1}^{k+1}$ of columns of the Hankel matrix $[\gamma_{i+j}]_{i,j=0}^k$ is called the *Hankel rank* of $\gamma = \{\gamma_n\}_{n=0}^{2k}$ and denoted by $\mathsf{r}(\gamma)$ (cf. [**3**]).

THEOREM 2.1 (Even Case). *If $\gamma = \{\gamma_n\}_{n=0}^{2k}$ is a finite sequence of real numbers with $k \geqslant 1$ and $\gamma_0 > 0$, then the following conditions are equivalent*:

(i) *there exists a Hausdorff moment sequence $\hat{\gamma} = \{\hat{\gamma}_n\}_{n=0}^\infty$ such that $\gamma \subseteq \hat{\gamma}$*;
(ii) *γ has an $\mathfrak{H}$-representing measure whose support consists of $\mathsf{r}(\gamma)$ points*;
(iii) *there exists $\gamma_{2k+1} \in \mathbb{R}$ such that $[\gamma_{i+k+1}]_{i=0}^k$ is a linear combination of $\{[\gamma_{i+j}]_{i=0}^k\}_{j=0}^k$, and $[\gamma_{i+j}]_{i,j=0}^k \geqslant [\gamma_{i+j+1}]_{i,j=0}^k \geqslant 0$*;
(iv) *$[\gamma_{i+j}]_{i,j=0}^k \geqslant 0$ and $[\gamma_{i+j+1}]_{i,j=0}^{k-1} \geqslant [\gamma_{i+j+2}]_{i,j=0}^{k-1}$.*

We now turn to the odd case. Let $\tilde{\gamma} = \{\gamma_n\}_{n=0}^{2k+1}$ be a finite sequence of real numbers with $k \geqslant 0$ and $\gamma_0 > 0$. The generating function of the sequence $\{[\gamma_{i+j-1}]_{i=0}^{k}\}_{j=1}^{k+2}$ will be called the *generating function* of $\tilde{\gamma}$ and denoted by $g_{\tilde{\gamma}}$ (cf. [**3**]).

THEOREM 2.2 (Odd Case). *If $\tilde{\gamma} = \{\gamma_n\}_{n=0}^{2k+1}$ is a finite sequence of real numbers with $k \geqslant 0$ and $\gamma_0 > 0$, then the following conditions are equivalent:*

(i) *there exists a Hausdorff moment sequence $\widehat{\gamma} = \{\hat{\gamma}_n\}_{n=0}^{\infty}$ such that $\tilde{\gamma} \subseteq \widehat{\gamma}$;*

(ii) *$\tilde{\gamma}$ has an $\mathfrak{H}$-representing measure whose support consists of $\mathsf{r}(\gamma)$ points which are roots of $g_{\tilde{\gamma}}$ with $\gamma = \{\gamma_n\}_{n=0}^{2k}$;*

(iii) *$[\gamma_{i+k+1}]_{i=0}^{k}$ is a linear combination of $\{[\gamma_{i+j}]_{i=0}^{k}\}_{j=0}^{k}$, and $[\gamma_{i+j}]_{i,j=0}^{k} \geqslant [\gamma_{i+j+1}]_{i,j=0}^{k} \geqslant 0$;*

(iv) *$[\gamma_{i+j}]_{i,j=0}^{k} \geqslant [\gamma_{i+j+1}]_{i,j=0}^{k} \geqslant 0$.*

We next consider truncations of completely alternating sequences. Following [**2**], we say that a sequence $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{\infty}$ of real numbers is *completely alternating* if $(\nabla^m \boldsymbol{\zeta})_k \leqslant 0$ for all integers $k \geqslant 0$ and $m \geqslant 1$ (see (2.2) for the definition of ∇). Recall that a sequence $\{\zeta_n\}_{n=0}^{\infty}$ of real numbers is completely alternating if and only if there exists a positive Borel measure τ on the closed interval $[0, 1]$ such that for all $n \geqslant 1$

$$\zeta_n = \zeta_0 + \int_{[0,1]} (1 + \ldots + s^{n-1}) \mathrm{d}\tau(s). \tag{2.3}$$

The measure τ is unique (cf. [**6**, Lemma 4.1]) and finite. We call it a $\mathfrak{ca}$-*representing measure* for $\{\zeta_n\}_{n=0}^{\infty}$. If $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{2k+1}$ is a finite sequence of real numbers with $k \geqslant 0$ and $\zeta_1 > \zeta_0$, then the affine rank of the sequence $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+1}$ will be called the *Hankel affine rank* of $\boldsymbol{\zeta}$ and denoted by $\mathsf{ar}(\boldsymbol{\zeta})$. In turn, if $\tilde{\boldsymbol{\zeta}} = \{\zeta_n\}_{n=0}^{2k+2}$ is a sequence of real numbers with $k \geqslant 0$ and $\zeta_1 > \zeta_0$, then the affine generating function of the sequence $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+2}$ will be called the *affine generating function* of $\tilde{\boldsymbol{\zeta}}$ and denoted by $G_{\tilde{\boldsymbol{\zeta}}}$.

