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SOME RESULTS ON GENERALIZED QUADRATIC OPERATORS

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ABSTRACT. A bounded linear operator acting on a Hilbert space is a generalized quadratic operator if it has an operator matrix of the form

$$\begin{bmatrix} aI & cT \\ dT^* & bI \end{bmatrix}.$$

It reduces to a quadratic operator if $d=0$. In this paper, norms and numerical ranges of generalized quadratic operators are determined. Some operator inequalities are also obtained. Moreover we consider q-numerical range.

1. INTRODUCTION

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with M_n if $\mathcal{H}$ has dimension n. An operator $A \in \mathcal{B}(\mathcal{H})$ is a *generalized quadratic operators* if it has an operator matrix of the form

$$\begin{bmatrix} aI & cT \\ dT^* & bI \end{bmatrix} \tag{1.1}$$

where T is an operator from $\mathcal{K}_2$ to $\mathcal{K}_1$ ($\mathcal{K}_1, \mathcal{K}_2$: Hilbert spaces), and a, b, c, d are complex numbers. [In the following discussion, we will not distinguish the operator and its operator matrix if there is no ambiguity.] When $d=0$, such an operator A satisfies condition

$$(aI-A)(bI-A)=0 \tag{1.2}$$

and is known as a *quadratic operator*. In fact, it is known that an operator A satisfies (1.2) if and only if it has an operator matrix of the form (1.1) with $d=0$.

In this paper, a complete description is given to the norm and ranges of an operator of the form (1.1). In particular, the norm of A is the same as that of A_p with $p=\|T\|$. **We always assume that** $cdT \neq 0$ in the following discussion.

In Section 2, we obtain a different operator matrix for an generalized quadratic operator A. In Section 3, we determine the numerical range and the norm of generalized quadratic operators. Furthermore, we obtain some operator inequalities concerning generalized quadratic operators that extend some results of Furuta [1] and Garcia [2]. We then give the description of q-numerical ranges of A in Section 4.

We will use the following notations in our discussion. For $S \subseteq \mathbb{C}$, denote by $\mathbf{int}(S)$, $\mathbf{cl}\,(S)$ and $\mathbf{conv}(S)$ the relative interior, the closure and the convex hull of S, respectively.

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Note that in our discussion, it may happen that $S = \mathbf{conv}\{\mu_1, \mu_2\}$ is a line segment in $\mathbb{C}$ so that $\mathbf{int}(S) = S \setminus \{\mu_1, \mu_2\}$.

For $A \in \mathcal{B}(\mathcal{H})$, let $\ker A$ and range$A$ denote the null space and range space of A, respectively. Let V be a closed subspace of $\mathcal{H}$ and Q the embedding of V into $\mathcal{H}$. Then $B = Q^*AQ$ is the *compression* of A onto V.

2. A DIFFERENT OPERATOR MATRIX REPRESENTATION

First, we obtain a different operator matrix for A of the form (1.1). The special form reduces to that of quadratic operators in [8, Theorem 1.1] if $d = 0$.

Theorem 2.1. *Let $A \in \mathcal{B}(\mathcal{H})$ $(\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2)$ be an operator with an operator matrix*

$$\begin{bmatrix} aI & cT \\ dT^* & bI \end{bmatrix} \tag{1.1}$$

where $a, b, c, d \in \mathbb{C}$ and $T \in \mathcal{B}(\mathcal{K}_2, \mathcal{K}_1)$ with $cdT \neq 0$. Let $\mathcal{H}_1 = \overline{\text{range}T^*}$ (the closure of rangeT^**), $\tilde{\mathcal{H}}_1 = \overline{\text{range}T}$, $\mathcal{H}_2 = \ker T^*$, $\mathcal{H}_3 = \ker T$. Let T_0 be a restriction of T to $\mathcal{H}_1$ with the polar decomposition $T_0 = U|T_0|$ where $U \in \mathcal{B}(\mathcal{H}_1, \tilde{\mathcal{H}}_1)$ is a unitary. Then the operator matrix (1.1) is unitarily similar to*

$$aI_{\mathcal{H}_2} \oplus \begin{bmatrix} aI_{\mathcal{H}_1} & c|T_0| \\ d|T_0| & bI_{\mathcal{H}_1} \end{bmatrix} \oplus bI_{\mathcal{H}_3} \in \mathcal{B}(\mathcal{H}) \quad (\mathcal{H} = \mathcal{H}_2 \oplus (\mathcal{H}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_3) \tag{2.1}$$

by the unitary

$$I_{\mathcal{H}_2} \oplus (U \oplus I_{\mathcal{H}_1}) \oplus I_{\mathcal{H}_3}$$

from $\mathcal{H}_2 \oplus (\mathcal{H}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_3$ to $\mathcal{H}_2 \oplus (\tilde{\mathcal{H}}_1 \oplus \mathcal{H}_1) \oplus \mathcal{H}_3$.

