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Riemannian mean and matrix inequalities related to chaotic order and Ando-Hiai inequality

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Abstract. Riemannian mean is a kind of geometric mean of n-matrices which is an extension of geometric mean of 2-matrices. In this paper, we shall show some matrix inequalities via Riemannian mean which are extensions of well-known matrix inequalities via geometric mean of 2-matrices. Exactly, we shall show extensions of so-called Ando-Hiai inequality and a characterization of chaotic order. Lastly, we shall discuss about the problem whether the same results are satisfied or not for other geometric means of n-matrices.

1. Introduction

For positive invertible matrices A and B, their weighted geometric mean $A\sharp_{\alpha}B$ is well known as

$$A\sharp_{\alpha}B=A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\alpha}A^{\frac{1}{2}}\quad\text{for }\alpha\in[0,1]. \tag{1.1}$$

Especially, in the case $\alpha=\frac{1}{2}$, we say $A\sharp_{\frac{1}{2}}B$ just a geometric mean, and denote it by $A\sharp B$, simply. If A and B be non-invertible positive matrices, their geometric mean can be defined by

$$A\sharp_{\alpha}B=\lim_{\varepsilon\to+0}(A+\varepsilon I)\sharp_{\alpha}(B+\varepsilon I)\quad\text{for }\alpha\in[0,1].$$

To extend the definition of $A\sharp B$ into geometric mean of n-matrices was a long standing problem. Recently, a nice definition of geometric mean of n-matrices was given in [3]. Since then, many authors study geometric mean of n-matrices, and we know three kind of definitions of geometric means. The one is defined by Ando-Li-Mathias in [3], the second one is defined in [9, 6] which is a modification of geometric mean by Ando-Li-Mathias. The third one is called Riemannian mean or the least squares mean defined in [5, 10, 12]. These geometric means have the same 10 properties including monotonicity and arithmetic-geometric means inequality (which will be introduced in the later).

On the other hands, there are many results on geometric mean of 2-matrices. Especially, the following result is well known as Ando-Hiai inequality [2]: Let $\alpha\in[0,1]$. Then for positive matrices A and B,

$$A\sharp_{\alpha}B\leq I\quad\text{implies}\quad A^{p}\sharp_{\alpha}B^{p}\leq I\quad\text{for all }p\geq 1,$$

where the order is defined by positive definiteness in the whole paper.

For positive invertible matrices A and B, the order $\log A\geq\log B$ is called chaotic order. It is a weaker order than the usual order $A\geq B$

since $\log t$ is an operator monotone function. As a characterization of chaotic order, it is well known that the following statements are mutually equivalent [1, 7, 8, 15]:

(1) $\log A \geq \log B$,
(2) $A^p \geq (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geq 0$,
(3) $A^r \geq (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ for all $p, r \geq 0$.

Ando-Hiai inequality and the above characterization of chaotic order are well known and important in the theory of matrix (operator) inequalities.

In this paper, we shall show some matrix inequalities via Riemannian mean. Some of them are extensions of Ando-Hiai inequality and characterization of chaotic order introducing in the above. In Section 2, we shall introduce the definition of Riemannian mean and its basic properties. In Section 3, we will show some matrix inequalities of Riemannian mean which include extensions of Ando-Hiai inequality and characterization of chaotic order. In Section 4, we will discuss whether our results hold for other two geometric means or not.