THEOREM 2.3 (Even Case). *If $\tilde{\boldsymbol{\zeta}} = \{\zeta_n\}_{n=0}^{2k+2}$ is a finite sequence of real numbers with $k \geqslant 0$ and $\zeta_1 > \zeta_0$, then the following conditions are equivalent:*

(i) *there exists a completely alternating sequence $\widehat{\boldsymbol{\zeta}} = \{\hat{\zeta}_n\}_{n=0}^{\infty}$ such that $\tilde{\boldsymbol{\zeta}} \subseteq \widehat{\boldsymbol{\zeta}}$;*

(ii) *$\tilde{\boldsymbol{\zeta}}$ has a $\mathfrak{ca}$-representing measure whose support consists of $\mathsf{ar}(\boldsymbol{\zeta})$ points which are roots of $G_{\tilde{\boldsymbol{\zeta}}}$ with $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{2k+1}$;*

(iii) *$[\zeta_{i+k+2}]_{i=0}^{k}$ is an affine combination of $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+1}$,*

$$[\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k} \geqslant 0 \textit{ and } [-\zeta_{i+j+2} + 2\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0; \tag{2.4}$$

(iv) *the condition (2.4) holds.*

A similar reasoning enables as to deduce Theorem 2.4 from Theorem 2.1.

THEOREM 2.4 (Odd Case). *If $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{2k+1}$ is a finite sequence of real numbers with $k \geqslant 1$ and $\zeta_1 > \zeta_0$, then the following conditions are equivalent:*

(i) *there exists a completely alternating sequence $\widehat{\boldsymbol{\zeta}} = \{\hat{\zeta}_n\}_{n=0}^{\infty}$ such that $\boldsymbol{\zeta} \subseteq \widehat{\boldsymbol{\zeta}}$;*

(ii) *$\boldsymbol{\zeta}$ has a $\mathfrak{ca}$-representing measure whose support consists of $\mathsf{ar}(\boldsymbol{\zeta})$ points;*

(iii) *there exists* $\zeta_{2k+2} \in \mathbb{R}$ *such that* $[\zeta_{i+k+2}]_{i=0}^{k}$ *is an affine combination of* $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+1}$,

$[\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k} \geqslant 0$ *and* $[-\zeta_{i+j+2} + 2\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0$;

(iv) $[\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0$ *and* $[-\zeta_{i+j+3} + 2\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k-1} \geqslant 0$.

3. Completely hyperexpansive completion problem

Given a bounded sequence $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers, we denote by $W_{\boldsymbol{\alpha}}$ the weighted shift with the weight sequence $\boldsymbol{\alpha}$, i.e., $W_{\boldsymbol{\alpha}}$ is a unique bounded linear operator on ℓ^2 such that $W_{\boldsymbol{\alpha}} e_n = \alpha_n e_{n+1}$ for all $n \geqslant 0$, where $\{e_n\}_{n=0}^{\infty}$ is the standard orthonormal basis of ℓ^2.

We now recall a well-known characterization of the complete hyperexpansivity of weighted shifts (see [**1**, Proposition 3] and [**6**, Lemma 4.1]).

PROPOSITION 3.1. *Let* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ *be a bounded sequence of positive real numbers. A weighted shift* $W_{\boldsymbol{\alpha}}$ *is completely hyperexpansive if and only if there exists a (unique) finite positive Borel measure* τ *on* $[0,1]$ *such that*

$$\alpha_0^2 \cdots \alpha_{n-1}^2 = 1 + \int_{[0,1]} (1 + \ldots + s^{n-1}) \mathrm{d}\tau(s), \quad n \geqslant 1. \tag{3.1}$$

The correspondence $W_{\boldsymbol{\alpha}} \longleftrightarrow \tau$ *is one-to-one.*

If (3.1) holds, then we say that the measure τ is *associated* with the weighted shift $W_{\boldsymbol{\alpha}}$ or that $W_{\boldsymbol{\alpha}}$ is *associated* with τ. Let $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{m}$ be a finite sequence of positive real numbers with $m \geqslant 0$. A weighted shift $W_{\widehat{\boldsymbol{\alpha}}}$ with positive weights $\widehat{\boldsymbol{\alpha}}$ is called a *completely hyperexpansive completion* of $\boldsymbol{\alpha}$ if $W_{\widehat{\boldsymbol{\alpha}}}$ is completely hyperexpansive and $\boldsymbol{\alpha} \subseteq \widehat{\boldsymbol{\alpha}}$.