Proof. The operator matrix (1.1) has the following form by the direct sum decomposition $\mathcal{H}(= \mathcal{K}_1 \oplus K_2) = (\mathcal{H}_2 \oplus \tilde{\mathcal{H}}_1) \oplus (\mathcal{H}_1 \oplus \mathcal{H}_3)$

$$\left[\begin{array}{cc|cc} aI_{\mathcal{H}_2} & 0 & 0 & 0 \\ 0 & aI_{\mathcal{H}_1} & cT_0 & 0 \\ \hline 0 & dT_0^* & bI_{\mathcal{H}_1} & 0 \\ 0 & 0 & 0 & bI_{\mathcal{H}_3} \end{array}\right].$$

So we may only consider the part $\begin{bmatrix} aI_{r_1} & cT_0 \\ dT_0^* & bI_{r_1} \end{bmatrix}$. Indeed, we have

$$\begin{bmatrix} U^* & 0 \\ 0 & I_{r_1} \end{bmatrix}^* \begin{bmatrix} aI_{r_1} & c|T_0| \\ d|T_0| & bI_{r_1} \end{bmatrix} \begin{bmatrix} U^* & 0 \\ 0 & I_{r_1} \end{bmatrix} = \begin{bmatrix} aI_{r_1} & cT_0 \\ dT_0^* & bI_{r_1} \end{bmatrix}.$$

It completes this theorem. □

Remark 2.2. *We have $\langle |T_0|x, x\rangle \neq 0$ for all nonzero $x \in \mathcal{H}_1$. That is, $|T_0|$ is injection.*

By Theorem 2.1, we can focus on an operator A with an operator matrix of the form (2.1) with $cd|T_0| \neq 0$. Also, the family of matrices

$$A_p = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}, \qquad p \geq 0, \tag{2.2}$$

will be very useful in our discussion.

3. Numerical range and operator inequalities

Recall that the *numerical range* of $A \in \mathcal{B}(\mathcal{H})$ is defined by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\};$$

see [3], [4], [5]. The numerical range is useful in studying matrices and operators. One of the basic properties of the numerical range is that $W(A)$ is always convex; for example, see [4]. In particular, we have the following result, e.g., see [5, Theorem 1.3.6] and [6].

Elliptical Range Theorem. *If $A \in M_2$ has eigenvalues μ_1 and μ_2, then $W(A)$ is an elliptical disk with μ_1, μ_2 as foci and $\sqrt{\operatorname{tr}(A^*A) - |\mu_1|^2 - |\mu_2|^2}$ as the length of minor axis. Furthermore, if $\hat{A} = A - (\operatorname{tr} A)I/2$, then the lengths of minor and major axis of $W(A)$ are, respectively,*

$$\{\operatorname{tr}(\hat{A}^*\hat{A}) - 2|\det \hat{A}|\}^{1/2} \quad \text{and} \quad \{\operatorname{tr}(\hat{A}^*\hat{A}) + 2|\det \hat{A}|\}^{1/2}.$$

Using this theorem, one can deduce the convexity of the numerical range of a general operator; e.g., see [6]. It turns out that for an operator A in Theorem 2.1, $W(A)$ is also an elliptical disk with all the boundary points, two boundary points, or none of its boundary points as shown in the following.

Theorem 3.1. *Suppose $A \in \mathcal{B}(\mathcal{H})$ has the operator matrix in Theorem 2.1. Let $\tilde{p} = \|T_0\|$, $\tilde{A} = \begin{bmatrix} a & c\tilde{p} \\ d\tilde{p} & b \end{bmatrix}$ so that $\tilde{A}$ has eigenvalues $\mu_\pm = \frac{1}{2}\left\{(a+b) \pm \sqrt{(a-b)^2 + 4cd\tilde{p}^2}\right\}$ and $W(\tilde{A})$ is the elliptical disk with foci μ_+, μ_- and minor axis of length*

$$\sqrt{|a|^2 + |b|^2 + \tilde{p}^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$

If $\|T_0 x\| = \|T_0\|$ for some unit vector $x \in \mathcal{H}_1$, then

$$W(A) = W(\tilde{A}).$$

Otherwise, $W(A) = \mathbf{int}(W(\tilde{A})) \cup \{a, b\}$. More precisely, one of the following holds:

(1) *If $|c| = |d|$ and $\bar{d}(a-b) = c(\bar{a} - \bar{b})$, then both A and $\tilde{A}$ are normal, and*

$$W(A) = W(\tilde{A}) \setminus \sigma(\tilde{A}) = \mathbf{conv}\{\mu_+, \mu_-\} \setminus \{\mu_+, \mu_-\}.$$

(2) *If $|c| = |d|$ and there is $\zeta \in (0, \pi)$ such that $\bar{d}(a-b) = e^{i2\zeta}c(\bar{a} - \bar{b}) \neq 0$, then both numbers a, b lie on the boundary $\partial W(A)$ of $W(A)$, and*

$$W(A) = \mathbf{int}(W(\tilde{A})) \cup \{a, b\}.$$

(3) *If $|c| \neq |d|$, then $W(A) = \mathbf{int}(W(\tilde{A}))$.*

To prove Theorem 3.1, we need the following lemma, which will also be useful for later discussion.