2. Riemannian mean and its basic properties

In this section, we shall introduce the definition of Riemannian mean and its basic properties. In what follows let $M_m(\mathbb{C})$ be the set of all $m \times m$ matrices on $\mathbb{C}$, and let $P_m(\mathbb{C})$ be the set of all $m \times m$ positive invertible matrices. For $A, B \in M_m(\mathbb{C})$, define an inner product $\langle A, B \rangle$ by $\langle A, B \rangle = \mathrm{tr} A^* B$. Then $M_m(\mathbb{C})$ is an inner product space equipped with the norm $\|A\|_2 = (\mathrm{tr} A^* A)^{\frac{1}{2}}$, moreover $P_m(\mathbb{C})$ is a differential manifold, and we can consider the geodesic $[A, B] \subset P_m(\mathbb{C})$ which includes $A, B \in P_m(\mathbb{C})$. It can be parameterized as follows:

Theorem A ([4, 5]). *Let $A, B \in P_m(\mathbb{C})$. Then there exists a unique geodesic $[A, B]$ joining A and B. It has a parametrization*

$$\gamma(t) = A \natural_t B = A^{\frac{1}{2}} (A^{\frac{-1}{2}} B A^{\frac{-1}{2}})^t A^{\frac{1}{2}}, \quad t \in [0, 1].$$

Furthermore, we have a distance $\delta_2(A, B)$ between A and B along the geodesic $[A, B]$ as

$$\delta_2(A, B) = \| \log A^{\frac{-1}{2}} B A^{\frac{-1}{2}} \|_2.$$

We call the metric $\delta_2(A, B)$ between A and B by Riemannian metric.

A vector $\omega = (w_1, w_2, \cdots, w_n)$ is called a probability vector if and only if its components satisfy $\sum_i w_i = 1$ and $w_i > 0$ for $i = 1, 2, \cdots, n$. Then weighted Riemannian mean is defined as follows:

Definition 1 ([4, 5, 10, 12]). *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$, and $\omega = (w_1, \cdots, w_n)$ be a probability vector. Then weighted Riemannian mean*

$\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n) \in P_m(\mathbb{C})$ *is defined by*

$$\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n) = \operatorname*{argmin}_{X\in P_m(\mathbb{C})} \sum_{i=1}^{n} w_i \delta_2^2(A_i, X),$$

where argmin$f(X)$ *means the point* X_0 *which attains minimum value of the function* $f(X)$.

It is easy to see that weighted Riemannian mean of 2-matrices just coincides with geometric mean in (1.1) by the following property of Riemannian metric.

$$\delta_2(A, A\sharp_\alpha B) = \alpha\delta_2(A, B) \quad \text{for } \alpha \in [0, 1].$$

This definition is firstly introduced in [5, 12] for the case of $\omega = (\frac{1}{n}, \cdots, \frac{1}{n})$. In this case, we denote weighted Riemannian mean by $\mathfrak{G}_\delta(A_1, \cdots, A_n)$, simply, and we call it just a Riemannian mean. Existence and uniqueness of Riemannian mean have been already shown in [5, 12]. Recently, Lawson and Lim defined weighted Riemannian mean in [10], generally.

It is known that Riemannian mean satisfies the following 10 properties: Let $A_i \in P_m(\mathbb{C})$, $i = 1, 2, \cdots, n$, and $\omega = (w_1, \cdots, w_n)$ be a probability vector. Then

(P1) If $A_1, \ldots, A_n$ commute with each other, then

$$\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) = A_1^{w_1} \cdots A_n^{w_n}.$$

(P2) Joint homogeneity.

$$\mathfrak{G}_\delta(\omega; a_1A_1, \ldots, a_nA_n) = a_1^{w_1} \cdots a_n^{w_n} \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n)$$

for positive numbers $a_i > 0$ $(i = 1, \ldots, n)$.

(P3) Permutation invariance. For any permutation π on $\{1, 2, \cdots, n\}$,

$$\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) = \mathfrak{G}_\delta(\pi(\omega); A_{\pi(1)}, \ldots, A_{\pi(n)}),$$

where $\pi(\omega) = (w_{\pi(1)}, \cdots, w_{\pi(n)})$.

