A CORRESPONDENCE OF CANONICAL Bases IN THE q-DEFORMED HIGHER LEVEL FOck SPACES

KAZUTO IJJIMA

ABSTRACT. The q-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado. The q-decomposition matrix is a transition matrix from the standard basis to the canonical basis defined by Uglov in the q-deformed Fock space. In this paper, we show that parts of q-decomposition matrices of level \( \ell \) coincide with that of level \( \ell - 1 \) under certain conditions of multi charge.

1. Introduction

The q-deformed Fock spaces of higher levels were introduced by Jimbo-Misra-Miwa-Okado [JMM09]. For a multi charge \( s = (s_1, \ldots, s_\ell) \in \mathbb{Z}_{\ell}^\ell \), the q-deformed Fock space \( F_q[s] \) of level \( \ell \) is the \( \mathbb{Q}(q) \)-vector space whose basis are indexed by \( \ell \)-tuples of Young diagrams, i.e. \( \{ |\lambda; s\rangle | \lambda \in \Pi^\ell \} \), where \( \Pi \) is the set of Young diagrams.

The canonical bases \( \{ G^+(\lambda; s) | \lambda \in \Pi^\ell \} \) and \( \{ G^-(\lambda; s) | \lambda \in \Pi^\ell \} \) are bases of the Fock space \( F_q[s] \) that are invariant under a certain involution \( - \) [Ug00]. Define matrices \( \Delta^+(q) = (\Delta^+_{\lambda,\mu}(q))_{\lambda,\mu} \) and \( \Delta^-(q) = (\Delta^-_{\lambda,\mu}(q))_{\lambda,\mu} \) by

\[
G^+(\lambda; s) = \sum_\mu \Delta^+_{\lambda,\mu}(q) |\mu; s\rangle, \quad G^-(\lambda; s) = \sum_\mu \Delta^-_{\lambda,\mu}(q) |\mu; s\rangle.
\]

We call \( \Delta^+_{\lambda,\mu}(q) \) and \( \Delta^-_{\lambda,\mu}(q) \) q-decomposition numbers. These q-decomposition matrices play an important role in representation theory. However, it is difficult to compute q-decomposition matrices.

In the case of \( \ell = 1 \), Varagnolo-Vasserot [VV99] proved that \( \Delta^+(q) \) coincides with the decomposition matrix of \( \nu \)-Schur algebra. For \( \ell \geq 2 \), Yvonne [Yvo07] conjectured that the matrix \( \Delta^+(q) \) coincides with the \( q \)-analogue of the decomposition matrix of cyclotomic Schur algebras at a primitive \( n \)-th root of unity under a suitable condition of multi charge. Rouquier [Rou08, Theorem 6.8, \S 6.5] conjectured that, for arbitrary multi charge, the multiplicities of simple modules in standard modules in the category \( \mathcal{O} \) of rational Cherednik algebras are equal to the corresponding coefficients \( \Delta^+_s(q) \).

We say that the \( j \)-th component \( s_j \) of the multi charge is sufficiently large for \( |\lambda; s\rangle \) if \( s_j - s_i \geq \lambda^{(0)}_i \) for any \( i = 1, 2, \ldots, \ell \), and that \( s_j \) is sufficiently small for \( |\lambda; s\rangle \) if \( s_i - s_j \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \) for any \( i = 1, 2, \ldots, \ell \) (see Definition 3.1). If \( s_j \) is sufficiently large for \( |\lambda; s\rangle \) and \( |\lambda; s\rangle > |\mu; s\rangle \), then the \( j \)-th components of \( \lambda \) and \( \mu \) are both the empty Young diagram \( \emptyset \) (Lemma 3.2). On the other hand, if \( s_j \) is sufficiently small for \( |\lambda; s\rangle \) and \( |\lambda; s\rangle \geq |\mu; s\rangle \), then \( \mu^{(j)} = \emptyset \) implies \( \lambda^{(j)} = \emptyset \) (Lemma 3.3).

Our main results are as follows.

