数理解析研究所講究録
第 1740 巻 2011 年 1-10

Generalized minimal surfaces in Minkowski spaces

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Abstract

In this note we discuss some properties, proved in [2], of timelike minimal surfaces in the Minkowski space. We also review a theory of generalized timelike minimal submanifolds proposed in [3], [4], in analogy to the varifold theory in the Euclidean space.

1 Introduction

This paper deals with the minimal surface equation for timelike submanifolds of the Minkowski space, which can be written as

$$a=(1-v^{2})\kappa, \tag{1}$$

where a, v, κ denote respectively the acceleration, the velocity and the mean curvature of a time-slice of the submanifold. Equation (1) is also known as Born-Infeld equation in the case of two-dimensional graphs, and corresponds to the evolution of a classical relativistic string [14].

Solutions to (1) are tipically not regular and develop various types of singularities [14], [12]. Moreover, differently from the Euclidean case, the space of such solutions is not compact under uniform convergence, and a satisfactory characterization of its closure is still missing for $n>2$.

This paper is organized as follows: in Section 2 we discuss the two-dimensional case (minimal surfaces), corresponding to the evolution of relativistic strings [14], [15]. In particular, thanks to a representation formula valid only in two-dimensions, in Section 2.1 we completely characterize the closure of such surfaces under uniform convergence (see Theorem 2.2), introducing the notion of subrelativistic string.

In Section 2.2 we consider the convexity preserving properties of solutions, and analyze the asymptotic profile near a collapsing time. In Proposition 2.8 we show that a relativistic string, which is smooth and uniformly convex and has zero initial velocity, remains uniformly convex for subsequent times, and shrinks to a point while its shape approaches a round circle. This result is analogous to the one proven by Gage and Hamilton

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in [9] for curvature flow of plane curves, and to the one proven in [11] for the hyperbolic curvature flow (non relativistic case). However, differently from the parabolic case [9], here the collapsing singularity is nongeneric, the generic singularity being the formation of a cusp, as discussed in [1, 8]. Adopting as a definition of weak solution the one given by D'Alambert formula for the linear wave system (4), it follows that after the collapse the solution restarts, and the motion is continued in a periodic way, in accordance with the conservative character of the problem. In Example 2.10 we discuss a solution corresponding to a homotetically shrinking square. Such example shows that, in contrast with the case of smooth strings, for Lipschitz strings the collapsing profile is not necessarily circular.

In Section 3 we introduce the concept of rectifiable and weakly rectifiable lorentzian varifold, which generalizes the notion of timelike submanifold in the spirit of geometric measure theory (see [13]). We also introduce the notion of stationarity, which corresponds to the minimality condition in this generalized setting. Finally, in Propositions 3.3 and 3.4 we show that it is possible to canonically associate to a relativistic (resp. subrelativistic) string a stationary rectifiable (resp. weakly rectifiable) varifold, supported on the evolving string.

2 Relativistic strings

In the following we shall denote by $\langle \cdot, \cdot \rangle_m$ the Minkowskian scalar product in $\mathbb{R}^{1+n}$, associated with the metric tensor $\eta = \mathrm{diag}(-1, +1, \dots, +1)$. We shall also denote by $|\cdot|_m$ the Minkowskian norm on timelike vertors, that is,

$$|\xi|_m := \sqrt{-\langle \xi, \xi \rangle_m} \qquad \text{for all } \xi \text{ such that } \langle \xi, \xi \rangle_m \leq 0.$$

An important example of minimal submanifolds is given by the so-called relativistic strings, which correspond to minimal surfaces in the Minkowski space $\mathbb{R}^{1+n}$. We recall (see for instance [15, Chapter 6]) that the Minkowski area $\mathcal{S}(X)$ of a timelike map $X : [0, T] \times [0, E] \to \mathbb{R}^{1+n}$ of class $\mathbf{C}^1$ is given by

$$\mathcal{S}(X) = \int_{[0,T]\times[0,E]} \sqrt{\langle X_t, X_x \rangle_m^2 - \langle X_t, X_t \rangle_m \langle X_x, X_x \rangle_m} \, dtdx, \tag{2}$$