Lemma 3.2. *Let $A_p = \begin{bmatrix} a & cp \\ dp & b \end{bmatrix}$ for $p \geq 0$ so that $W(A_p)$ is the closed elliptical disk with foci $\mu_\pm = \frac{1}{2}\{(a+b) \pm \sqrt{(a-b)^2 + 4cdp^2}\}$ and minor axis of length*

$$\sqrt{|a|^2 + |b|^2 + p^2(|c|^2 + |d|^2) - |\mu_+|^2 - |\mu_-|^2}.$$

Then

$$W(A_p) \subseteq W(A_q) \qquad \text{for } p < q.$$

More precisely, one of the following holds:

(1) *If* $|c| = |d|$ *and* $\bar{d}(a-b) = c(\bar{a}-\bar{b})$, *then* $W(A_p) = \mathbf{conv}\sigma(A_p)$ *and* $W(A_q) = \mathbf{conv}\sigma(A_q)$ *are line segments such that* $W(A_p)$ *is a subset of the relative interior of* $W(A_q)$.

(2) *If* $|c| = |d|$ *and there is* $\zeta \in (0,\pi)$ *such that* $\bar{d}(a-b) = e^{i2\zeta}c(\bar{a}-\bar{b}) \neq 0$, *then* $\{a,b\} = \partial W(A_p) \cap \partial W(A_q)$, *and*

$$W(A_p) \subseteq \mathbf{int}(W(A_q)) \cup \{a,b\}.$$

(3) *If* $|c| \neq |d|$, *then* $W(A_p) \subseteq \mathbf{int}W(A_q)$.

Proof. All numerical ranges $W(A_p)$ have the same center $\alpha = (a+b)/2$. Suppose $\beta = (a-b)/2$. Denote by $\lambda_1(X)$ the largest eigenvalue of a self-adjoint matrix X. Then

$$W(A_p) = \bigcap_{\xi\in[0,2\pi)} \Pi_\xi(A_p)$$

where

$$\Pi_\xi(A_p) = \{\mu \in \mathbb{C} : e^{i\xi}\mu + e^{-i\xi}\bar{\mu} \leq \lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*)\}$$

is a half space in $\mathbb{C}$. Since

$$\lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\bar{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\bar{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}\bar{d}|^2}$$

is an increasing function of p, we see that $\Pi_\xi(A_p) \subseteq \Pi_\xi(A_q)$ and hence $W(A_p) \subseteq W(A_q)$ if $p \leq q$.

Case 1. Suppose a,b,c,d satisfy condition (1). Then A_p is normal and $A_p = \alpha I_2 + B_p$, where $W(B_p) = \mathbf{conv}\{\pm\sqrt{-\det(B_p)}\}$ is a line segment of length $2\sqrt{|\beta|^2 + p^2|c|^2} = 2\sqrt{|\beta|^2 + p^2|d|^2}$. Thus, the conclusion of (1) holds.

Case 2. Suppose a,b,c,d satisfy condition (2). Then $A_p = \alpha I_2 + \beta B_p$ with

$$e^{i\zeta}B_p = \begin{bmatrix} e^{i\zeta} & \delta p \\ \bar{\delta} p & -e^{i\zeta} \end{bmatrix}, \qquad \delta = e^{i\zeta}\frac{2c}{a-b} = e^{-i\zeta}\frac{2\bar{d}}{\bar{a}-\bar{b}}.$$

Using the elliptical range theorem, one readily checks that $W(e^{i\zeta}B_p)$ is a nondegenerate elliptical disk. Since $B_p = \begin{bmatrix} 1 & \delta p e^{-i\zeta} \\ \bar{\delta} p e^{-i\zeta} & -1 \end{bmatrix}$ and

$$e^{i\xi}B_p + e^{-i\xi}B_p^* = 2\begin{bmatrix} \cos\xi & \delta p\cos(\xi-\zeta) \\ \bar{\delta} p\cos(\xi-\zeta) & -\cos\xi \end{bmatrix},$$

we have

$$\lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*) = 2\sqrt{\cos^2\xi + |\delta|^2p^2\cos^2(\xi-\zeta)} \geq \pm 2\cos\xi = \pm\left(e^{i\xi} + e^{-i\xi}\right)$$

where equality holds only for $\xi = \zeta \pm \pi/2$. Therefore $\lambda_1(e^{i\xi}B_p + e^{-i\xi}B_p^*)$ is a strictly increasing function for $p \geq 0$, except for $\xi = \zeta \pm \pi/2$. Moreover 1 and -1 are on the boundary of $W(B_p)$ for $\xi = \zeta \pm \pi/2$. From this, we get the conclusion of (2).