**Theorem A.** (Theorem 3.4) [Iij]

Let \( \epsilon \in \{ +, - \} \). If \( s_j \) is sufficiently large for \( |\lambda; s\rangle \), then

\[
\Delta^\epsilon_{\lambda,\mu;\beta}(q) = \Delta^\epsilon_{\lambda,\mu;\beta}(q).
\]
where $\lambda$ (resp. $\mu, s$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$), $\Delta_{\lambda,\mu,s}^\epsilon(q)$ is the $q$-decomposition number of level $\ell$ and $\Delta_{\lambda,\mu,s}^{\epsilon}(q)$ is the $q$-decomposition number of level $\ell - 1$.

**Theorem B.** (Theorem 3.5) [Iij]

Let $s \in \{+,-\}$. If $s_j$ is sufficiently small for $|\mu; s)$ and $\mu^{(i)} = 0$, then

$$\Delta_{\lambda,\mu,s}^\epsilon(q) = \Delta_{\lambda,\mu,s}^{\epsilon}(q),$$

where $\lambda$ (resp. $\mu, s$) is obtained by omitting the $j$-th component of $\lambda$ (resp. $\mu, s$).

This paper is organized as follows. In Section 2, we review the $q$-deformed Fock spaces of higher levels and its canonical bases. In Section 3, we state the main results.

**Acknowledgments.** I am deeply grateful to Hyohe Miyachi and Soichi Okada for their advice.

**Notations.** For a positive integer $N$, a partition of $N$ is a non-increasing sequence of non-negative integers summing to $N$. We write $|\lambda| = N$ if $\lambda$ is a partition of $N$. The length $l(\lambda)$ of $\lambda$ is the number of non-zero components of $\lambda$. And we use the same notation $\lambda$ to represent the Young diagram corresponding to $\lambda$. For an $\ell$-tuple $\lambda = (\lambda^{(1)}, \lambda^{(2)}, \cdots, \lambda^{(\ell)})$ of Young diagrams, we put $|\lambda| = |\lambda^{(1)}| + |\lambda^{(2)}| + \cdots + |\lambda^{(\ell)}|.$

### 2. The $q$-deformed Fock spaces of higher levels

#### 2.1. $q$-wedge products and straightening rules.

Let $n, \ell, s$ be integers such that $n \geq 2$ and $\ell \geq 1$. We define $P(s)$ and $P^{++}(s)$ as follows;

1. $P(s) = \{ k = (k_1, k_2, \cdots) \in \mathbb{Z}^\infty \mid k_r = s - r + 1 \text{ for any sufficiently large } r \} ,$
2. $P^{++}(s) = \{ k = (k_1, k_2, \cdots) \in P(s) \mid k_1 > k_2 > \cdots \} .$

Let $\Lambda^s$ be the $\mathbb{Q}(q)$ vector space spanned by the $q$-wedge products

$$u_k = u_{k_1} \wedge u_{k_2} \wedge \cdots , \quad (k \in P(s))$$

subject to certain commutation relations, so-called straightening rules. Note that the straightening rules depend on $n$ and $\ell$. [Ugl00, Proposition 3.16].

**Example 2.1.** (i) For every $k_1 \in \mathbb{Z}$, $u_{k_1} \wedge u_{k_1} = -u_{k_1} \wedge u_{k_1}$. Therefore $u_{k_1} \wedge u_{k_1} = 0$.

(ii) Let $n = 2$, $\ell = 2$, $k_1 = -2$, and $k_2 = 4$. Then

$$u_{-2} \wedge u_4 = q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0 .$$

(iii) Let $n = 2$, $\ell = 2$, $k_1 = -1$, $k_2 = -2$ and $k_3 = 4$. Then

$$u_{-1} \wedge u_{-2} \wedge u_4 = u_{-1} \wedge (u_{-2} \wedge u_4) = u_{-1} \wedge (q u_4 \wedge u_{-2} + (q^2 - 1) u_2 \wedge u_0)$$

$$= q u_{-1} \wedge u_4 \wedge u_{-2} + (q^2 - 1) u_{-1} \wedge u_2 \wedge u_0 .$$

By applying the straightening rules, every $q$-wedge product $u_k$ is expressed as a linear combination of so-called ordered $q$-wedge products, namely $q$-wedge products $u_k$ with $k \in P^{++}(s)$. The ordered $q$-wedge products $\{ u_k \mid k \in P^{++}(s) \}$ form a basis of $\Lambda^s$ called the *standard basis*. 
2.2. Abacus. It is convenient to use the abacus notation for studying various properties in straightening rules.