We shall always assume that X has the form

$$X(t, x) := (t, \gamma(t, x)), \qquad (t, x) \in [0, T] \times [0, E], \tag{3}$$

and that $\gamma(t, \cdot)$ parametrizes a closed curve in $\mathbb{R}^n$. It is well known (see for instance [14, 15]) that the critical points of $\mathcal{S}$, which corresponds to minimal surfaces in $\mathbb{R}^{1+n}$, can be described by the constrained wave system

$$\begin{cases} \gamma_{tt} = \gamma_{xx} \\ \langle \gamma_t, \gamma_x \rangle = 0 \\ |\gamma_t|^2 + |\gamma_x|^2 = 1. \end{cases} \tag{4}$$

Notice that, given a regular solution γ to (4) in $[0,T] \times \mathbb{R}$, we can find $a, b \in \mathbf{C}^2(\mathbb{R};\mathbb{R}^n)$ with

$$|a'| = 1 \qquad |b'| = 1 \qquad \text{a.e. in } \mathbb{R} \tag{5}$$

such that γ has the representation

$$\gamma(t,x) = \frac{1}{2}\left[a(x+t) + b(x-t)\right] \qquad (t,x) \in [0,T] \times \mathbb{R}. \tag{6}$$

Solutions to (4) have been considered by Born and Infeld [6] in the case of graphs, and analyzed later on by many authors. A large number of explicit solutions is discussed in the literature [1, Chapter 4], [14], [12]. Such solutions may develop singularities in the image of the parametrization, such as cusps or collapse singularities, when the regularity condition $\gamma_x \neq 0$ fails. The evolving curve parametrized by a solution γ to system (4) is called a *relativistic string.*

Mathematical questions related to (4) include the qualitative properties of solutions for special initial data, and their asymptotic shape near a singularity time, for instance near a collapse. This latter problem is, in turn, intimately related to the existence of weak global solutions, defined also after the onset of a singularity.

Example 2.1 (Kink). For all $R > 0$, the map

$$\gamma(t,x) := R(\cos(x/R), \sin(x/R))\cos(t/R)$$

is a regular solution to (4) for all $t \neq R\pi(1/2 + z)$, $z \in \mathbb{Z}$. Such solution corresponds to a pulsating circular string with collapsing singularities at $t = R\pi(1/2 + z)$.

2.1 A closure result

It is easy to see that the nonlinear constraint in (4) is not closed under uniform convergence. Indeed, many examples in the physics literature [7, 14, 12] show that the limit of a convergent sequence of relativistic strings is not, in general, a relativistic string (such limits are often called *wiggly* or *subrelativistic strings*). A natural question is then to characterize the closure of relativistic strings. This issue is discussed in the literature [14] and, in the case of strings which are entire graphs, an answer was provided by Y. Brenier [5] who showed that the nonlinear constraint is essentially convexified, and limit solutions have in general only Lipschitz regularity. We state an analogous result for the case of closed relativistic strings.

Theorem 2.2 ([2]). *Let $E_k \to E \in [0,+\infty)$ as $k \to +\infty$, and let $\{\gamma_k\} \subset \mathbf{C}^2([0,T]\times\mathbb{R};\mathbb{R}^n)$ be a sequence of E_k-periodic solutions to* (4)*. Assume further that $\{\gamma_k\}$ converges to a map $\gamma \in \mathrm{Lip}([0,T]\times\mathbb{R};\mathbb{R}^n)$ uniformly in $[0,T]\times\mathbb{R}$ as $k \to +\infty$. Then there exist E-periodic maps $a, b \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^n)$ with*

$$|a'| \le 1 \qquad |b'| \le 1 \qquad \text{a.e. in } \mathbb{R} \tag{7}$$

such that γ has the representation

$$\gamma(t,x) = \frac{1}{2}\left[a(x+t) + b(x-t)\right], \qquad (t,x) \in [0,T]\times\mathbb{R}. \tag{8}$$

