数理解析研究所講究録
第 1737 巻 2011 年 111-114

On a reverse of Cauchy-Schwarz inequalities in pre-inner product C*-modules

芝浦工業大学工学部　　瀬尾祐貴 (Yuki Seo)
Faculty of Engineering, Shibaura Institute of Technology

1. Introduction

This report is based on [3].

Let A be a positive operator on a Hilbert space H such that $mI \leq A \leq MI$ for some scalars $0 < m < M$. Then Kantorovich inequality [6, 4] says that

$$(1) \qquad (Ax, x)(A^{-1}x, x) \leq \frac{(M+m)^2}{4Mm}$$

for every unit vector $x \in H$. This inequality (1) can be rephrased as follows:

$$\| Ax \| \| x \| \leq \frac{M+m}{2\sqrt{Mm}}(Ax, x)$$

for every vector $x \in H$. Therefore, Kantorovich inequality is just regarded as a reverse of Cauchy-Schwarz inequality

$$(Ax, x) \leq \| Ax \| \| x \| .$$

Dragomir [1] considered Kantorovich inequality (1) in the framework of an inner product space: Let $(H, \langle \cdot, \cdot \rangle)$ be an inner prodct space. Cauchy-Schwarz inequality says that

$$(2) \qquad |\langle x, y \rangle| \leq \langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \qquad \text{for all } x, y \in H.$$

Dragomir showed the following Kantorovich type inequality for Cauchy-Schwarz inequality (2): If $x, y \in H$ and $\alpha, \beta \in \mathbb{C}$ satisfy the condition

$$\mathrm{Re}\langle \alpha y - x, x - \beta y \rangle \geq 0,$$

then

$$\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \leq \frac{|\alpha + \beta|}{2\sqrt{\mathrm{Re}(\alpha\overline{\beta})}} |\langle x, y \rangle|$$

and

$$(x, x)^{\frac{1}{2}} (y, y)^{\frac{1}{2}} - |(x, y)| \leq \frac{|\alpha - \beta|^2}{4|\alpha + \beta|}(y, y).$$

In this report, by virtue of the operator geometric mean and by using some ideas of [2], we shall consider Kantorovich type inequalities for Cauchy-Schwarz inequality in the framework of a pre-inner product C*-module over a unital C*-algebra, also see [9].

2. Pre-inner product C*-modules

Let $\mathscr{A}$ be a unital C^*-algebra with the unit element e and the center $\mathcal{Z}(\mathscr{A})$. For $a \in \mathscr{A}$, we denote the real part of a by $\text{Re}\, a = \frac{1}{2}(a + a^*)$. If $a \in \mathscr{A}$ is positive (that is selfadjoint with positive spectrum), then $a^{\frac{1}{2}}$ denotes a unique positive $b \in \mathscr{A}$ such that $b^2 = a$. For $a \in \mathscr{A}$, we denote the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$. If $a \in \mathcal{Z}(\mathscr{A})$ is positive, then $a^{\frac{1}{2}} \in \mathcal{Z}(\mathscr{A})$. If $a, b \in \mathscr{A}$ are positive and $ab = ba$, then ab is positive and $(ab)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}$.

Let $\mathscr{X}$ be an algebraic left $\mathscr{A}$-module which is a complex linear space fulfilling $a(\lambda x) = (\lambda a)x = \lambda(ax)$ $(x \in \mathscr{X}, a \in \mathscr{A}, \lambda \in \mathbb{C})$. The space $\mathscr{X}$ is called a *(left) pre-inner product $\mathscr{A}$-module* (or an *pre-inner product C^*-module over the unital C^*-algebra $\mathscr{A}$*) if there exists a mapping $\langle \cdot, \cdot \rangle \colon \mathscr{X} \times \mathscr{X} \to \mathscr{A}$ satisfying

(i) $\langle x, x \rangle \geq 0$,
(ii) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$,
(iii) $\langle ax, y \rangle = a\langle x, y \rangle$,
(iv) $\langle y, x \rangle = \langle x, y \rangle^*$,

for all $x, y, z \in \mathscr{X}$, $a \in \mathscr{A}$, $\lambda \in \mathbb{C}$. Moreover, if

(v) $x = 0$ whenever $\langle x, x \rangle = 0$,

then $\mathscr{X}$ is called an *inner product $\mathscr{A}$-module.* In this case $\| x \| := \sqrt{\| \langle x, x \rangle \|}$, where the latter norm denotes the C^*-norm on $\mathscr{A}$. If this norm is complete, then $\mathscr{X}$ is called a *Hilbert $\mathscr{A}$-module.* Any inner product space is an inner product $\mathbb{C}$-module and any C^*-algebra $\mathscr{A}$ is a Hilbert C^*-module over itself via $\langle a, b \rangle = ab^*$ $(a, b \in \mathscr{A})$. For more details on Hilbert C^*-modules, see [8]. Notice that (iii) and (iv) imply $\langle x, ay \rangle = \langle x, y \rangle a^*$ for all $x, y \in \mathscr{X}, a \in \mathscr{A}$.