(P4) Monotonicity. For each $i = 1, 2, \ldots, n$, if $B_i \le A_i$, then

$$\mathfrak{G}_\delta(\omega; B_1, \ldots, B_n) \le \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n).$$

(P5) Continuity. For each $i = 1, 2, \ldots, n$, let $\{A_i^{(k)}\}_{k=1}^{\infty}$ be positive invertible matrix sequences such that $A_i^{(k)} \to A_i$ as $k \to \infty$. Then

$$\mathfrak{G}_\delta(\omega; A_1^{(k)}, \ldots, A_n^{(k)}) \to \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \quad \text{as } k \to \infty.$$

(P6) Congruence invariance. For any invertible matrix S,

$$\mathfrak{G}_\delta(\omega; S^*A_1S, \ldots, S^*A_nS) = S^*\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n)S.$$

(P7) Joint concavity.

$$\mathfrak{G}_\delta(\omega; \lambda A_1 + (1-\lambda)A_1', \ldots, \lambda A_n + (1-\lambda)A_n') \ge \lambda\mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) + (1-\lambda)\mathfrak{G}_\delta(\omega; A_1', \ldots, A_n') \quad \text{for } 0 \le \lambda \le 1.$$

(P8) Self-duality.

$$\mathfrak{G}_\delta(\omega; A_1^{-1}, \ldots, A_n^{-1})^{-1} = \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n).$$

(P9) Determinantial identity.

$$\det \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) = \prod_{i=1}^{n} (\det A_i)^{w_i}.$$

(P10) Arithmetic-geometric-harmonic means inequalities.

$$\left(\sum_{i=1}^{n} w_i A_i^{-1} \right)^{-1} \leq \mathfrak{G}_\delta(\omega; A_1, \ldots, A_n) \leq \sum_{i=1}^{n} w_i A_i.$$

Moreover, instead of continuity (P5), weighted Riemannian mean satisfies non-expansive property as follows:

(P5') $$\delta_2(\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n), \mathfrak{G}_\delta(\omega; B_1, \cdots, B_n)) \leq \sum_{i=1}^{n} w_i \delta_2(A_i, B_i).$$

(P3), (P5), (P6) and (P8) follow from the definition of weighted Riemannian mean and properties of Riemannian metric [4, 5, 10]. (P1), (P2), (P9) and (P10) follow from the following characterization of weighted Riemannian mean [10, 12, 16].

Theorem B ([10, 12]). *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$, and $\omega = (w_1, \cdots, w_n)$ be a probability vector. Then $X = \mathfrak{G}_\delta(\omega; A_1, \cdots, A_n)$ is a unique positive solution of the following matrix equation:*

$$w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}} = 0.$$

(P4) and (P7) are not easy consequences. But very recently, Lawson and Lim have given a proof of (P4) and (P7) in [10] by using Sturm's result [14], and then Lawson and Lim showed that weighted Riemannian mean satisfied (P5') in [10]. Theorem B has been obtained by Moakher in [12] in the case of just a Riemannian mean, and then Lawson and Lim obtained Theorem B in [10], completely.

3. Main results

In this section, we shall show further properties of weighted Riemannian mean. Almost these results are matrix inequalities, and some of them extends well-known matrix (operator) inequalities introduced in Section 1.

Theorem 1. *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$, and $\omega = (w_1, \cdots, w_n)$ be a probability vector. Then $w_1 \log A_1 + \cdots + w_n \log A_n \leq 0$ implies $\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n) \leq I$.*

Proof. If $w_1 \log A_1 + \cdots + w_n \log A_n \leq 0$, then there exists a matrix $A \in P_m(\mathbb{C})$ such that $A \geq I$ and

$$\frac{w_1}{2}\log A_1 + \cdots + \frac{w_n}{2}\log A_n + \frac{1}{2}\log A = 0.$$

Then $\omega_1 = (\frac{w_1}{2}, \cdots, \frac{w_n}{2}, \frac{1}{2})$ is a probability vector, and by Theorem B, we have

$$\mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, A) = I.$$

Define a matrix sequence $\{G_n\}_{n=0}^{\infty}$ by

$$G_{n+1} = \mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, G_n) \quad \text{and} \quad G_0 = \mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, I).$$

Then by $A \geq I$ and monotonicity (P4) of weighted Riemannian mean, we have

$$I = \mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, A) \geq \mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, I) = G_0,$$

and hence we obtain

$$I \geq G_0 \geq G_1 \geq \cdots \geq G_n \geq \cdots \geq 0.$$

Therefore a matrix sequence $\{G_n\}_{n=0}^{\infty}$ converges to a positive semidefinite matrix.