Before investigating solutions of the completely hyperexpansive completion problem, we introduce two transformations acting on sequences (finite or not) of real numbers. Fix $m \in \{0,1,2,\ldots\} \cup \{\infty\}$. Denote by Π_m the bijection between the set of all sequences $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{m} \subseteq (0,\infty)$ and the set of all sequences $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{m+1} \subseteq (0,\infty)$ with $\zeta_0 = 1$ that maps $\boldsymbol{\alpha}$ to $\boldsymbol{\zeta}$ via

$$\boldsymbol{\zeta} = \Pi_m(\boldsymbol{\alpha}): \quad \zeta_n = \begin{cases} 1 & \text{if } n = 0, \\ \alpha_0^2 \cdots \alpha_{n-1}^2 & \text{otherwise,} \end{cases} \tag{3.2}$$

for $n \in \lfloor 0, m+1 \rfloor$. Its inverse Π_m^{-1} which maps $\boldsymbol{\zeta}$ to $\boldsymbol{\alpha}$ is given by

$$\boldsymbol{\alpha} = \Pi_m^{-1}(\boldsymbol{\zeta}): \quad \alpha_n = \sqrt{\frac{\zeta_{n+1}}{\zeta_n}}, \tag{3.3}$$

for $n \in \lfloor 0, m \rfloor$. Denote by Δ_m the bijection between the set of all sequences $\boldsymbol{\zeta} = \{\zeta_n\}_{n=0}^{m+1} \subseteq \mathbb{R}$ with $\zeta_0 = 1$ and the set of all sequences $\boldsymbol{\gamma} = \{\gamma_n\}_{n=0}^{m} \subseteq \mathbb{R}$ that maps $\boldsymbol{\zeta}$ to $\boldsymbol{\gamma}$ via

$$\boldsymbol{\gamma} = \Delta_m(\boldsymbol{\zeta}): \quad \gamma_n = \zeta_{n+1} - \zeta_n, \tag{3.4}$$

for $n \in \lfloor 0, m \rfloor$. Its inverse Δ_m^{-1} which maps $\boldsymbol{\gamma}$ to $\boldsymbol{\zeta}$ is given by

$$\boldsymbol{\zeta} = \Delta_m^{-1}(\boldsymbol{\gamma}): \quad \zeta_n = 1 + \sum_{i=0}^{n-1} \gamma_i, \tag{3.5}$$

for $n \in \lfloor 1, m+1 \rfloor$.

PROPOSITION 3.2. *Suppose that* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^m$ *is a finite sequence of positive real numbers, with* $m \geqslant 1$, *such that either two of its successive terms coincide or one of them is equal to* 1. *Then the following conditions are equivalent:*

(i) $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion;*
(ii) $\alpha_0 \geqslant 1$ *and* $\alpha_n = 1$ *for* $n \in \lfloor 1, m \rfloor$.

Moreover, if (i) *holds, then there exists a unique completely hyperexpansive weighted shift* $W_{\hat{\boldsymbol{\alpha}}}$ *such that* $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$; *its weights are given by:* $\hat{\alpha}_0 = \alpha_0$ *and* $\hat{\alpha}_n = 1$ *for* $n \geqslant 1$.

For definitions of transformations Π_m and Δ_m that are used below, we refer the reader to (3.2) and (3.4).

THEOREM 3.3 (Even Case). *Suppose that* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{2k}$ *is a finite sequence of positive real numbers with* $k \geqslant 1$ *and* $\alpha_0 > 1$. *Let* $\boldsymbol{\zeta} = \Pi_{2k}(\boldsymbol{\alpha})$. *Then the following conditions are equivalent:*

(i) $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion;*
(ii) $[\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0$ *and* $[-\zeta_{i+j+3} + 2\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k-1} \geqslant 0$;
(iii) *there exists* $\zeta_{2k+2} \in \mathbb{R}$ *such that* $[\zeta_{i+k+2}]_{i=0}^{k}$ *is an affine combination of* $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+1}$,

$$[\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k} \geqslant 0 \text{ and } [-\zeta_{i+j+2} + 2\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0.$$

Moreover, if (i) *holds, then there exists a bounded sequence* $\hat{\boldsymbol{\alpha}} = \{\hat{\alpha}_n\}_{n=0}^{\infty}$ *of positive real numbers such that* $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$ *and* $W_{\hat{\boldsymbol{\alpha}}}$ *is a completely hyperexpansive weighted shift with associated measure whose support consists of* $\mathsf{ar}(\boldsymbol{\zeta})$ *points.*

For clarity of presentation, we formulate Theorem 3.4 without using the tilde notation that has appeared in Theorem 2.3.