Conversely, if $\gamma \in \mathrm{Lip}([0,T]\times\mathbb{R};\mathbb{R}^n)$ can be represented as in (8) *where $a, b \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^n)$ are E-periodic maps satisfying* (7)*, then there exists a sequence $\{\gamma_k\} \subset \mathbf{C}^2([0,T]\times\mathbb{R};\mathbb{R}^n)$ of E-periodic maps solving* (4) *in $[0,T]\times\mathbb{R}$, such that $\{\gamma_k\}$ converges to γ uniformly in $[0,T]\times\mathbb{R}$ as $k \to +\infty$.*

Definition 2.3. *We say that $\gamma : [0,T]\times\mathbb{R} \to \mathbb{R}^{1+n}$ is a subrelativistic string if there exist E-periodic maps a, $b \in \mathrm{Lip}(\mathbb{R};\mathbb{R}^{1+n})$ satisfying* (7) *and* (8)*.*

2.2 Convex strings

Let $\bar{t} > 0$ and let $\gamma \in \mathbf{C}^2([0,\bar{t})\times\mathbb{R};\mathbb{R}^n)$ be a E-periodic regular solution of (4). We know that there exist E-periodic maps $a, b \in C^2(\mathbb{R};\mathbb{R}^n)$ such that

$$\gamma(t,x) = \frac{1}{2}\left[a(x+t) + b(x-t)\right]$$

for any $(t,x) \in [0,\bar{t})\times\mathbb{R}$. As a consequence γ can be extended to a global solution $\gamma \in \mathbf{C}^2(\mathbb{R}\times\mathbb{R};\mathbb{R}^n)$. Adopting this as definition of global solution, we show in this section that initial convex curves may shrink to a point, and then continue the motion in a periodic way.

Definition 2.4. *Let $\bar{t} > 0$ and $p \in \mathbb{R}^n$. We say that $\bar{t}$ is a collapsing time, and that γ has a collapsing singularity at $\bar{t}$ with p as collapsing point, if $\gamma(\bar{t},x) = p$ for any $x \in \mathbb{R}$.*

At a collapsing time $\bar{t}$ we have

$$0 = \gamma_x(\bar{t},x) = \frac{1}{2}\left[a'(x+\bar{t}) + b'(x-\bar{t})\right], \qquad x \in \mathbb{R}. \tag{9}$$

Remark 2.5. Suppose $n = 2$ and that the regular curve γ of class $\mathbf{C}^2$ has a collapsing singularity at time $\bar{t} > 0$, with $p \in \mathbb{R}^2$ as collapsing point. From the representation formula (8) with $a = b$, and from Taylor's formula we get, for $t < \bar{t}$,

$$\begin{aligned} \gamma(t,x) &= \frac{1}{2}\left[a(x+\bar{t}) + a(x-\bar{t})\right] + \frac{1}{2}\left[a'(x-\bar{t}) - a'(x+\bar{t})\right](\bar{t}-t) \\ &\quad + O(|\bar{t}-t|^2) \\ &= p + a'(x-\bar{t})(\bar{t}-t) + O(|\bar{t}-t|^2), \end{aligned}$$

where in the last equality we use $a'(x+\bar{t}) + a'(x-\bar{t}) = 0$. It follows that

$$|\gamma(t,x) - p| = |\bar{t} - t| + O(|\bar{t}-t|^2). \tag{10}$$

In particular, the asymptotic shape near the collapse is circular, and the blow-up shape of the image of the corresponding map X (see (3)) at $(\bar{t}, p)$ is half a light cone.