Fix an integer $N \geq 2$, and form an infinite abacus with $N$ runners labeled $1, 2, \cdots N$ from left to right. The positions on the $i$-th runner are labeled by the integers having residue $i$ modulo $N$.

\begin{align*}
\cdots & \cdots \cdots \cdots \cdots \\
-N + 1 & -N + 2 & \cdots & -1 & 0 \\
1 & 2 & \cdots & N - 1 & N \\
N + 1 & N + 2 & \cdots & 2N - 1 & 2N \\
\cdots & \cdots \cdots \cdots \cdots
\end{align*}

Each $k \in P^{++}(s)$ (or the corresponding $q$-wedge product $u_k$) can be represented by a bead-configuration on the abacus with $n\ell$ runners and beads put on the positions $k_1, k_2, \cdots$. We call this configuration the abacus presentation of $u_k$.

**Example 2.2.** If $n = 2$, $\ell = 3$, $s = 0$, and $k = (6, 3, 2, -2, -4, -5, -7, -8, -9, \cdots)$, then the abacus presentation of $u_k$ is

\begin{align*}
d = 1 & | d = 2 & | d = 3 \\
\cdots & \cdots & \cdots \\
1 & 1 & 1 & 1 & 1 & \cdots m = 3 \\
1 & 1 & -9 & -8 & -7 & -6 & \cdots m = 2 \\
-5 & -4 & -3 & -2 & -1 & 0 & \cdots m = 1 \\
1 & 2 & 3 & 4 & 5 & 6 & \cdots m = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{align*}

We use another labeling of runners and positions. Given an integer $k$, let $c, d$ and $m$ be the unique integers satisfying

\begin{equation}
k = c + n(d - 1) - n\ell m , \quad 1 \leq c \leq n \quad \text{and} \quad 1 \leq d \leq \ell.
\end{equation}

Then, in the abacus presentation, the position $k$ is on the $c + n(d - 1)$-th runner (see the previous example). Relabeling the position $k$ by $c - nm$, we have $\ell$ abaci with $n$ runners.

**Example 2.3.** In the previous example, relabeling the position $k$ by $c - nm$, we have

\begin{align*}
d = 1 & | d = 2 & | d = 3 \\
\cdots & \cdots & \cdots \\
-5 & -4 & -5 & -4 & -5 & -4 & \cdots m = 3 \\
-3 & -2 & -3 & -2 & -3 & -2 & \cdots m = 2 \\
-1 & 0 & -1 & 0 & -1 & 0 & \cdots m = 1 \\
1 & 2 & 1 & 2 & 1 & 2 & \cdots m = 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{align*}
We assign to each of \( \ell \) abacus presentations with \( n \) runners a \( q \)-wedge product of level 1. In fact, straightening rules in each “sector” are the same as those of level 1 by identifying the abacus in the sector with that of level 1. (see Example 2.5 below)

We introduce some notation.

**Definition 2.4.** For an integer \( k \), let \( c, d \) and \( m \) be the unique integers satisfying (4), and write

\[
(5) \quad u_k = u_{c-nm}^{(d)}.
\]

Also we write \( u_{c_{1} - m_{1}}^{(d_{1})} > u_{c_{2} - m_{2}}^{(d_{2})} \) if \( k_{1} > k_{2} \), where \( k_{i} = c_{i} + n(d_{i} - 1) - n\ell m_{i} \), \( (i = 1, 2) \).

We regard \( u_{c-nm}^{(d)} \) as \( u_{c-nm} \) in the case of \( \ell = 1 \).