THEOREM 3.4 (Odd Case). *Suppose that* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{2k+1}$ *is a finite sequence of positive real numbers with* $k \geqslant 0$ *and* $\alpha_0 > 1$. *Let* $\boldsymbol{\zeta} = \Pi_{2k+1}(\boldsymbol{\alpha})$. *Then the following conditions are equivalent:*

(i) $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion;*
(ii) $[\zeta_{i+j+2} - \zeta_{i+j+1}]_{i,j=0}^{k} \geqslant 0$ *and* $[-\zeta_{i+j+2} + 2\zeta_{i+j+1} - \zeta_{i+j}]_{i,j=0}^{k} \geqslant 0$.

Moreover, if (i) *holds, then* $[\zeta_{i+k+2}]_{i=0}^{k}$ *is an affine combination of* $\{[\zeta_{i+j}]_{i=0}^{k}\}_{j=0}^{k+1}$, *and there exists a bounded sequence* $\hat{\boldsymbol{\alpha}} = \{\hat{\alpha}_n\}_{n=0}^{\infty}$ *of positive real numbers such that* $\boldsymbol{\alpha} \subseteq \hat{\boldsymbol{\alpha}}$ *and* $W_{\hat{\boldsymbol{\alpha}}}$ *is a completely hyperexpansive weighted shift with associated measure whose support consists of* $\mathsf{ar}(\{\zeta_n\}_{n=0}^{2k+1})$ *points which are roots of* $G_{\boldsymbol{\zeta}}$.

We write down Theorems 3.3 and 3.4 in a particularly useful determinant form below.

THEOREM 3.5 (Even Case - determinant test). *Suppose that* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{2k}$ *is a finite sequence of positive real numbers with* $k \geqslant 1$ *and* $\alpha_0 > 1$. *Let* $\boldsymbol{\zeta} = \Pi_{2k}(\boldsymbol{\alpha})$. *Then* $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion if and only if one of the following two disjunctive conditions holds:*

(i) $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion and at least one of the determinants* $\det \Omega_0(k-1)$ *and* $\det \Theta_1(k-1)$ *vanishes;*

(ii) $\det \Omega_0(n) > 0$ *and* $\det \Theta_1(n) > 0$ *for all* $n \in \lfloor 1, k-1 \rfloor$, $\det \Omega_0(k) \geqslant 0$ *and* $\det \Theta_1(k) \geqslant 0$, *where*

$$\Omega_0(n) := \begin{bmatrix} \zeta_1 - \zeta_0 & \cdots & \zeta_{n+1} - \zeta_n \\ \vdots & \ddots & \vdots \\ \zeta_{n+1} - \zeta_n & \cdots & \zeta_{2n+1} - \zeta_{2n} \end{bmatrix}, \quad n \in \lfloor 0, k \rfloor,$$

$$\Theta_1(n) := \begin{bmatrix} -\zeta_3 + 2\zeta_2 - \zeta_1 & \cdots & -\zeta_{n+2} + 2\zeta_{n+1} - \zeta_n \\ \vdots & \ddots & \vdots \\ -\zeta_{n+2} + 2\zeta_{n+1} - \zeta_n & \cdots & -\zeta_{2n+1} + 2\zeta_{2n} - \zeta_{2n-1} \end{bmatrix}, \quad n \in \lfloor 1, k \rfloor.$$

THEOREM 3.6 (Odd Case - determinant test). *Suppose that* $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{2k+1}$ *is a finite sequence of positive real numbers with* $k \geqslant 0$ *and* $\alpha_0 > 1$. *Let* $\boldsymbol{\zeta} = \Pi_{2k+1}(\boldsymbol{\alpha})$. *Then* $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion if and only if one of the following two disjunctive conditions holds:*

(i) $\boldsymbol{\alpha}$ *has a completely hyperexpansive completion and at least one of the determinants* $\det \Omega_1(k)$ *and* $\det \Theta_0(k-1)$ *vanishes;*

(ii) $\det \Omega_1(n) > 0$ *for all* $n \in \lfloor 1, k \rfloor$, $\det \Theta_0(n) > 0$ *for all* $n \in \lfloor 0, k-1 \rfloor$, $\det \Omega_1(k+1) \geqslant 0$ *and* $\det \Theta_0(k) \geqslant 0$, *where*

$$\Omega_1(n) := \begin{bmatrix} \zeta_2 - \zeta_1 & \cdots & \zeta_{n+1} - \zeta_n \\ \vdots & \ddots & \vdots \\ \zeta_{n+1} - \zeta_n & \cdots & \zeta_{2n} - \zeta_{2n-1} \end{bmatrix}, \quad n \in \lfloor 1, k+1 \rfloor,$$

$$\Theta_0(n) := \begin{bmatrix} -\zeta_2 + 2\zeta_1 - \zeta_0 & \cdots & -\zeta_{n+2} + 2\zeta_{n+1} - \zeta_n \\ \vdots & \ddots & \vdots \\ -\zeta_{n+2} + 2\zeta_{n+1} - \zeta_n & \cdots & -\zeta_{2n+2} + 2\zeta_{2n+1} - \zeta_{2n} \end{bmatrix}, \quad n \in \lfloor 0, k \rfloor.$$