Case 3. Suppose a,b,c,d do not satisfy the conditions in (1) or (2). Since $|c| \neq |d|$, for every $\xi \in [0,2\pi)$,

$$\lambda_1(e^{i\xi}A_p + e^{-i\xi}A_p^*) = e^{i\xi}\alpha + e^{-i\xi}\bar{\alpha} + \sqrt{|e^{i\xi}\beta + e^{-i\xi}\bar{\beta}|^2 + p^2|e^{i\xi}c + e^{-i\xi}\bar{d}|^2}$$

is a strictly increasing function for $p \geq 0$. Thus, the conclusion of (3) holds. $\square$

Proof of Theorem 3.1. Since $W(X \oplus Y) = \mathbf{conv}\{W(X) \cup W(Y)\} = W(X)$ if $W(Y) \subseteq W(X)$, we may assume that γI_s is vacuous. Let $P = |T_0|$.

Suppose $x \in \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$ is a unit vector and $\mu = \langle Ax, x \rangle \in W(A)$. Let $x = \begin{bmatrix} \cos\theta x_1 \\ \sin\theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. Let $\langle Px_1, x_2 \rangle = pe^{-i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$. Then

$$\mu = [\cos\theta \mid e^{-i\phi}\sin\theta] A_p \begin{bmatrix} \cos\theta \\ e^{i\phi}\sin\theta \end{bmatrix} \in W(A_p) \subseteq W(\tilde{A})$$

by Lemma 3.2.

If there is a unit vector $x \in \mathcal{H}_1$ such that $\|P\| = \|Px\|$, then

$$\|P\|^2 = \langle P^2 x, x \rangle \leq \|P^2 x\| \|x\| \leq \|P^2\| = \|P\|^2.$$

Thus, $P^2 x = \|P\|^2 x$ and hence $Px = \|P\| x$ as P is positive semi-definite. Then the operator matrix of A with respect to $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^{\perp}$, where

$$\mathcal{H}_0 = \text{span}\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix} \right\}$$

has the form $\tilde{A} \oplus \tilde{A}' \in \mathcal{B}(\mathcal{H})$. Thus, $W(\tilde{A}) \subseteq W(A)$, and the equality holds.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|P\| = \|Pz\|$. Then for any unit vector $x \in \mathcal{H}$, let $x = \begin{bmatrix} \cos\theta x_1 \\ \sin\theta x_2 \end{bmatrix}$ for some unit vectors $x_1, x_2 \in \mathcal{H}_1$. If $\langle Px_1, x_2 \rangle = pe^{i\phi}$ with $p \in [0, \tilde{p}]$ and $\phi \in [0, 2\pi)$, then $p < \tilde{p}$. By Lemma 3.2, we see that $\mu \in \mathbf{int}(W(\tilde{A}))$ if (a) or (c) holds, and $\mu \in \mathbf{int}(W(\tilde{A})) \cup \{a, b\}$ if (b) holds.

To prove the reverse set equalities, note that there is a sequence of unit vectors $\{x_m\}$ in $\mathcal{H}_1$ such that $\langle Px_m, x_m \rangle = p_m$ converges to $\tilde{p}$. Then the compression of A on the subspace

$$V_m = \text{span}\left\{ \begin{bmatrix} x_m \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x_m \end{bmatrix} \right\} \subseteq \mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1$$

has the form A_{p_m}. Since $W(A_{p_m}) \to W(\tilde{A})$, we see that $\mathbf{int}(W(\tilde{A})) \subseteq W(A)$. It is also clear that $\{a, b\} \subseteq W(A)$. Thus, the set equalities in (1) – (3) hold. $\square$

We consider some operator inequalities. Denote by

$$w(A) = \sup\{|\mu| : \mu \in W(A)\}$$

the *numerical radius* of $A \in \mathcal{B}(\mathcal{H})$. It follows readily from Theorem 3.1 that $w(A) = w(\tilde{A})$ if A and $\tilde{A}$ are defined as in Theorem 3.1. Since A has a dilation of the form $\tilde{A} \otimes I$, we have $\|A\| \leq \|\tilde{A}\|$. As shown in the proof of Theorem 3.1, there is a sequence of two dimensional subspaces $\{V_m\}$ such that the compression of A on V_m is A_{p_m} which converges to $\tilde{A}$. Thus, we have $\|A\| = \|\tilde{A}\|$. Suppose $\tilde{A}$ has singular values $s_1 \geq s_2$. Then $\|\tilde{A}\| = s_1$, $\operatorname{tr}(\tilde{A}^*\tilde{A}) = s_1^2 + s_2^2$ and $|\det(\tilde{A})| = s_1 s_2$. Hence, for $\tilde{p} = \|P\|$,

$$\begin{aligned} \|\tilde{A}\| &= \frac{1}{2}\left\{ \sqrt{\operatorname{tr}(\tilde{A}^*\tilde{A}) + 2|\det(\tilde{A})|} + \sqrt{\operatorname{tr}(\tilde{A}^*\tilde{A}) - 2|\det(\tilde{A})|} \right\} \\ &= \frac{1}{2}\left\{ \sqrt{|a|^2 + |b|^2 + (|c|^2 + |d|^2)\tilde{p}^2 + 2|ab - cd\tilde{p}^2|} \right. \end{aligned}$$