**Example 2.5.** If \( n = 2, \ell = 3 \), then we have

\[
(6) \quad u_{-10} \wedge u_{1} = -q^{-1} u_{1} \wedge u_{-10} + (q^{-2} - 1) u_{-4} \wedge u_{-5},
\]

that is,

\[
(7) \quad u_{-2}^{(1)} \wedge u_{1}^{(1)} = -q^{-1} u_{1}^{(1)} \wedge u_{-2}^{(1)} + (q^{-2} - 1) u_{0}^{(1)} \wedge u_{-1}^{(1)}.
\]

On the other hand, in the case of \( n = 2, \ell = 1 \),

\[
(8) \quad u_{-2} \wedge u_{1} = -q^{-1} u_{1} \wedge u_{-2} + (q^{-2} - 1) u_{0} \wedge u_{-1}.
\]

**2.3. \( \ell \)-tuples of Young diagrams.** Another indexation of the ordered \( q \)-wedge products is given by the set of pairs \((\lambda, s)\) of \( \ell \)-tuples of Young diagrams \( \lambda = (\lambda^{(1)}, \cdots , \lambda^{(t)}) \) and integer sequences \( s = (s_{1}, \cdots , s_{t}) \) summing up to \( s \). Let \( k = (k_{1}, k_{2}, \cdots) \in P^{++}(s) \), and write

\[
k_{r} = c_{r} + n(d_{r} - 1) - n\ell m_{r} \quad , \quad 1 \leq c_{r} \leq n \quad , \quad 1 \leq d_{r} \leq \ell \quad , \quad m_{r} \in \mathbb{Z}.
\]

For \( d \in \{1, 2, \cdots , \ell\} \), let \( k_{1}^{(d)}, k_{2}^{(d)}, \cdots \) be integers such that

\[
(9) \quad \beta^{(d)} = \{c_{r} - nm_{r} \mid d_{r} = d\} = \{k_{1}^{(d)}, k_{2}^{(d)}, \cdots\} \quad \text{and} \quad k_{1}^{(d)} > k_{2}^{(d)} > \cdots
\]

Then we associate to the sequence \((k_{1}^{(d)}, k_{2}^{(d)}, \cdots)\) an integer \( s_{d} \) and a partition \( \lambda^{(d)} \) by

\[
k_{r}^{(d)} = s_{d} - r + 1 \quad \text{for sufficiently large } r \quad \text{and} \quad \lambda_{r}^{(d)} = k_{r}^{(d)} - s_{d} + r - 1 \quad \text{for } r \geq 1.
\]

In this correspondence, we also write

\[
(10) \quad u_{k} = |\lambda; s\rangle \quad (k \in P^{++}(s)).
\]

**Example 2.6.** If \( n = 2, \ell = 3, s = 0, \) and \( k = (6, 3, 2, 1, -2, -4, -5, -7, -8, -9, \cdots) \), then

\[
k_{1} = 6 = 2 + 2(3 - 1) - 6 \cdot 0 \quad , \quad k_{2} = 3 = 1 + 2(2 - 1) - 6 \cdot 0 \quad ,
\]

\[
k_{3} = 2 = 2 + 2(1 - 1) - 6 \cdot 0 \quad , \quad \cdots \quad \text{and so on.}
\]

Hence,

\[
\beta^{(1)} = \{2, 1, 0, -1, -2, \cdots\} \quad , \quad \beta^{(2)} = \{1, 0, -2, -3, -4, \cdots\} \quad , \quad \beta^{(3)} = \{2, -3, -4, -5, \cdots\}.
\]

Thus, \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)). \)

Note that we can read off \( s = (2, 0, -2) \) and \( \lambda = (\emptyset, (1, 1), (4)) \) from the abacus presentation. (see Example 2.3)
2.4. The $q$-deformed Fock spaces of higher levels.

**Definition 2.7.** For $s \in \mathbb{Z}^\ell$, we define the $q$-deformed Fock space $F_q[s]$ of level $\ell$ to be the subspace of $\Lambda^\ell$ spanned by $|\lambda; s\rangle$ ($\lambda \in \Pi^\ell$):

\[
F_q[s] = \bigoplus_{\lambda \in \Pi^\ell} Q(q) |\lambda; s\rangle.
\]

We call $s$ a multi-charge.