4. Solutions for low numbers of weights

4.1. Two-, three- and four weights: 2-isometries. Let us start with one weight α_0. It follows from Proposition 3.2 applied to α_0 and $\alpha_1 := 1$ that a one-term sequence $\{\alpha_0\}$ has a completely hyperexpansive completion if and only if $\alpha_0 \geqslant 1$.

PROPOSITION 4.1 (Two weights). *A sequence* $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^{1}$ *of positive real numbers such that* $\alpha_0 > 1$ *and* $\alpha_1 \geqslant 1$ *has a completely hyperexpansive completion if and only if* $\alpha_0^2\alpha_1^2 - 2\alpha_0^2 + 1 \leqslant 0$.

Note that the assumption $\alpha_0 > \alpha_1 > 1$ does not guarantee that $\boldsymbol{\alpha}$ has a completely hyperexpansive completion, e.g. this is the case for $\alpha_0 = 2$ and $4 > \alpha_1^2 > 7/4$.

PROPOSITION 4.2 (Three weights). *A sequence* $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^{2}$ *of positive real numbers with* $\alpha_0 > 1$ *has a completely hyperexpansive completion if and only if the following two conditions hold:*

(i) $\alpha_1^2\alpha_2^2 - 2\alpha_1^2 + 1 \leqslant 0$;

(ii) $\alpha_0^2(\alpha_1^2 - 1)^2 \leqslant (\alpha_0^2 - 1)\alpha_1^2(\alpha_2^2 - 1)$.

Before proving the next result, we recall that a weighted shift $W_{\boldsymbol{\alpha}}$ with positive weights $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ is 2-isometric if and only if there exists $q \in [0,\infty)$ such that

$$\alpha_n = \sqrt{\frac{1+(n+1)q}{1+nq}}, \quad n \geqslant 0.$$

(see [**7**, Lemma 6.1 (ii)]). The measure associated with such $W_{\boldsymbol{\alpha}}$ is equal to $q \cdot \delta_1$. If $q = 1$, then $W_{\boldsymbol{\alpha}}$ is called the *Dirichlet weighted shift*.

PROPOSITION 4.3 (Four weights). *A sequence $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^{3}$ of positive real numbers such that $\alpha_0 > 1$ and $\alpha_1 > 1$ has a completely hyperexpansive completion if and only if one of the following two disjunctive conditions holds*:

(i) *$\boldsymbol{\alpha}$ has a 2-isometric completion*;
(ii) *the following three inequalities hold*:
(ii-a) $\alpha_0^2\alpha_1^2 - 2\alpha_0^2 + 1 < 0$,
(ii-b) $\alpha_1^2(\alpha_2^2-1)^2 \leqslant (\alpha_1^2-1)\alpha_2^2(\alpha_3^2-1)$,
(ii-c) $\alpha_0^2(\alpha_1^2\alpha_2^2 - 2\alpha_1^2+1)^2 \leqslant (\alpha_0^2\alpha_1^2 - 2\alpha_0^2+1)\alpha_1^2(\alpha_2^2\alpha_3^2 - 2\alpha_2^2+1)$.

Moreover, if (i) *holds, then $\boldsymbol{\alpha}$ has a unique completely hyperexpansive completion.*

4.2. Five weights: quasi- and nearly 2-isometries. A completely hyperexpansive weighted shift $W_{\boldsymbol{\alpha}}$ is said to be *quasi-2-isometric* if it is associated with a measure of the form $c \cdot \delta_\lambda$, where $\lambda \in [0,1]$ and $c \in [0,\infty)$. Owing to Proposition 3.1, the weights $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{\infty}$ of a quasi-2-isometric weighted shift $W_{\boldsymbol{\alpha}}$ associated with the measure $c \cdot \delta_\lambda$ are given by

$$\alpha_n = \begin{cases} \sqrt{\frac{(1-\lambda)+c(1-\lambda^{n+1})}{(1-\lambda)+c(1-\lambda^n)}} & \text{if } \lambda \in [0,1), \\ \sqrt{\frac{1+c(n+1)}{1+cn}} & \text{if } \lambda = 1, \end{cases} \qquad n \geqslant 0. \tag{4.1}$$

A completely hyperexpansive weighted shift $W_{\boldsymbol{\alpha}}$ is said to be *nearly 2-isometric* if it is associated with a measure of the form $c \cdot \delta_0 + d \cdot \delta_1$, where $c, d \in [0,\infty)$.