Let us still suppose $n = 2$ and $\gamma_t(0,\cdot) = 0$, so that we can choose $a = b \in \mathbf{C}^2(\mathbb{R};\mathbb{R}^2)$. We suppose in addition that a parametrizes a closed uniformly convex curve of class $\mathbf{C}^2$. Since the initial curve $\gamma(0,\cdot)$ is uniformly convex, for any $x \in [0,E]$ there exists a unique $\mathrm{t}(x) \in (0, E/2)$ such that

$$\gamma_x(\mathrm{t}(x), x) = \frac{1}{2}\left[a'(x+\mathrm{t}(x)) + a'(x-\mathrm{t}(x))\right] = 0, \tag{11}$$

and the function t belongs to $\mathbf{C}^1([0,E];(0,E/2))$. Moreover, if we set

$$\mathrm{t}_{\min} := \min_{x\in[0,E]} \mathrm{t}(x) \qquad \mathrm{t}_{\max} := \max_{x\in[0,E]} \mathrm{t}(x),$$

we have $t_{\min} > 0$, $t_{\max} < E/2$, and $\gamma(t,\cdot)$ is a regular parametrization for all $t \in [0, \mathrm{t}_{\min}) \cup (\mathrm{t}_{\max}, E/2]$. We can think of $\mathrm{t}_{\min}$ (resp. $\mathrm{t}_{\max}$) as the first (resp. last) singularity time in the periodicity interval $[0,E]$.

Proposition 2.6. *Let $\gamma \in \mathbf{C}^2([0,\mathrm{t}_{\min}) \times \mathbb{R}; \mathbb{R}^2)$ be a E-periodic solution of (4) given by (8). Assume that $\gamma(0,[0,E])$ is regular, embedded, encloses a compact centrally symmetric uniformly convex body $K(0)$, and $\gamma_t(0,\cdot) = 0$. Then γ has a collapsing singularity at time $t_{\min} = E/4$ with the origin as collapsing point.*

Proof. The assertion follows by observing that $K(0)$ is centrally symmetric, and the function t defined in (11) is constant and equals $E/4 = t_{\min}$. □

Remark 2.7. Generically, one can assume that

- the last equality in (9) does not hold;
- the set $\{x \in [0,E] : \mathrm{t}(x) = t\}$ is finite for all $t \in [\mathrm{t}_{\min}, \mathrm{t}_{\max}]$, and consists of a single point $x_{\min}$ (resp. $x_{\max}$) for $t = \mathrm{t}_{\min}$ (resp. $t = \mathrm{t}_{\max}$).

From the condition $\mathrm{t}'(x_{\min}) = \mathrm{t}'(x_{\max}) = 0$ we get

$$\begin{aligned} a''(x_{\min} + \mathrm{t}_{\min}) &= -a''(x_{\min} - \mathrm{t}_{\min}) \\ a''(x_{\max} + \mathrm{t}_{\max}) &= -a''(x_{\max} - \mathrm{t}_{\max}), \end{aligned}$$

which implies that the images $\gamma(\mathrm{t}_{\min}, [0,E])$ and $\gamma(\mathrm{t}_{\max}, [0,E])$ are of class $\mathbf{C}^1$. In this generic setting, the formation of singularities has been discussed in [8] (see also [14], [1]), where it is shown that $t_{\min}$ is the first singular time, the singularity has the asymptotic behavior $y \sim x^{\frac{4}{3}}$ in graph coordinates, and two cusps $y \sim x^{\frac{2}{3}}$ appear from the point $x_{\min}$ at time $t_{\min}$, persist for some positive time, and eventually disappear.

We now show that the convexity of the curve is preserved before the onset of singularities, that is on the time interval $[0, t_{\min})$.

Proposition 2.8 ([2]). *Let $\gamma \in \mathbf{C}^2([0, t_{\min}) \times \mathbb{R}; \mathbb{R}^2)$ be a E-periodic solution of* (4) *given by* (8). *Assume that $\gamma(0, [0, E])$ is regular, embedded, encloses a compact uniformly convex body $K(0)$, and $\gamma_t(0, \cdot) = 0$. Then $\gamma(t, \cdot)$ is the regular parametrization of a closed uniformly convex embedded curve for all $t \in [0, t_{\min})$. Moreover, letting $K(t)$ the compact convex set enclosed by $\gamma(t, \cdot)$, we have*

$$t_1, t_2 \in [0, t_{\min}),\ t_1 \le t_2 \quad \Rightarrow \quad K(t_1) \subseteq K(t_2), \tag{12}$$

with strict inclusion if $0 < t_1 < t_2 < t_{\min}$.