2.5. The bar involution.

**Definition 2.8.** The involution $\overline{\cdot}$ of $\Lambda^\ell$ is the $\mathbb{Q}$-vector space automorphism such that $\overline{q} = q^{-1}$ and

\[
\overline{u_k} = \overline{u_{k_1} \wedge \cdots \wedge u_{k_r} \wedge \cdots} = (-q)^{\kappa(c_1, \cdots, c_r)} q^{-\kappa(c_1, \cdots, c_r)} (u_{k_r} \wedge \cdots \wedge u_{k_1} \wedge \cdots),
\]

where $c_i, d_i$ are defined by $k_i$ as in (4), $r$ is an integer satisfying $k_r = s - r + 1$. And $\kappa(a_1, \cdots, a_r)$ is defined by

\[
\kappa(a_1, \cdots, a_r) = \#\{(i, j) | i < j, a_i = a_j\}.
\]

**Remarks** (i) The involution is well defined, i.e. it doesn’t depend on $r$ [Ugl00].

(ii) The involution comes from the bar involution of affine Hecke algebra $H_r$. (see [Ugl00] for more detail.)

(iii) The involution preserves the $q$-deformed Fock space $F_q[s]$ of higher level.

2.6. The dominance order. We define a partial ordering $|\lambda; s\rangle \geq |\mu; s\rangle$. For $|\lambda; s\rangle$ and $|\mu; s\rangle$, we define multi-sets $\overline{\lambda}$ and $\overline{\mu}$ as

\[
\overline{\lambda} = \{\lambda_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\},
\]

\[
\overline{\mu} = \{\mu_a^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)}))\}.
\]

We denote by $(\overline{\lambda}_1, \overline{\lambda}_2, \cdots)$ (resp. $(\overline{\mu}_1, \overline{\mu}_2, \cdots)$) the sequence obtained by rearranging the elements in the multi-set $\overline{\lambda}$ (resp. $\overline{\mu}$) in decreasing order.

**Definition 2.9.** Let $|\lambda; s\rangle = u_{k_1} \wedge u_{k_2} \wedge \cdots$ and $|\mu; s\rangle = u_{g_1} \wedge u_{g_2} \wedge \cdots$. We define $|\lambda; s\rangle \geq |\mu; s\rangle$ if $|\lambda| = |\mu|$ and

\[
\left\{ \begin{array}{ll}
(\text{a}) & \overline{\lambda} \neq \overline{\mu}, \quad \sum_{j=1}^{r} \lambda_j \geq \sum_{j=1}^{r} \mu_j \quad (\text{for all} \quad r = 1, 2, 3, \cdots) \quad \text{or} \\
(\text{b}) & \overline{\lambda} = \overline{\mu}, \quad \sum_{j=1}^{r} k_j \geq \sum_{j=1}^{r} g_j \quad (\text{for all} \quad r = 1, 2, 3, \cdots)
\end{array} \right.
\]

**Remark.** The order in Definition 2.9 is different from the order in [Ugl00] (see Example 2.10 below). However, the unitriangularity in (11) holds for both of them.

**Example 2.10.** Let $n = \ell = 2$, $s = (1, -1)$, $\lambda = ((1, 1), \emptyset)$, and $\mu = (\emptyset, (2))$. Then, $|\lambda; s\rangle = u_2 \wedge u_1 \wedge u_{-1} \wedge u_{-3} \wedge \cdots$ and $|\mu; s\rangle = u_{3} \wedge u_1 \wedge u_{-2} \wedge u_{-3} \wedge \cdots$. In Uglov’s order, $|\mu; s\rangle$ is greater than $|\lambda; s\rangle$. However, $|\lambda; s\rangle > |\mu; s\rangle$ under our order since $\{\overline{\lambda}_1, \overline{\lambda}_2, \overline{\lambda}_3\} = \{2, 2, -1\}$ and $\{\overline{\mu}_1, \overline{\mu}_2, \overline{\mu}_3\} = \{1, 1, 1\}$.
We define a matrix \((a_{\lambda,\mu}(q))_{\lambda,\mu}\) by

\[
\overline{\langle \lambda; s \rangle} = \sum_{\mu} a_{\lambda,\mu}(q) \langle \mu; s \rangle.
\]

Then the matrix \((a_{\lambda,\mu}(q))_{\lambda,\mu}\) is unitriangular with respect to \(\geq\), that is

\[
\begin{cases} 
(a) & \text{if } a_{\lambda,\mu}(q) \neq 0, \text{ then } \langle \lambda; s \rangle \geq \langle \mu; s \rangle, \\
(b) & a_{\lambda,\lambda}(q) = 1.
\end{cases}
\]

Thus, by the standard argument, the unitriangularity implies the following theorem.