We now consider the case of five weights.

THEOREM 4.4 (Five weights). *A sequence $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^{4}$ of positive real numbers with $\alpha_0 > 1$ has a completely hyperexpansive completion if and only if one of the following two disjunctive conditions holds*:

(i) *$\boldsymbol{\alpha}$ has either a quasi-2-isometric completion or a nearly-2-isometric completion*;
(ii) *the following four inequalities hold*:
(ii-a) $\alpha_0^2(\alpha_1^2-1)^2 < (\alpha_0^2-1)\alpha_1^2(\alpha_2^2-1)$;
(ii-b) $\alpha_1^2\alpha_2^2 - 2\alpha_1^2 + 1 < 0$;
(ii-c) $\alpha_1^2(\alpha_2^2\alpha_3^2 - 2\alpha_2^2+1)^2 \leqslant (\alpha_1^2\alpha_2^2 - 2\alpha_1^2+1)\alpha_2^2(\alpha_3^2\alpha_4^2 - 2\alpha_3^2+1)$;
(ii-d) $(\alpha_0^2-1)(\alpha_3^2-1)^2\alpha_1^2\alpha_2^4 + (\alpha_1^2-1)^2(\alpha_4^2-1)\alpha_0^2\alpha_2^2\alpha_3^2 + (\alpha_2^2-1)^3\alpha_0^2\alpha_1^4 \leqslant$
$(\alpha_0^2-1)(\alpha_2^2-1)(\alpha_4^2-1)\alpha_1^2\alpha_2^2\alpha_3^2 + 2(\alpha_1^2-1)(\alpha_2^2-1)(\alpha_3^2-1)\alpha_0^2\alpha_1^2\alpha_2^2$.

Moreover, if (i) *holds, then $\boldsymbol{\alpha}$ has a unique completely hyperexpansive completion.*

4.3. Six weights: almost and pseudo-2-isometries. A completely hyperexpansive weighted shift $W_{\boldsymbol{\alpha}}$ is said to be *almost 2-isometric* if it is associated with a measure of the form $c \cdot \delta_\lambda + d \cdot \delta_1$, where $c, d \in [0,\infty)$ and $\lambda \in [0,1)$. A completely hyperexpansive weighted shift $W_{\boldsymbol{\alpha}}$ is said to be *pseudo-2-isometric* if it is associated with a measure of the form $c \cdot \delta_0 + d \cdot \delta_\lambda$, where $c, d \in [0,\infty)$ and $\lambda \in (0,1]$.

THEOREM 4.5 (Six weights). *A sequence $\boldsymbol{\alpha} = \{\alpha_i\}_{i=0}^5$ of positive real numbers such that $\alpha_0 > 1$ and $\alpha_1 > 1$ has a completely hyperexpansive completion if and only if one of the following two disjunctive conditions holds*:

(i) *$\boldsymbol{\alpha}$ has either an almost 2-isometric or a pseudo-2-isometric completion*;

(ii) *the following four inequalities hold*:

(ii-a) $\alpha_1^2(\alpha_2^2-1)^2 < \alpha_2^2(\alpha_1^2-1)(\alpha_3^2-1)$;

(ii-b) $\alpha_0^2\alpha_1^2 - 2\alpha_0^2 + 1 < 0$;

(ii-c) $\alpha_0^2(\alpha_1^2\alpha_2^2 - 2\alpha_1^2+1)^2 < \alpha_1^2(\alpha_2^2\alpha_3^2 - 2\alpha_2^2+1)(\alpha_0^2\alpha_1^2-2\alpha_0^2+1)$;

(ii-d) $\det \Omega_1(3) \geqslant 0$ *and* $\det \Theta_0(2) \geqslant 0$ (*see Theorem* 3.6 *for definitions*).

Moreover, if (i) *holds, then $\boldsymbol{\alpha}$ has a unique completely hyperexpansive completion.*

5. Applications to the subnormal completion problem

We begin by relating the contractive subnormal completion problem to the completely hyperexpansive completion problem. Fix $m \in \{0,1,2,\ldots\} \cup \{\infty\}$. Let $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{m+1}$ be a sequence of real numbers such that $\alpha_0 = \sqrt{2}$ and $\alpha_n > 1$ for all $n \in \lfloor 1, m+1 \rfloor$. Set $\boldsymbol{\zeta} = \Pi_{m+1}(\boldsymbol{\alpha})$ and $\boldsymbol{\gamma} = \Delta_{m+1}(\boldsymbol{\zeta})$ (cf. (3.2) and (3.4) for definitions). Note that $\gamma_0 = 1$ and $\gamma_n > 0$ for all $n \in \lfloor 1, m+1 \rfloor$. Set $\boldsymbol{\beta} = \Pi_m^{-1}(\boldsymbol{\gamma})$, i.e. (cf. (3.3)),

$$\beta_n = \sqrt{\frac{\gamma_{n+1}}{\gamma_n}} = \alpha_n \sqrt{\frac{\alpha_{n+1}^2 - 1}{\alpha_n^2 - 1}}, \quad n \in \lfloor 0, m \rfloor. \tag{5.1}$$