A result analogous to Proposition 2.8 has been obtained in [11] for the nonrelativistic equation $\mathrm{a} = \kappa$.

Remark 2.9. As a consequence of Remark 2.5 and Proposition 2.8, it follows that an initial uniformly convex curve of class $\mathbf{C}^2$ with zero initial velocity shrinks to a point, the asymptotic shape near the collapse is circular, and the blow-up shape is half a light cone.

The conclusion on the asymptotic shape in Remark 2.5 is not true if we drop the regularity assumption on the initial convex set, as shown in the following example.

Example 2.10 (Square). Assume $n = 2$, let $L > 0$ and let $a = b : \mathbb{R} \to \mathbb{R}^2$ be $4L$-periodic maps, such that $a_{|[0,4L]}$ parametrizes the boundary of the square $Q_0 = [-L/2, L/2]^2$ (sending for instance $\{0\}$ into the point $x^1 = -L/2$, $x^2 = -L/2$). Obviously $a_{|[0,4L]} \in \mathbf{C}^2([0, 4L] \setminus \{0, L, 2L, 3L\}; \mathbb{R}^2)$, and a is Lipschitz continuous in $[0, 4L]$. Define $\gamma(t, x) := \frac{1}{2}[a(x + t) + a(x - t)]$ for any $(t, x) \in \mathbb{R} \times \mathbb{R}$. Then $\gamma(t, \cdot)$ is a Lipschitz parametrization of $\partial Q(t)$, where $Q(t)$ is defined as

$$Q(t) := Q_0 \cap \left\{(x^1, x^2) \in \mathbb{R}^2 : |x^1| + |x^2| \le L - t\right\}, \qquad t \in [0, L],$$

and continued periodically for times larger than L (see Fig. 1).

Observe that

(i) the map $X(t, x) := (t, \gamma(t, x))$ is Lipschitz continous, and at those points of $\mathrm{Im}X$ where there exists the tangent plane such plane is timelike.

(ii) For $t \in [0, L/2)$ the set $Q(t)$ is a shrinking octagon with vertices $p_1(t), \ldots, p_8(t)$. Moreover, for $t \in [0, L/2)$ we have $|\gamma_x| > 0$ almost everywhere.

(iii) For $t \in [L/2, L)$ the set $Q(t)$ is a shrinking rotated square of side $\sqrt{2}(L - t)$ (depicted in bold in Fig. 1). It shrinks to the point $(0, 0)$ at $t = L$ (collapsing singularity). Its normal velocity is constantly equal to $\frac{1}{\sqrt{2}}$.

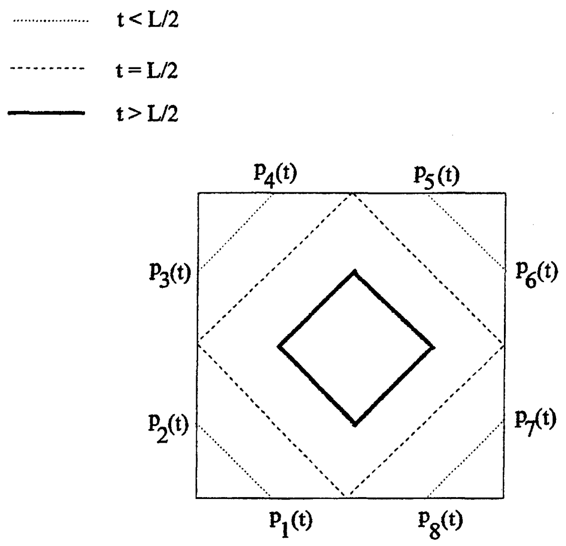

Figure 1: The evolution of a square starting with zero initial velocity.