Let $X = \mathfrak{G}_\delta(\omega; A_1, \cdots, A_n)$. We shall show that G_n converges to X. Noting that by Theorem B, we have

$$0 = \sum_{i=1}^{n} w_i \log X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}} = \sum_{i=1}^{n} \frac{w_i}{2} \log X^{\frac{-1}{2}} A_i X^{\frac{-1}{2}} + \frac{1}{2}\log X^{\frac{-1}{2}} X X^{\frac{-1}{2}},$$

and hence

$$\mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, X) = X.$$

Then by non-expansive property (P5'), we have

$$\begin{aligned}
\delta_2(X, G_k) &= \delta_2(\mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, X), \mathfrak{G}_\delta(\omega_1; A_1, \cdots, A_n, G_{k-1})) \\
&\leq \frac{1}{2}\delta_2(X, G_{k-1}) \\
&\leq \cdots \\
&\leq \left(\frac{1}{2}\right)^k \delta_2(X, G_0) \to 0 \quad \text{as } k \to +\infty,
\end{aligned}$$

and hence $G_k \to X$ as $k \to +\infty$. Since $\{G_k\}_{k=0}^{\infty}$ is contractive and decreasing sequence, we have

$$\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n) = X \leq I.$$

□

Theorem 2. *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$. If $\mathfrak{G}_\delta(\omega; A_1, \cdots, A_n) \leq I$ holds for a probability vector ω, then $\mathfrak{G}_\delta(\omega; A_1^p, \cdots, A_n^p) \leq I$ holds for all $p \geq 1$.*

Theorem 2 is an extension of the following Ando-Hiai inequality, because $\mathfrak{G}_\delta(1-\alpha,\alpha;A,B)=A\sharp_\alpha B$.

Theorem C (Ando-Hiai inequality [2]). *Let A and B be positive matrices. For any $\alpha\in[0,1]$, $A\sharp_\alpha B\le I$ implies $A^p\sharp_\alpha B^p\le I$ for all $p\ge 1$.*

Proof of Theorem 2. Let $\omega=(w_1,\cdots,w_n)$ and $X=\mathfrak{G}_\delta(\omega;A_1,\cdots,A_n)\le I$. Then for $p\in[1,2]$, we have

$$\begin{aligned}0&=p(w_1\log X^{\frac12}A_1^{-1}X^{\frac12}+\cdots+w_n\log X^{\frac12}A_n^{-1}X^{\frac12})\\&=w_1\log(X^{\frac12}A_1^{-1}X^{\frac12})^p+\cdots+w_n\log(X^{\frac12}A_n^{-1}X^{\frac12})^p\\&\le w_1\log X^{\frac12}A_1^{-p}X^{\frac12}+\cdots+w_n\log X^{\frac12}A_n^{-p}X^{\frac12},\end{aligned}$$

where the last inequality holds since $\log t$ is operator monotone and Hansen's inequality for $p\in[1,2]$ and $X\le I$. It is equivalent to

$$w_1\log X^{\frac{-1}{2}}A_1^pX^{\frac{-1}{2}}+\cdots+w_n\log X^{\frac{-1}{2}}A_n^pX^{\frac{-1}{2}}\le 0,$$

and by Theorem 1, we have

$$\mathfrak{G}_\delta(\omega;X^{\frac{-1}{2}}A_1^pX^{\frac{-1}{2}},\cdots,X^{\frac{-1}{2}}A_n^pX^{\frac{-1}{2}})\le I,$$

and then

$$\mathfrak{G}_\delta(\omega;A_1^p,\cdots,A_n^p)\le X=\mathfrak{G}_\delta(\omega;A_1,\cdots,A_n)\le I$$

for $p\in[1,2]$ by (P6). Repeating this procedure for $\mathfrak{G}_\delta(\omega;A_1^p,\cdots,A_n^p)\le I$, the proof is complete. □

Let $p_1,\cdots,p_n$ be positive numbers. For $i=1,2,\cdots,n$, we denote $\prod_{j\ne i}p_j$ by $p_{\ne i}$.