Then $\beta_n > 0$ for all $n \in \lfloor 0, m \rfloor$. Conversely, if $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^m$ is a sequence of positive real numbers, then $\boldsymbol{\alpha} := (\Pi_{m+1}^{-1} \circ \Delta_{m+1}^{-1} \circ \Pi_m)(\boldsymbol{\beta})$ is a sequence of real numbers such that $\alpha_0 = \sqrt{2}$ and $\alpha_n > 1$ for all $n \in \lfloor 1, m+1 \rfloor$ (cf. (3.5)). The transformation

$$\boldsymbol{\alpha} \mapsto \boldsymbol{\beta} = (\Pi_m^{-1} \circ \Delta_{m+1} \circ \Pi_{m+1})(\boldsymbol{\alpha}) \tag{5.2}$$

is a bijection between the set of all sequences $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{m+1}$ of real numbers such that $\alpha_0 = \sqrt{2}$ and $\alpha_n > 1$ for all $n \in \lfloor 1, m+1 \rfloor$, and the set of all sequences $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^m$ of positive real numbers.

LEMMA 5.1. *If $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^\infty$ is a bounded sequence of positive real numbers such that $\alpha_0 = \sqrt{2}$, $\alpha_1 > 1$ and the weighted shift $W_{\boldsymbol{\alpha}}$ is completely hyperexpansive, then $\alpha_n > 1$ for all $n \geqslant 1$, the sequence $\boldsymbol{\beta} := (\Pi_\infty^{-1} \circ \Delta_\infty \circ \Pi_\infty)(\boldsymbol{\alpha})$ is bounded and the weighted shift $W_{\boldsymbol{\beta}}$ is contractive and subnormal. Conversely, if $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^\infty$ is a bounded sequence of positive real numbers and the weighted shift $W_{\boldsymbol{\beta}}$ is contractive and subnormal, then the sequence $\boldsymbol{\alpha} := (\Pi_\infty^{-1} \circ \Delta_\infty^{-1} \circ \Pi_\infty)(\boldsymbol{\beta})$ is bounded, $\alpha_0 = \sqrt{2}$, $\alpha_n > 1$ for all $n \geqslant 1$, and the weighted shift $W_{\boldsymbol{\alpha}}$ is completely hyperexpansive.*

We are now ready to relate the contractive subnormal completion problem to the completely hyperexpansive completion problem.

PROPOSITION 5.2. *Fix a nonnegative integer m. Let $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^m$ be a sequence of positive real numbers and let $\boldsymbol{\alpha} := (\Pi_{m+1}^{-1} \circ \Delta_{m+1}^{-1} \circ \Pi_m)(\boldsymbol{\beta})$ (equivalently: $\boldsymbol{\alpha} = \{\alpha_n\}_{n=0}^{m+1}$ is a sequence of real numbers such that $\alpha_0 = \sqrt{2}$ and $\alpha_n > 1$ for all $n \in \lfloor 1, m+1 \rfloor$, and $\boldsymbol{\beta} = (\Pi_m^{-1} \circ \Delta_{m+1} \circ \Pi_{m+1})(\boldsymbol{\alpha})$). Then $\boldsymbol{\beta}$ has a contractive subnormal completion if and only if $\boldsymbol{\alpha}$ has a completely hyperexpansive completion. Moreover, if $m \geqslant 2$ and $\boldsymbol{\beta}$ has a contractive subnormal completion, then the numbers $\beta_0, \ldots, \beta_m$ are distinct if and only if $\boldsymbol{\alpha}$ has no pseudo-2-isometric completion.*

Next, we consider the ontractive subnormal completions for five weights.