(iv) Given $t \in (L/2, L)$, we have $\gamma_x(t, x) = 0$ when x belongs to the union $I(t)$ of four intervals of length $2t - L$, corresponding to intervals centered at the centers of the four sides of ∂Q_0. Indeed, $\gamma_x(t, x) = 0$ when $a'(x + t) = -a'(x - t)$, hence, for instance when x corresponds to the center of $[-L/2, L/2] \times \{-L/2\}$ and $x + t$ and $x - t$ belong to opposite vertical sides of ∂Q_0.

Note that the blow-up of X at $(L, 0)$ (for times smaller than L) is not half a light cone as in Remark 2.5, but is the half cone $\{(t, x_1, x_2) : |t - L| + |x^1| + |x^2| = 1\}$ with square section, inscribed in the light cone.

3 Rectifiable lorentzian varifolds

In this section, following the approach of [13] in the Euclidean setting, we intruduce the notion of *rectifiable lorentzian varifold*, corresponding to a generalized timelike submanifold in the Minkowski space $\mathbb{R}^{1+n}$.

Let us first introduce some notation. Given a h-dimensional subspace (h-plane for short) $\Pi \subset \mathbb{R}^{1+n}$ we set

$$\Pi^{\perp_m} := \{\zeta \in \mathbb{R}^{1+n} : \langle \zeta, \xi \rangle_m = 0 \quad \forall \xi \in \Pi\}.$$

We recall that Π is timelike if n is spacelike for all $\mathrm{n} \in \Pi^{\perp_m}$. The set of all timelike h-planes is open, and if Π is timelike we have $\dim(\Pi^{\perp_m}) = n + 1 - h$.
We embed the set of timelike h-planes into the vector space of $(n+1) \times (n+1)$-real matrices M_{n+1} as follows: we associate with a timelike h-plane Π the matrix P_Π corresponding to the orthogonal projection $\mathbb{R}^{1+n} \to \mathbb{R}^{1+n}$ onto Π with respect to the minkowskian scalar

product. More precisely, we let

$$P_\Pi := \mathrm{Id} - \sum_{j=1}^{n+1-h} \mathrm{n}_j \otimes \eta \mathrm{n}_j, \tag{13}$$

where Id is the identity matrix on $\mathbb{R}^{1+n}$ and the vectors $\{\mathrm{n}_1, \dots, \mathrm{n}_n\}$ satisfy the following properties:

- $\mathrm{n}_1, \dots \mathrm{n}_{n+1-h} \in \mathbb{R}^{1+n}$ are spacelike vectors belonging to $\Pi^{\perp_m}$,
- the time component n_1^0 of n_1 is nonnegative,
- $\mathrm{n}_2, \dots \mathrm{n}_{n+1-h}$ have vanishing time component,
- for $i, j \in \{1, \dots, n+1-h\}$

$$(\mathrm{n}_i, \mathrm{n}_j)_m = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{14}$$

In what follows, for $j \in \{1, \dots, n+1-h\}$ the components of the vector n_j are denoted by $(\mathrm{n}_j^0, \dots, \mathrm{n}_j^n)$. We assign greek letters to the coordinates indices, and latin letters to indicate a specific vector. We also adopt the convention of summation over repeated indices.

Definition 3.1. *We denote by $T_{h,n+1} \subset M_{n+1}$ the set of all $(n+1) \times (n+1)$-real matrices $P = P_\Pi$ corresponding to timelike h-planes Π in the sense described above.*

Notice that, differently from the Euclidean case, the set $T_{h,n+1}$ is not bounded, hence it is not compact. We denote by $\overline{T_{h,n+1}}$ its compactification (see [4] for details).