Theorem 3. *Let $A_1,\cdots,A_n\in P_m(\mathbb{C})$. Then the following assertions are mutually equivalent;*

(1) $\log A_1+\cdots+\log A_n\le 0$,
(2) $\mathfrak{G}_\delta(A_1^p,\cdots,A_n^p)\le I$ *for all* $p\ge 1$,
(3) $\mathfrak{G}_\delta(\omega';A_1^{p_1},\cdots,A_n^{p_n})\le I$ *for all* $p_i\ge 1$, $i=1,2,\cdots,n$,

where ω' is a probability vector defined by

$$\omega'=\Big(\frac{p_{\ne 1}}{\sum_i p_{\ne i}},\cdots,\frac{p_{\ne n}}{\sum_i p_{\ne i}}\Big).$$

Theorem 3 is an extension of the following characterization of chaotic order:

Theorem D (Characterization of chaotic order [1, 7, 8, 15]). *Let A and B be positive invertible matrices. Then the following assertions are mutually equivalent:*

(1) $\log A\ge\log B$,
(2) $A^p\ge(A^{\frac p2}B^pA^{\frac p2})^{\frac12}$ *for all* $p\ge 0$,
(3) $A^r\ge(A^{\frac r2}B^pA^{\frac r2})^{\frac{r}{p+r}}$ *for all* $p,r\ge 0$.

In fact, Theorem D can be rewritten in the following form:

Theorem D'. *Let A and B be positive invertible matrices. Then the following assertions are mutually equivalent:*

(1) $\log A + \log B \leq 0$,
(2) $A^p \natural B^p \leq I$ *for all* $p \geq 0$,
(3) $A^r \natural_{\frac{r}{p+r}} B^p \leq I$ *for all* $p, r \geq 0$.

To prove Theorem 3, we need the following result:

Theorem E ([13]). *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$. Then*

$$\lim_{p \to +0} \left(\frac{A_1^p + \cdots + A_n^p}{n} \right)^{\frac{1}{p}} = \exp \left(\frac{\log A_1 + \cdots + \log A_n}{n} \right),$$

uniformly.

Proof of Theorem 3. Proof of (1) $\to$ (3). If $\log A_1 + \cdots + \log A_n \leq 0$, then we have

$$\frac{\prod_i p_i}{\sum_i p_{\neq i}} (\log A_1 + \cdots + \log A_n) \leq 0,$$

i.e.,

$$\frac{p_{\neq 1}}{\sum_i p_{\neq i}} \log A_1^{p_1} + \cdots + \frac{p_{\neq n}}{\sum_i p_{\neq i}} \log A_n^{p_n} \leq 0.$$

Hence by Theorem 1, we have

$$\mathfrak{G}_\delta(\omega'; A_1^{p_1}, \cdots, A_n^{p_n}) \leq I$$

for all $p_i \geq 1$, $i = 1, 2, \cdots, n$.

Proof of (3) $\longrightarrow$ (2) is easy by putting $p_1 = \cdots = p_n = p$.