$$+\sqrt{|a|^2+|b|^2+(|c|^2+|d|^2)\tilde{p}^2-2|ab-cd\tilde{p}^2|}\Big\}.$$

By the fact that s_1^2 is the larger zero of $\det(\lambda I-\tilde{A}^*\tilde{A})$ and that $\det(\tilde{A}^*\tilde{A})=|\det(\tilde{A})|^2$, we have

$$\|\tilde{A}\|=\frac{1}{\sqrt{2}}\left\{\sqrt{\mathrm{tr}\,(\tilde{A}^*\tilde{A})+\sqrt{[\mathrm{tr}\,(\tilde{A}^*\tilde{A})]^2-4|\det(\tilde{A})|^2}}\right\}$$

$$=\frac{1}{\sqrt{2}}\sqrt{|a|^2+|b|^2+(|c|^2+|d|^2)\tilde{p}^2+\sqrt{(|a|^2+|b|^2+(|c|^2+|d|^2)\tilde{p}^2)^2-4|ab-cd\tilde{p}^2|^2}}$$

$$=\frac{1}{\sqrt{2}}\sqrt{|a|^2+|b|^2+(|c|^2+|d|^2)\tilde{p}^2+\sqrt{(|a|^2-|b|^2+(|c|^2-|d|^2)\tilde{p}^2)^2+4|a\bar{c}+\bar{b}d|^2\tilde{p}^2}}.$$

We summarize the above discussion in the following corollary, which also covers the result of Furuta [1] on $w(A)$ for A of the form (1.1) for $a,b,c,d\geq 0$.

Corollary 3.3. *Suppose A and $\tilde{A}$ satisfy the hypothesis of Theorem 3.1. Then $w(A)=w(\tilde{A})$ and $\|A\|=\|\tilde{A}\|$. In particular, if $a,b\in\mathbb{R}$ and $c,d\in\mathbb{C}$ satisfy $cd\geq 0$, then $\mathbf{cl}\,(W(A))=W(\tilde{A})$ is symmetric about the real axis, and*

$$\begin{aligned} w(A) &= w((A+A^*)/2)=w(\tilde{A})=w((\tilde{A}+\tilde{A}^*)/2)\\ &= \frac{1}{2}\left\{|a+b|+\sqrt{(a-b)^2+(|c|+|d|)^2\|P\|^2}\right\}\end{aligned}$$

and

$$\|A\|=\|\tilde{A}\|=\frac{1}{2}\left\{\sqrt{(a+b)^2+(|c|-|d|)^2\|P\|^2}+\sqrt{(a-b)^2+(|c|+|d|)^2\|P\|^2}\right\}.$$

Proof. The first assertion follows readily from Theorem 3.1. Suppose $a,b\in\mathbb{R}$ and $c,d\in\mathbb{C}$ with $cd\geq 0$. Then there is a diagonal unitary matrix $D=\mathrm{diag}\,(1,\mu)$ such that $D^*\tilde{A}D=\begin{bmatrix} a & |c|\|P\| \\ |d|\|P\| & b\end{bmatrix}$. It is then easy to get the equalities. □

Corollary 3.4. *Let A_i be self-adjoint operators on $\mathcal{H}_i$ with $\sigma(A_i)\subseteq[m,M]$ for $i=1,2$, and let T be an operator from $\mathcal{H}_2$ to $\mathcal{H}_1$. Then*

$$w\left(\begin{bmatrix} A_1 & T \\ T^* & -A_2\end{bmatrix}\right)\leq\frac{1}{2}(M-m)+\frac{1}{2}\sqrt{(M+m)^2+4\|T\|^2}. \tag{3.1}$$

Proof. For two self-adjoint operators $X,Y\in\mathcal{B}(\mathcal{H})$, we write $X\leq Y$ if $Y-X$ is positive semidefinite. Since $mI\leq A_i\leq MI$ for $i=1,2$, we have

$$\begin{bmatrix} mI & T \\ T^* & -MI\end{bmatrix}\leq\begin{bmatrix} A_1 & T \\ T^* & -A_2\end{bmatrix}\leq\begin{bmatrix} MI & T \\ T^* & -mI\end{bmatrix}.$$

By Theorem 3.1,

$$\left\|\begin{bmatrix} mI & T \\ T^* & -MI\end{bmatrix}\right\|=\left\|\begin{bmatrix} MI & T \\ T^* & -mI\end{bmatrix}\right\|=\frac{1}{2}(M-m)+\frac{1}{2}\sqrt{(M+m)^2+4\|T\|^2}.$$

The desired inequality holds. □

Note that if $X, Y \in \mathcal{B}(\mathcal{H})$, then we have the unitary similarity relations

$$\begin{aligned}\begin{bmatrix} X+iY & 0 \\ 0 & X-iY \end{bmatrix} &= \frac{1}{\sqrt{2}}\begin{bmatrix} I & iI \\ iI & I \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\begin{bmatrix} I & -iI \\ -iI & I \end{bmatrix}\frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}\begin{bmatrix} X & iY \\ iY & X \end{bmatrix}\begin{bmatrix} I & -I \\ I & I \end{bmatrix}\frac{1}{\sqrt{2}}.\end{aligned}$$