**Theorem 2.11.** [Ugl00] There exist unique bases \(\{G^{+}(\lambda; s)|\lambda \in \Pi^{\ell}\}\) and \(\{G^{-}(\lambda; s)|\lambda \in \Pi^{\ell}\}\) of \(F_{q}[s]\) such that

\[
\begin{align*}
(1) \quad & G^{+}(\lambda; s) = G^{+}(\lambda; s) \mod qL^{+}, \\
& G^{-}(\lambda; s) = G^{-}(\lambda; s) \mod q^{-1}L^{-}
\end{align*}
\]

where

\[
L^{+} = \bigoplus_{\lambda \in \Pi^{\ell}} \mathbb{Q}[q] \langle \lambda; s \rangle, \quad L^{-} = \bigoplus_{\lambda \epsilon \Pi^{\ell}} \mathbb{Q}[q^{-1}] \langle \lambda; s \rangle.
\]

**Definition 2.12.** Define matrices \(\Delta^{+}(q) = (\Delta_{\lambda,\mu}^{+}(q))_{\lambda,\mu}\) and \(\Delta^{-}(q) = (\Delta_{\lambda,\mu}^{-}(q))_{\lambda,\mu}\) by

\[
G^{+}(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^{+}(q) \langle \mu; s \rangle, \quad \quad \quad \quad G^{-}(\lambda; s) = \sum_{\mu} \Delta_{\lambda,\mu}^{-}(q) \langle \mu; s \rangle.
\]

The entries \(\Delta_{\lambda,\mu}^{\pm}(q)\) are called \(q\)-decomposition numbers. Note that \(q\)-decomposition numbers \(\Delta^{\pm}(q)\) depend on \(n, \ell\) and \(s\). The matrices \(\Delta^{+}(q)\) and \(\Delta^{-}(q)\) are also unitriangular with respect to \(\geq\).

It is known [Ugl00, Theorem 3.26] that the entries of \(\Delta^{-}(q)\) are Kazhdan-Lusztig polynomials of parabolic submodules of affine Hecke algebras of type \(A\), and that they are polynomials in \(p = -q\) with non-negative integer coefficients (see [KT02]).

3. A Comparison of \(q\)-Decomposition Numbers

3.1. Sufficiently large and sufficiently small.

**Definition 3.1.** Let \(s = (s_{1}, s_{2}, \cdots, s_{\ell}) \in \mathbb{Z}^{\ell}\) be a multi charge and \(1 \leq j \leq \ell\).

(i). We say that the \(j\)-th component \(s_{j}\) of the multi charge \(s\) is sufficiently large for \(\langle \lambda; s \rangle \in F_{q}[s]\) if

\[
s_{j} - s_{i} \geq \lambda_{1}^{(i)} \quad \text{for all} \quad i = 1, 2, \cdots, \ell.
\]

More generally, we say that \(s_{j}\) is sufficiently large for a \(q\)-wedge \(u_{k}\) if

\[
s_{j} \geq c_{r} - nm_{r} \quad \text{for all} \quad r = 1, 2, \cdots,
\]

where \(k_{r} = c_{r} + n(d_{r} - 1) - n\ell m_{r}, (r = 1, 2, \cdots), 1 \leq c \leq n\) and \(1 \leq d \leq \ell\) (see §2).

(ii). We say that \(s_{j}\) is sufficiently small for \(\langle \lambda; s \rangle\) if

\[
s_{i} - s_{j} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(t)}| \quad \text{for all} \quad i \neq j.
\]
Note that the definition of sufficiently small depends only on the size of \( \lambda \) and the multi charge \( s \). When we fix the multi charge \( s \), we say that \( s_j \) is sufficiently small for \( N \) if
\[
s_i - s_j \geq N \quad \text{for all } i \neq j.
\]

**Remark.** If \( |\lambda; s| \) is 0-dominant in the sense of [Ugl00], that is
\[
s_i - s_{i+1} \geq |\lambda| = |\lambda^{(1)}| + \cdots + |\lambda^{(\ell)}| \quad \text{for all } i = 1, 2, \ldots, \ell - 1,
\]
then \( s_1 \) is sufficiently large for \( |\lambda; s| \) and \( s_\ell \) is sufficiently small for \( |\lambda; s| \).