Proof of (2) $\longrightarrow$ (1). By geometric-harmonic means inequality, we have

$$I \geq \mathfrak{G}_\delta(A_1^p, \cdots, A_n^p) \geq \left(\frac{A_1^{-p} + \cdots + A_n^{-p}}{n} \right)^{-1}.$$

By Theorem E, we have

$$\begin{aligned} I &\geq \lim_{p \to +0} \left(\frac{A_1^{-p} + \cdots + A_n^{-p}}{n} \right)^{\frac{-1}{p}} \\ &= \left(\exp \frac{\log A_1^{-1} + \cdots + \log A_n^{-1}}{n} \right)^{-1} = \exp \frac{\log A_1 + \cdots + \log A_n}{n}. \end{aligned}$$

Hence we have (1). $\square$

4. Other geometric means

In the previous section, we showed further properties of weighted Riemannian mean. Here one might expect that other geometric means satisfy the same properties stated in the previous section. In this section, we shall discuss about this problem, and we will give a negative answer for it.

It is known that there are two types of geometric means of n-matrices except weighted Riemannian mean which satisfy 10 properties (P1)–(P10) stated in Section 2. The most famous one has been defined by Ando-Li-Mathias in [3]. In this paper, we call it ALM mean. The other one is defined by Bini-Meini-Poloni and Izumino-Nakamura, independently in [6, 9]. We call it BMP mean in this paper. Weighted BMP mean has been considered in [6, 9], and recently weighted interpolation mean between ALM and BMP means has been defined in [11].

Theorem 4. *Let $A_1, \cdots, A_n \in P_m(\mathbb{C})$, ω be a probability vector, and $\mathfrak{G}(\omega; A_1, \cdots, A_n)$ be a weighted geometric mean satisfying properties (P1)-(P10). If the weighted geometric mean satisfies Theorem 2, then the weighted geometric mean $\mathfrak{G}$ coincides with weighted Riemannian mean.*

Proof. Let $\omega = (w_1, \cdots, w_n)$. If $w_1 \log A_1 + \cdots + w_n \log A_n \leq 0$ is satisfied, then by arithmetic-geometric means inequality, we have

$$I \geq w_1(I+\frac{\log A_1}{k})+\cdots+w_n(I+\frac{\log A_n}{k}) \geq \mathfrak{G}\left(\omega; I+\frac{\log A_1}{k}, \cdots, I+\frac{\log A_n}{k}\right)$$

hold for sufficiently large k. Since the weighted geometric mean $\mathfrak{G}$ satisfies Theorem 2, we have

$$\mathfrak{G}\left(\omega; (I+\frac{\log A_1}{k})^k, \cdots, (I+\frac{\log A_n}{k})^k\right) \leq I.$$

By well-known formula $\lim_{k\to+\infty}(I+\frac{\log A_i}{k})^k = A_i$ and (P5), we have

$$\mathfrak{G}(\omega; A_1, \cdots, A_n) \leq I,$$

i.e., weighted geometric mean $\mathfrak{G}$ satisfies Theorem 1.

If geometric mean satisfies Theorem 1, we have

$$\begin{aligned} \sum_{i=1}^{n} w_i \log A_i \geq 0 &\iff \sum_{i=1}^{n} w_i \log A_i^{-1} \leq 0 \\ &\implies \mathfrak{G}(\omega; A_1^{-1}, \cdots, A_n^{-1}) \leq I \\ &\iff \mathfrak{G}(\omega; A_1, \cdots, A_n) \geq I \quad \text{by (P8)}. \end{aligned}$$

Hence we obtain

$$w_1 \log A_1 + \cdots + w_n \log A_n = 0 \implies \mathfrak{G}(\omega; A_1, \cdots, A_n) = I. \tag{4.1}$$

Let $X = \mathfrak{G}_\delta(\omega; A_1, \cdots, A_n)$ be a weighted Riemannian mean. Then by Theorem B, we have

$$w_1 \log X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}} + \cdots + w_n \log X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}} = 0,$$

and by (4.1) and (P6),

$$\begin{aligned}&\mathfrak{G}(\omega; X^{\frac{-1}{2}} A_1 X^{\frac{-1}{2}}, \cdots, X^{\frac{-1}{2}} A_n X^{\frac{-1}{2}}) = I \\ &\Longleftrightarrow \mathfrak{G}(\omega; A_1, \cdots, A_n) = X = \mathfrak{G}_\delta(\omega; A_1, \cdots, A_n).\end{aligned}$$

It completes the proof. □

Generally, ALM, BMP and Riemannian means are different from each other. Here we shall introduce a concrete example. Before introducing an example, we shall introduce definitions of ALM and BMP means in the 3-matrices cases, briefly.