Definition 3.2. *A h-dimensional* weakly rectifiable lorentzian varifold *is a Radon measure $V = \mu_V \otimes V_z$ on $\mathbb{R}^{1+n} \times \overline{T_{h,n+1}}$ such that*

1. *$\Sigma = \mathrm{spt}(\mu_V) \subset \mathbb{R}^{1+n}$ is an h-rectifiable set whose tangent space is timelike $\mathcal{H}^h$-almost everywhere,*
2. *V_z is a probability measure on $\overline{T_{h,n+1}}$.*

We say that V is rectifiable *if $V_z = \delta_{P_\Sigma(z)}$, where $P_\Sigma(z)$ denotes the projection onto the tangent space $T_z\Sigma$.*
We say that V is stationary *if*

$$\int_\Sigma \mathrm{tr}\left(\overline{P}\ \nabla \mathbf{X}\right)\, d\mu_V = 0 \tag{15}$$

for all vector fields $\mathbf{X} \in [\mathrm{C}_c^1(\mathbb{R}^{1+n})]^{n+1}$, where $\overline{P}(z)$ is the barycenter of V_z. In particular, $\overline{P} = P_\Sigma$ if V is rectifiable.

When V is rectifiable and Σ is smooth, a direct computation [13] shows that (15) implies that Σ is a timelike minimal submanifold of dimension h and μ_V coincides, up to a multiplicative constant, with the h-dimensional Minkowski area σ^h restricted to Σ.

3.1 Two-dimensional varifolds and relativistic strings

Given a solution $\gamma \in \mathbf{C}^2(\mathbb{R}^2; \mathbb{R}^n)$ to (4), we define $\Phi_\gamma : \mathbb{R}^2 \to \mathbb{R}^{1+n}$ as

$$\Phi_\gamma(t, x) := (t, \gamma(t, x)) \qquad (t, x) \in \mathbb{R}^2,$$

and we let $\Sigma_\gamma := \Phi_\gamma(\mathbb{R}^2)$ be the image of Φ_γ. Then $\Sigma_\gamma \setminus K$ is a lorentzian minimal surface (see for instance [14]), where

$$K := \Phi_\gamma(\{(t, x) : |\gamma_x(t, x)| = 0\}) \subset \Sigma_\gamma$$

is a closed set of zero $\mathcal{H}^2$-measure. Explicit examples (see for instance Example 2.1) exhibit singularities of Σ_γ so that in general the singular set K is nonempty.

To any γ as above we canonically associate the rectifiable lorentzian varifold

$$V_\gamma := \theta_\gamma \sigma^2 \llcorner \Sigma_\gamma \otimes \delta_{P_{\Sigma_\gamma}},$$

where σ^2 is the 2-dimensional Minkowski area, and the multiplicity θ_γ is defined as

$$\theta_\gamma(t, x) := \#\{y : \ \gamma(t, y) = x\} \qquad (t, x) \in \Sigma_\gamma. \tag{16}$$

Despite the possible presence of singular points of Σ_γ, as in Example 2.1, the varifold V_γ corresponds to a generalized minimal submanifold of $\mathbb{R}^{1+n}$ in the following sense.

Proposition 3.3 ([4]). *V_γ is a stationary varifold.*

To any subrelativistic string $\gamma \in \mathrm{Lip}(\mathbb{R}^2; \mathbb{R}^n)$ we can still associate the Lipschitz map $\Phi_\gamma(t, x) := (t, \gamma(t, x))$ and its image $\Sigma_\gamma := \Phi_\gamma(\mathbb{R}^2)$ as above. From the inequality $|\gamma_t|^2 + |\gamma_x|^2 \leq 1$ it follows that Σ_γ is timelike, so that $V_\gamma := \theta_\gamma \sigma^2 \llcorner \Sigma_\gamma \otimes \delta_{P_{\Sigma_\gamma}}$, with θ_γ as in (16), is still a rectifiable lorentzian varifold. Unfortunately, it is not difficult to show that in general V_γ is not a stationary varifold, as it happens for instance in the case of the shrinking square discussed in Example 2.10. However, in [4] we prove the following result.

Proposition 3.4 ([4]). *Let $\gamma \in \mathrm{Lip}(\mathbb{R}^2; \mathbb{R}^n)$ be a subrelativistic string in the sense of Definition 2.3. Then there exists a weakly rectifiable lorentzian varifold $\widetilde{V}_\gamma$ such that $\Sigma_\gamma = \mathrm{spt}(\mu_{\widetilde{V}_\gamma})$. Moreover, the varifold $\widetilde{V}_\gamma$ is stationary.*

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