Thus,

$$\max\{\|X+iY\|, \|X-iY\|\} = \left\|\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right\| = \left\|\begin{bmatrix} X & iY \\ iY & X \end{bmatrix}\right\|.$$

Consequently, if $X, Y \in \mathcal{B}(\mathcal{H})$ are self-adjoint with $\sigma(X) \subseteq [m, M]$, then using Corollary 3.4, we have

$$\begin{aligned}\|X+iY\| &= \|X-iY\| = \left\|\begin{bmatrix} X & iY \\ iY & X \end{bmatrix}\right\| = \left\|\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right\| = \left\|\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}\right\| \\ &\le \frac{1}{2}(M-m) + \frac{1}{2}\sqrt{(M+m)^2 + 4\|Y\|^2}.\end{aligned}$$

This covers a result in [2].

4. q-NUMERICAL RANGE

For $q \in [0,1]$, the *q-numerical range* of A is the set

$$W_q(A) := \{\langle Ax, y\rangle : x, y \in \mathcal{H}, \|x\| = \|y\| = 1, \langle x, y\rangle = q\}. \tag{4.1}$$

It is known [7], [9] that

$$W_q(A) = \left\{q\langle Ax, x\rangle + \sqrt{1-q^2}\langle Ax, y\rangle : \exists \text{ orthonormal } \{x,\ y\} \subseteq \mathcal{H}\right\}, \tag{4.2}$$

and also
(4.3)

$$W_q(A) = \left\{q\mu + \sqrt{1-q^2}\nu : \exists\, x \in \mathcal{H} \text{ with } \|x\| = 1,\ \mu = \langle Ax, x\rangle,\ |\mu|^2 + |\nu|^2 \le \|Ax\|^2\right\}.$$

If $q = 1$, then $W_q(A) = W(A)$. For $0 \le q < 1$, we have the following description of $W_q(A)$ for a generalized quadratic operator $A \in \mathcal{B}(\mathcal{H})$. In particular, $W_q(A)$ will always be an open or closed elliptical disk, which may degenerate to a line segment or a point.

Theorem 4.1. *Suppose A and $\tilde{A}$ satisfy the condition in Theorem 3.1. For any $q \in [0,1)$, if there is a unit vector $z \in \mathcal{H}_1$ such that $\|T_0 z\| = \|T_0\|$, then $W_q(A) = W_q(\tilde{A})$; otherwise $W_q(A) = \mathbf{int}\left(W_q(\tilde{A})\right)$.*

We need the following lemma:

Lemma 4.2. *Let A_p be defined as in (2.2). If $p < q$, then for any unit vector $x \in \mathbb{C}^2$ there is a unit vector $x' \in \mathbb{C}^2$ such that $\langle A_p x, x\rangle = \langle A_q x', x'\rangle$ and $\|A_p x\| < \|A_q x'\|$.*

Proof. Choose a unit vector y orthogonal to x such that $A_px = \mu_1 x + \nu_1 y$. Let $U = [x\,|\,y]$. Then U is a unitary in $M_2(\mathbb{C})$. So A_p is unitarily similar to a matrix of the following form by U

$$\hat{A}_p = \begin{bmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{bmatrix}$$
$$\left(= U^* A_p U = \begin{bmatrix} x^* \\ y^* \end{bmatrix} A_p [x\,|\,y] = \begin{bmatrix} x^* \\ y^* \end{bmatrix} [A_px \mid A_py] = \begin{bmatrix} \langle A_px, x\rangle & \langle A_py, x\rangle \\ \langle A_px, y\rangle & \langle A_py, y\rangle \end{bmatrix} \right).$$

Here we remark that $\mu_1 = \langle A_px, x\rangle$ and $\|A_px\|^2 = |\mu_1|^2 + |\nu_1|^2$. Since the condition $p < q$ implies $W(A_p) \subseteq W(A_q)$ by Lemma 3.2, there exists a unit vector $x' \in W_q(A)$ such that $\langle A_px, x\rangle = \langle A_qx', x'\rangle$. Moreover there exists a unit vector y' orthogonal to x' such that $A_qx' = \mu_1 x' + \hat{\nu}_1 y'$. Then $V = [x'|y']$ is a unitary in $M_2(\mathbb{C})$. Since $\operatorname{tr} A_p = \operatorname{tr} A_q\,(= a + b = \operatorname{tr}(U^*A_pU) = \operatorname{tr}(V^*A_qV))$ and $V^*A_qV = \begin{bmatrix} \langle A_qx', x'\rangle & \langle A_qy', x'\rangle \\ \langle A_qx', y'\rangle & \langle A_qy', y'\rangle \end{bmatrix}$, we have $\langle A_px, x\rangle + \langle A_py, y\rangle = \langle A_qx', x'\rangle + \langle A_qy', y'\rangle$. It implies $\nu_2 = \langle A_py, y\rangle = \langle A_qy', y'\rangle$. Hence A_q is unitarily similar to a matrix of the following form by V

$$\hat{A}_q = \begin{bmatrix} \mu_1 & \hat{\mu}_2 \\ \hat{\nu}_1 & \nu_2 \end{bmatrix} = V^* A_q V.$$

Since $\|A_qx'\|^2 = |\mu_1|^2 + |\hat{\nu}_1|^2$, we may show $|\nu_1| < |\hat{\nu}_1|$ for this lemma.