Let $A, B, C \in P_m(\mathbb{C})$. Define matrices sequences $\{A_n\}_{n=0}^\infty$, $\{B_n\}_{n=0}^\infty$, $\{C_n\}_{n=0}^\infty$ as follows: $A_0 = A$, $B_0 = B$, $C_0 = C$ and

$$A_{n+1} = B_n \sharp C_n, \quad B_{n+1} = C_n \sharp A_n, \quad C_{n+1} = A_n \sharp B_n.$$

Then we can obtain the same limit, and we define it as ALM mean [3] (denoted by $\mathfrak{G}_{alm}(A, B, C)$), i.e.,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \mathfrak{G}_{alm}(A, B, C).$$

On the other hand, BMP mean is defined as follows: $A_0 = A$, $B_0 = B$, $C_0 = C$ and

$$A_{n+1} = (B_n \sharp C_n) \sharp_{\frac{1}{3}} A_n, \quad B_{n+1} = (C_n \sharp A_n) \sharp_{\frac{1}{3}} B_n, \quad C_{n+1} = (A_n \sharp B_n) \sharp_{\frac{1}{3}} C_n.$$

Then we can obtain the same limit, and we define it as BMP mean [6, 9] (denoted by $\mathfrak{G}_{bmp}(A, B, C)$), i.e.,

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = \mathfrak{G}_{bmp}(A, B, C).$$

Example. Let

$$A = \begin{pmatrix} 18 & 5 \\ 5 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 200 \end{pmatrix}, \quad C = \begin{pmatrix} 75 & 54 \\ 54 & 40 \end{pmatrix}.$$

Then

$$G_1 = \mathfrak{G}_{alm}(A, B, C) = \begin{pmatrix} 9.06732 & 4.86436 \\ 4.86436 & 8.89146 \end{pmatrix}$$

and

$$\begin{aligned}&\log G_1^{\frac{1}{2}} A^{-1} G_1^{\frac{1}{2}} + \log G_1^{\frac{1}{2}} B^{-1} G_1^{\frac{1}{2}} + \log G_1^{\frac{1}{2}} C^{-1} G_1^{\frac{1}{2}} \\ &\qquad = \begin{pmatrix} -0.263706 & -0.0340424 \\ -0.0340424 & 0.263706 \end{pmatrix} \neq O.\end{aligned}$$

Hence by Theorem B, $\mathfrak{G}_\delta(A,B,C) \neq G_1 = \mathfrak{G}_{alm}(A,B,C)$. On the other hand,

$$G_2 = \mathfrak{G}_{bmp}(A,B,C) = \begin{pmatrix} 9.39875 & 4.91569 \\ 4.91659 & 8.63133 \end{pmatrix}$$

and

$$\log G_2^{\frac{1}{2}} A^{-1} G_2^{\frac{1}{2}} + \log G_2^{\frac{1}{2}} B^{-1} G_2^{\frac{1}{2}} + \log G_2^{\frac{1}{2}} C^{-1} G_2^{\frac{1}{2}}$$
$$= \begin{pmatrix} -0.101249 & -0.0568546 \\ -0.0568546 & 0.101249 \end{pmatrix} \neq O.$$

Hence by Theorem B, $\mathfrak{G}_\delta(A,B,C) \neq G_2 = \mathfrak{G}_{bmp}(A,B,C)$.

Corollary 5. *ALM and BMP means do not satisfy Theorems 1 and 2.*

Proof. ALM and BMP means do not coincide with Riemannian mean. Hence by Theorem 4 and its proof, the proof is complete. □

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