We discuss the Cauchy-Schwarz inequality and its reverse in a pre-inner product C*-module over a unital C*-algebra $\mathscr{A}$. Since the product of $\langle x, x \rangle$ and $\langle y, y \rangle$ are not selfadjoint in general, we would expect that the following Cauchy-Schwarz inequalities hold:

$$|\langle x, y \rangle|^2 \leq \text{Re}\langle x, x \rangle \langle y, y \rangle \qquad \text{for } x, y \in \mathscr{X}$$

and

$$\text{Re}\langle x, y \rangle \leq \text{Re}\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}} \qquad \text{for } x, y \in \mathscr{X}.$$

But we have a counterexample. As a matter of fact, let $\mathscr{A} = M_2(\mathbb{C})$ be the C*-albegra of 2×2 matrices with an inner product $\langle x, y \rangle = xy^*$ for $x, y \in \mathscr{A}$. Put $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$. Then we have $|\langle x, y \rangle|^2 \not\leq \text{Re}\langle x, x \rangle \langle y, y \rangle$ and $\text{Re}\langle x, y \rangle \not\leq \text{Re}\langle x, x \rangle^{\frac{1}{2}} \langle y, y \rangle^{\frac{1}{2}}$.

In a pre-inner product C^*-module, the Cauchy-Schwarz inequality is firstly established by Lance [8]:

$$|\langle y, x \rangle|^2 = \langle x, y \rangle \langle y, x \rangle \leq \| \langle y, y \rangle \| \langle x, x \rangle$$

for $x, y \in \mathscr{X}$. Afterwards, Ilisević and Varosanec [5] showed another version:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

for $x, y \in \mathscr{X}$ and $\langle x, x \rangle \in \mathcal{Z}(\mathscr{A})$.

3. Cauchy-Schwarz inequality and its reverse

Let A and B be positive operators on a Hilbert space. Then the operator geometric mean $A \sharp B$ is defined by

$$A \sharp B = A^{\frac{1}{2}} \left(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}\right)^{\frac{1}{2}} A^{\frac{1}{2}}$$

if A is invertible, see [7]. The operator geometric mean has the symmetric property: $A \sharp B = B \sharp A$. If A commutes with B, then $A \sharp B = A^{\frac{1}{2}} B^{\frac{1}{2}}$. From viewpoint of (2), we would expect the following Cauchy-Schwarz inequality in a pre-inner product C*-module:

$$|\langle x, y\rangle| \le \langle x, x\rangle \sharp \langle y, y\rangle \tag{3}$$

holds for $x, y \in \mathscr{X}$. Unfortunately we also have a counterexample. If $x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $y = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ mentioned above, then we have $|\langle x, y\rangle| = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\langle x, x\rangle \sharp \langle y, y\rangle = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, we have $|\langle x, y\rangle| \not\le \langle x, x\rangle \sharp \langle y, y\rangle$.

However, we have the following Cauchy-Schwarz type inequality:

Theorem 1. *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ and $u \in \mathscr{A}$. Then*

$$|\langle x, y\rangle| \le u^*\langle x, x\rangle u \sharp \langle y, y\rangle.$$

To prove a reverse of Cauchy-Schwarz type inequality in Theorem 1, we need the following lemma:

Lemma 2. *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that there exist a partial isometry $u \in \mathscr{A}$ such that a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ and*

$$\mathrm{Re}\langle Ay - u^*x, u^*x - ay\rangle \ge 0 \tag{4}$$

for some $a, A \in \mathscr{Z}(\mathscr{A})$. Then

$$u^*\langle x, x\rangle u + \mathrm{Re}(Aa^*)\langle y, y\rangle \le \mathrm{Re}(A + a)|\langle x, y\rangle|.$$

Remark 3. *The condition (4) in Lemma 2 is equivalent to*

$$\langle u^*x - \frac{A+a}{2}y, u^*x - \frac{A+a}{2}y\rangle \le \frac{|A-a|^2}{4}\langle y, y\rangle.$$