Since a matrix $X \in M_2$ is unitarily similar to ${}^tX$ in general, we may assume that $|\hat{\nu}_1| \geq |\hat{\mu}_2|$. By basic calculations we have

$$\begin{aligned} |\hat{\nu}_1|^2 + |\hat{\mu}_2|^2 - |\nu_1|^2 - |\mu_2|^2 &= \operatorname{tr}(\hat{A}_q^*\hat{A}_q - \hat{A}_p^*\hat{A}_p) = \operatorname{tr}(A_q^*A_q - A_p^*A_p) \\ &= (|c|^2 + |d|^2)(q^2 - p^2) > 0, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \big|\,|\hat{\nu}_1\hat{\mu}_2| - |\nu_1\mu_2|\,\big| &\leq |\hat{\nu}_1\hat{\mu}_2 - \nu_1\mu_2| = |\det(\hat{A}_p) - \det(\hat{A}_q)| \\ &= |\det(A_p) - \det(A_q)| = |cd|(q^2 - p^2). \end{aligned} \tag{4.5}$$

The above two inequalities (4.4) and (4.5) implies

$$(|\hat{\nu}_1| + |\hat{\mu}_2|)^2 - (|\nu_1| + |\mu_2|)^2 \geq (|c| - |d|)^2(q^2 - p^2) \geq 0$$

and

$$(|\hat{\nu}_1| - |\hat{\mu}_2|)^2 - (|\nu_1| - |\mu_2|)^2 \geq (|c| - |d|)^2(q^2 - p^2) \geq 0.$$

So we have

$$|\hat{\nu}_1| + |\hat{\mu}_2| \geq |\nu_1| + |\mu_2| \quad \text{and} \quad |\hat{\nu}_1| - |\hat{\mu}_2| \geq \big||\nu_1| - |\mu_2|\big| \geq |\nu_1| - |\mu_2| \tag{4.6}$$

which implies that $|\hat{\nu}_1| \geq |\nu_1|$. From the proof, we can see that if $|\hat{\nu}_1| = |\nu_1|$, then we have $|\hat{\mu}_2| = |\mu_2|$ by (4.6). Then the left hand side of (4.4) is 0, a contradiction. Therefore, we must have $|\hat{\nu}_1| > |\nu_1|$ and the result follows. □

Proof of Theorem 4.1. Since the operator A has a dilation of the form $\tilde{A} \otimes I$, we have

$$W_q(A) \subseteq W_q(\tilde{A} \otimes I) = W_q(\tilde{A}).$$

Let $P = |T_0|$ and $\{z_m\}$ be a sequence of unit vectors in $\mathcal{H}_1$ such that $\langle Pz_m, z_m\rangle = p_m \longrightarrow \|P\| = p$. The compression of A on the subspace $V_m = \text{span}\left\{\begin{bmatrix} z_m \\ 0\end{bmatrix}, \begin{bmatrix} 0 \\ z_m\end{bmatrix}\right\}$ equals A_{p_m} as defined in (2.2). Indeed, we have $\left\langle A\begin{bmatrix}\alpha z_m \\ \beta z_m\end{bmatrix}, \begin{bmatrix}\alpha z_m \\ \beta z_m\end{bmatrix}\right\rangle = \left\langle A_{p_m}\begin{bmatrix}\alpha \\ \beta\end{bmatrix}, \begin{bmatrix}\alpha \\ \beta\end{bmatrix}\right\rangle$ for any $\begin{bmatrix}\alpha z_m \\ \beta z_m\end{bmatrix} \in V_m$. Thus, $W_q(A_{p_m}) \subseteq W_q(A)$ for all m.

Suppose that there is a unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\| = p$. Then we may assume that $z_m = z$ for each m so that $W_q(\tilde{A})\,(= W_q(A_p)) \subseteq W_q(A)$. So we have $W_q(A) = W_q(\tilde{A})$.