THEOREM 5.3. *A sequence $\{\beta_n\}_{n=0}^4$ of distinct positive real numbers has a contractive subnormal completion if and only if the following two disjunctive conditions hold:*

(i) *there exist $c \in (0,\infty)$ and $\lambda \in (0,1)$ such that*

$$\beta_n = \sqrt{\frac{c\lambda^{n+1}+1}{c\lambda^n+1}}, \quad n \in \lfloor 0,4 \rfloor; \tag{5.3}$$

(ii) *the following inequalities hold:*

(ii-a) $\beta_1 < \beta_2$;

(ii-b) $\beta_0 < 1$;

(ii-c) $(\beta_1^2 - \beta_0^2) + \beta_1^2(\beta_0^2 - \beta_2^2) + \beta_0^2\beta_1^2(\beta_2^2 - \beta_1^2) > 0$;

(ii-d) $\eta_4 \geqslant 0$ *and* $\eta_1 + \beta_2^2\eta_2 + \beta_1^2\beta_2^2\eta_3 - \beta_0^2\beta_1^2\beta_2^2\eta_4 \geqslant 0$, *where*

$$\begin{cases} \eta_1 = 2\beta_0^2\beta_1^2\beta_2^2 - \beta_0^2\beta_1^4 - \beta_0^2\beta_2^2\beta_3^2 + \beta_1^2\beta_2^2\beta_3^2 - \beta_1^2\beta_2^4, \\ \eta_2 = -\beta_0^2\beta_1^2\beta_2^2 - \beta_0^2\beta_1^2\beta_3^2 + \beta_0^2\beta_1^4 + \beta_0^2\beta_3^2\beta_4^2 + \beta_1^2\beta_2^2\beta_3^2 - \beta_1^2\beta_3^2\beta_4^2, \\ \eta_3 = -\beta_0^2\beta_1^2\beta_2^2 + \beta_0^2\beta_1^2\beta_3^2 + \beta_0^2\beta_2^2\beta_3^2 - \beta_0^2\beta_3^2\beta_4^2 + \beta_2^2\beta_3^2\beta_4^2 - \beta_2^2\beta_3^4, \\ \eta_4 = 2\beta_1^2\beta_2^2\beta_3^2 - \beta_1^2\beta_2^4 - \beta_1^2\beta_3^2\beta_4^2 + \beta_2^2\beta_3^2\beta_4^2 - \beta_2^2\beta_3^4. \end{cases} \tag{5.4}$$

Finally, we discuss the subnormal completions for five weights.

THEOREM 5.4. *A sequence $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^4$ of distinct positive real numbers has a subnormal completion if and only if the following requirements are satisfied:*

(i) $\beta_0 < \beta_1 < \beta_2$;

(ii) *one of the following two disjunctive conditions holds:*

(ii-a) $\eta_1 > 0$ *and* $\eta_4 \geqslant 0$,

(ii-b) $\eta_1 = \eta_4 = 0$.

Moreover, if (ii-b) *holds, then* $\eta_2 = \eta_3 = 0$.

PROPOSITION 5.5. *A sequence $\boldsymbol{\beta} = \{\beta_n\}_{n=0}^4$ of distinct positive real numbers has a subnormal completion if and only if the following requirements are satisfied:*

(i) $\beta_0 < \beta_1 < \beta_2$;

(ii) *one of the following four disjunctive conditions holds:*

(ii-a) $\eta_1 > 0$ *and* $\eta_4 \geqslant 0$,

(ii-b) $\eta_1 = 0$, $\eta_2 > 0$ *and* $\eta_4 \geqslant 0$,

(ii-c) $\eta_1 = \eta_2 = 0$, $\eta_3 > 0$ *and* $\eta_4 \geqslant 0$,

(ii-d) $\eta_1 = \eta_2 = \eta_3 = \eta_4 = 0$.

We conclude this work by showing that the solution of the subnormal completion problem for five weights given in [**9**, page 45] is wrong. Indeed, this solution implies that a sequence $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$ of positive real numbers has a subnormal completion if and only if the sequences $\{\beta_n\}_{n=0}^3$ and $\{\beta_n\}_{n=1}^4$ have subnormal completions. However, as is justified below, this is not true.

Example 5.6. Set $\beta_0 = \sqrt{\frac{3}{4}}$, $\beta_1 = \sqrt{\frac{5}{6}}$, $\beta_2 = \sqrt{\frac{9}{10}}$, $\beta_3 = \sqrt{\frac{17}{18}}$ and $\beta_4 = 1$. Then $\beta_0 < \beta_1 < \beta_2 < \beta_3 < \beta_4$, $\eta_1 = 0$, $\eta_2 = -\frac{1}{432}$, $\eta_3 = \frac{1}{240}$ and $\eta_4 = \frac{1}{540}$. By Theorem 5.4, the sequence $\{\beta_n\}_{n=0}^4$ does not have subnormal completion. Since the inequalities $\eta_1 \geqslant 0$ and $\eta_4 \geqslant 0$ are equivalent respectively to the first and the second inequality in the assertion 3 of [**9**, Corollary 2.12], we infer from[**10**, Remark, p. 377] that the sequences $\{\beta_n\}_{n=0}^3$ and $\{\beta_n\}_{n=1}^4$ have subnormal completions.

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