Theorem 4. *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that there exist a partial isometry $u \in \mathscr{A}$ such that a polar decomposition $\langle x, y\rangle = u|\langle x, y\rangle|$ and (4) holds for some elements $a, A \in \mathscr{Z}(\mathscr{A})$ and* $\mathrm{Re}(Aa^*)$ *is positive invertible and* $\mathrm{Re}(A + a)$ *is invertible. Then*

$$u^*\langle x, x\rangle u \sharp \langle y, y\rangle \le \frac{\mathrm{Re}(A+a)}{2\sqrt{\mathrm{Re}(Aa^*)}}|\langle x, y\rangle|. \tag{i}$$

$$u^*\langle x, x\rangle u \sharp \langle y, y\rangle - |\langle x, y\rangle| \le \frac{(\mathrm{Re}(A+a))^2 - 4\mathrm{Re}(Aa^*)}{4\mathrm{Re}(A+a)}\langle y, y\rangle. \tag{ii}$$

$$\text{(iii)} \qquad u^*\langle x,x\rangle u \,\sharp\, \langle y,y\rangle - |\langle x,y\rangle| \le \frac{(\mathrm{Re}(A+a))^2 - 4\mathrm{Re}(Aa^*)}{4\mathrm{Re}(Aa^*)\mathrm{Re}(A+a)}\langle x,x\rangle.$$

Finally, though the inequality (3) does not hold in general, we have reverse types of (3):

Theorem 5. *Let $\mathscr{X}$ be a pre-inner product C^*-module over a unital C^*-algebra $\mathscr{A}$. Suppose that $x, y \in \mathscr{X}$ such that*

$$\langle Ay - x, x - ay\rangle \ge 0 \qquad \text{for some positive invertible } A, a \in \mathcal{Z}(\mathscr{A}).$$

Then

$$\text{(i)} \qquad \langle x,x\rangle \,\sharp\, \langle y,y\rangle \le \frac{A+a}{2\sqrt{Aa}}\mathrm{Re}\langle x,y\rangle.$$

$$\text{(ii)} \qquad \langle x,x\rangle \,\sharp\, \langle y,y\rangle - \mathrm{Re}\langle x,y\rangle \le \frac{(A-a)^2}{4(A+a)}\langle y,y\rangle.$$

$$\text{(ii)} \qquad \langle x,x\rangle \,\sharp\, \langle y,y\rangle - \mathrm{Re}\langle x,y\rangle \le \frac{(A-a)^2}{4Aa(A+a)}\langle x,x\rangle.$$

References

[1] S.S. Dragomir, *Reverses of Schwarz, triangle and bessel inequalities in inner product spaces*, J. Inequal. Pure Appl. Math., **5**, Issue 3, Article 76, 2004.

[2] N. Elezović, Lj. Marangunić and J.E. Pečarić, *Unified treatment of complemented Schwarz and Grüss inequalities in inner product spaces*, Math. Inequal. Appl., **8** (2005), no.2, 223–231.

[3] J.I.Fujii, M.Fujii, M.S.Moslehian, J.E.Pečarić and Y.Seo, *Reverse Cauchy-Schwarz type inequalities in pre-inner product C^*-modules*, preprint.

[4] W. Greub and W. Rheinboldt, *On a generalization of an inequality of L.V.Kantorovich*, Proc. Amer. Math. Soc., **10**(1959), 407–415.

[5] D. Ilisević and S. Varošanec, *On the Cauchy-Schwarz inequality and its reverse in semi-inner product C^*-modules*, Banach J. Math. Anal., **1** (2007), 78–84.

[6] L.V.Kantorovich, *Functional analysis and applied mathematics*, Uspehi Mat. Nauk., **3**(1948), pp.89-185. Translated from the Russian by Curtis D. Benster, National Bureau of Standards, Report 1509, March 7, 1952.

[7] F. Kubo and T. Ando, *Means of positive linear operators*, Math. Ann., **246**(1980), 205–224.

[8] E.C. Lance, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Series 210, Cambridge Univ. Press, 1995.

[9] M.S. Moslehian and L.-E. Persson, *Reverse Cauchy-Schwarz inequalities for positive C^*-valued sesquilinear forms*, Math. Inequal. Appl., **4** (2009), no.12, 701–709.

Faculty of Engineering, Shibaura Institute of Technology, 307 Fukasaku, Minuma-ku, Saitama-City, Saitama 337-8570, Japan.

E-mail address : yukis@sic.shibaura-it.ac.jp