Suppose there is no unit vector $z \in \mathcal{H}_1$ such that $\|Pz\| = \|P\|$. Since $A_{p_m} \longrightarrow \tilde{A}$, we see that $\mathbf{int}(W_q(\tilde{A})) \subseteq W_q(A)$. For any unit vectors $x, y \in \mathcal{H}$ with $\langle x, y\rangle = q$, we put $x = \begin{bmatrix}\alpha_1 u_1 \\ \alpha_2 u_2\end{bmatrix}, y = \begin{bmatrix}\beta_1 u_1 + \gamma_1 v_1 \\ \beta_2 u_2 + \gamma_2 v_2\end{bmatrix} \in \mathcal{H}_1 \oplus \mathcal{H}_1$ such that $u_1, u_2, v_1, v_2 \in \mathcal{H}_1$ are unit vectors with $u_i \perp v_i$ and $\alpha_i, \beta_i, \gamma_i \in \mathbb{C}$ for $i = 1, 2$. Then the compression of A on

$$V = \text{span}\left\{\begin{bmatrix}u_1 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ u_2\end{bmatrix}, \begin{bmatrix}v_1 \\ 0\end{bmatrix}, \begin{bmatrix}0 \\ v_2\end{bmatrix}\right\}$$

has the form

$$B = \begin{bmatrix} aI_2 & cS \\ dS^* & bI_2\end{bmatrix}$$

where $S \in M_2$ satisfies $\|S\| < \|P\|$. Let $\tilde{B} \equiv A_{\|S\|}$. Since $W(B) \subseteq W(\tilde{B})$ by Theorem 3.1, B has a dilation $\tilde{B} \otimes I$. Therefore, $W_q(B) \subseteq W_q(\tilde{B} \otimes I) = W_q(\tilde{B})$. Let $\zeta = \langle Ax, y\rangle \in W_q(A)$. Since B is a compression of A on V, we have $\zeta \in W_q(B)(\subset W_q(\tilde{B}))$. By the inequality (4.2), there exist orthogonal vectors $x', y' \in \mathbb{C}^2$ such that $\zeta = q\langle \tilde{B}x', x'\rangle + \sqrt{1-q^2}\langle \tilde{B}x', y'\rangle$. Moreover there exist μ_1, ν_1 in $\mathbb{C}$ such that $\tilde{B}x' = \mu_1 x' + \nu_1 y'$. We see $\mu_1 = \langle \tilde{B}x', x'\rangle$, $\nu_1 = \langle \tilde{B}x', y'\rangle$ and so $\zeta = q\mu_1 + \sqrt{1-q^2}\nu_1$. Let $U = [x'|y']$ be a unitary. Hence $\tilde{B}$ is unitarily similar to a matrix of the form

$$\hat{B} = \begin{bmatrix}\mu_1 & \mu_2 \\ \nu_1 & \nu_2\end{bmatrix}\left(= U^*\tilde{B}U = \begin{bmatrix}\langle \tilde{B}x, x\rangle & \langle \tilde{B}y, x\rangle \\ \langle \tilde{B}x, y\rangle & \langle \tilde{B}y, y\rangle\end{bmatrix}\right).$$

Hence we remark that $\tilde{B} = A_{\|S\|}$ and $\tilde{A} = A_{\|P\|}$ ($\|S\| < \|P\|$). By Lemma 4.2, there exists a unit vector y'' in $\mathbb{C}^2$ that $(\mu_1 =)\langle \tilde{B}x', x'\rangle = \langle \tilde{A}y'', y''\rangle$ and $\|\tilde{B}x'\| < \|\tilde{A}y''\|$. Let $z = \begin{bmatrix}1 \\ 0\end{bmatrix}$. Then we have $\|\hat{B}z\| = \|\tilde{B}x'\| = \sqrt{|\mu_1|^2 + |\nu_1|^2}$ and $\langle \hat{B}z, z\rangle = \langle \tilde{B}z, z\rangle = \mu_1$, and so

$$\begin{aligned}
\zeta = q\mu_1 + \sqrt{1-q^2}\nu_1 \quad &\in \left\{q\mu_1 + \sqrt{1-q^2}\nu : \mu_1 = \langle \hat{B}z, z\rangle,\ |\mu_1|^2 + |\nu|^2 \le \|\hat{B}z\|^2\right\} \\
&= \left\{q\mu_1 + \sqrt{1-q^2}\nu : \mu_1 = \langle \tilde{B}x', x'\rangle,\ |\mu_1|^2 + |\nu|^2 \le \|\tilde{B}x'\|^2\right\} \\
&\subsetneq \left\{q\mu_1 + \sqrt{1-q^2}\nu : \mu_1 = \langle \tilde{A}y'', y''\rangle,\ |\mu_1|^2 + |\nu|^2 < \|\tilde{A}y''\|^2\right\} \\
&\qquad\qquad\qquad\qquad (\text{by } \|\tilde{B}x''\| < \|\tilde{A}y''\|) \\
&\subseteq \mathbf{int}W_q(\tilde{A}).
\end{aligned}$$

In above, we remark that

$$\left\{(\mu_1,\nu) : |\mu_1|^2 + |\nu|^2 < \|\tilde{A}y''\|^2\right\} \subset \left\{(\mu,\nu) : |\mu|^2 + |\nu|^2 < \|\tilde{A}y''\|^2\right\}$$
$$\subset \mathbf{int}\left\{(\mu,\nu) : |\mu|^2 + |\nu|^2 \leq \|\tilde{A}y''\|^2\right\}.$$

Hence the proof is completed. □

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