**Lemma 3.2.** If \( s_j \) is sufficiently large for \( |\lambda; s| \) and \( |\lambda; s| \geq |\mu; s| \), then

(i) \( \lambda^{(j)} = \emptyset \),

(ii) \( s_j \) is also sufficiently large for \( |\mu; s| \). In particular, \( \mu^{(j)} = \emptyset \).

**Proof.** It is clear that \( \lambda^{(j)} = \emptyset \) by the definition.

Note that
\[
s_j \text{ is sufficiently large for } |\lambda; s| \iff s_j - s_i \geq \lambda^{(i)}_1 \quad \text{for all } i = 1, 2, \ldots, \ell
\]
\[
\iff s_j \geq \max(\lambda^{(1)}_1 + s_1, \cdots, \lambda^{(\ell)}_1 + s_\ell) = \lambda_1.
\]

If \( |\lambda; s| \geq |\mu; s| \), then \( \lambda_1 \geq \mu_1 \) and so \( s_j \geq \mu_1 \). It means that \( s_j \) is sufficiently large for \( |\mu; s| \). \( \square \)

**Lemma 3.3.** Suppose that \( s_j \) is sufficiently small for \( |\lambda; s| \). If \( |\lambda; s| \geq |\mu; s| \) and \( \mu^{(j)} = \emptyset \), then \( \lambda^{(j)} = \emptyset \).

**Proof.** Suppose that \( k(\lambda^{(j)}) \geq 1 \). Then \( s_j \) is the minimal integer in the set \( \{\mu_{a}^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) because \( \mu^{(j)} = \emptyset \) and \( s_j \) is the minimal integer in \( s \). On the other hand, the minimal integer in the set \( \{\lambda_{a}^{(d)} + s_d | 1 \leq d \leq \ell, 1 \leq a \leq \max(l(\lambda^{(d)}), l(\mu^{(d)})) \} \) is greater than \( s_j \) because \( s_j \) is sufficiently small for \( |\lambda; s| \). Therefore \( |\lambda; s| \not\geq |\mu; s| \). This is a contradiction. \( \square \)

### 3.2. Main results.

**Theorem 3.4 ([Iij]).** Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently large for \( |\lambda; s| \), then
\[
\Delta_{\lambda,s}^\varepsilon(q) = \Delta_{\lambda,s}^\varepsilon(q),
\]
where \( \lambda \) (resp. \( \mu, s \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

**Theorem 3.5 ([Iij]).** Let \( \varepsilon \in \{+, -\} \). If \( s_j \) is sufficiently small for \( |\mu; s| \) and \( \mu^{(j)} = \emptyset \), then
\[
\Delta_{\lambda,s}^\varepsilon(q) = \Delta_{\lambda,s}^\varepsilon(q),
\]
where \( \lambda \) (resp. \( \mu, s \)) is obtained by omitting the \( j \)-th component of \( \lambda \) (resp. \( \mu, s \)).

**Example 3.6.** (i) If \( n = \ell = 2, s = (3, -3) \) and \( \lambda = (\emptyset, (6)), \mu = (\emptyset, (5, 1)) \), then \( s_1 \) is sufficiently large for \( |\lambda; s| \). Hence
\[
\Delta_{\lambda,s}^+ (q) = \Delta_{\lambda,s}^+ (q) = \Delta_{(6), (5, 1); (3, -3)}^+ (q) = -q^{-1}.
\]

(ii) If \( n = \ell = 2, s = (3, -3) \) and \( \lambda = ((6), \emptyset), \mu = ((5, 1), \emptyset) \), then \( s_2 \) is sufficiently small for \( |\mu; s| \). Hence
\[
\Delta_{\lambda,s}^- (q) = \Delta_{\lambda,s}^- (q) = \Delta_{(6), (5, 1); (3, -3)}^- (q) = -q^{-1}.
\[ \Delta_{\lambda,\mu;\nu}^{-}(q) = \Delta_{\check{\lambda},\check{\mu};\check{\nu}}^{-}(q) = \Delta_{(6),(5,1);(-3)}^{-}(q) = -q^{-1}. \]

REFERENCES


Graduate School of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464-8602, Japan
E-mail address: kazuto.iijima@math.nagoya-u.ac.jp