数理解析研究所講究録
第 1740 巻 2011 年 74-88

A new two-phase fluid problem with surface energy

Yoshihiro Tonegawa

Abstract

We prove the existence of weak solution for the incompressible and viscous non-Newtonian two-phase fluid flow with surface tension when $d=2,3$. An approximation scheme combining the Galerkin method and the phase field method is adopted. This is a joint work with Chun Liu (Pen State) and Norifumi Sato (Furano IIS) and is the main part of Sato's doctoral thesis.

1 Introduction

In this paper we describe some existence results for incompressible viscous two-phase fluid flow with surface tension in the torus $\Omega=\mathbb{T}^{d}=(\mathbb{R}/\mathbb{Z})^{d}$, $d=2,3$. A freely moving $(d-1)$-dimensional phase boundary $\Gamma(t)$ separates the domain Ω into two domains $\Omega^{+}(t)$ and $\Omega^{-}(t)$, $t\geq 0$. The fluid flow is described by means of the velocity field $u:\Omega\times[0,\infty)\rightarrow \mathbb{R}^{d}$ and the pressure $\Pi:\Omega\times[0,\infty)\rightarrow \mathbb{R}$. We assume the stress tensor of the fluids is of the form $T^{\pm}(u,\Pi)=\nu^{\pm}(|e(u)|)\,e(u)-\Pi I$ on $\Omega^{\pm}(t)$, respectively. Here $2e(u)=\nabla u+\nabla u^{T}$ and I is the $d\times d$ identity matrix. We assume that the functions $\nu^{\pm}:\mathbb{R}^{+}\rightarrow \mathbb{R}$ is locally Lipschitz and satisfy for some $\nu_{0}>0$ and ν_{1}, $\nu_{2}\geq 0$

$$\nu_{0}s^{p-2}+\nu_{1}\leq\nu^{\pm}(s)\leq\nu_{0}^{-1}s^{p-2}+\nu_{2}, \quad (\nu^{\pm}(s)s)'\geq 0, \quad p>\frac{d+2}{2}. \tag{1.1}$$

A typical example is $\nu^{\pm}(s)=(a^{\pm}+b^{\pm}s^{\frac{p-2}{2}})^{2}$ with $a^{\pm}\geq 0$ and $b^{\pm}>0$. We set $\tau^{\pm}(e(u))=\nu^{\pm}(|e(u)|)\,e(u)$.

We assume that the velocity field $u(x,t)$ satisfies the following non-Newtonian fluid flow equation:

$$\frac{\partial u}{\partial t}+u\cdot\nabla u=\operatorname{div}\tau^{\pm}(e(u))-\nabla\Pi, \quad \operatorname{div}u=0 \qquad \text{in } \Omega^{+}(t)\cup\Omega^{-}(t),\ t>0, \tag{1.2}$$

$$u^{+}=u^{-}, \quad n\cdot(T^{+}(u,\Pi)-T^{-}(u,\Pi))=\kappa_{1}H \qquad \text{on } \Gamma(t),\ t>0. \tag{1.3}$$

The upper script $\pm$ indicates the limiting values approaching to $\Gamma(t)$ from $\Omega^{\pm}(t)$, respectively, n is the unit outer normal vector of $\partial\Omega^{+}(t)$, H is the mean curvature vector of $\Gamma(t)$ and $\kappa_{1}>0$ is a constant. The conditions (1.3) represents the force balance with an isotropic surface tension effect of the free boundary. The phase boundary $\Gamma(t)$ moves with the velocity given by

$$V_{\Gamma}=(u\cdot n)n+\kappa_{2}H \quad \text{on} \quad \Gamma(t), \quad t>0, \tag{1.4}$$

where $\kappa_{2}>0$ is a constant. This differs from the conventional kinematic condition $(\kappa_{2}=0)$ and is motivated from the phase boundary motion with hydrodynamic effect. The reader is referred to [22] and the references therein for the physical background. By setting $\varphi=1$ on $\Omega^{+}(t)$, $\varphi=-1$ on $\Omega^{-}(t)$ and

$$\tau(\varphi,e(u))=\frac{1+\varphi}{2}\tau^{+}(e(u))+\frac{1-\varphi}{2}\tau^{-}(e(u))$$

on $\Omega^+(t) \cup \Omega^-(t)$, the equations (1.2)-(1.3) are expressed in the distributional sense as

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = \operatorname{div} \tau(\varphi, e(u)) - \nabla \Pi + \kappa_1 H \mathcal{H}^{d-1} \lfloor_{\Gamma(t)} \quad \text{in } \Omega \times (0, \infty), \tag{1.5}$$

where $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure. We remark that the sufficiently smooth solutions of (1.2)-(1.4) satisfy the following energy equality,

$$\frac{d}{dt}\left\{\frac{1}{2}\int_\Omega |u|^2\,dx + \kappa_1 \mathcal{H}^{d-1}(\Gamma(t))\right\} = -\int_\Omega \tau(\varphi, e(u)) : e(u)\,dx - \kappa_1\kappa_2 \int_{\Gamma(t)} |H|^2\,d\mathcal{H}^{d-1}. \tag{1.6}$$

This follows from the first variation formula for the surface measure

$$\frac{d}{dt}\mathcal{H}^{d-1}(\Gamma(t)) = -\int_{\Gamma(t)} V_\Gamma \cdot H\,d\mathcal{H}^{d-1} \tag{1.7}$$

and by the equations (1.2)-(1.4).

In this paper we give an almost complete outline of [21] which shows the time-global existence of the weak solution for (1.2)-(1.4) (see Theorem 2.3 for the precise statement). In establishing (1.4) we adopt the formulation due to Brakke [7] where he proved the existence of moving varifolds by mean curvature. We have the extra transport effect $(u \cdot n)n$ which is not very regular in the present problem. Typically we would only have $u \in L^p_{loc}([0,\infty); W^{1,p}(\Omega)^d)$. This poses a serious difficulty in modifying Brakke's original construction in [7] which is already intricate and involved. Instead we take advantage of the recent progress on the understanding on the Allen-Cahn equation with transport term,

$$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = \kappa_2 \left(\Delta \varphi - \frac{W'(\varphi)}{\varepsilon^2}\right). \tag{ACT}$$

Here W is the equal depth double-well potential and we set $W(\varphi) = (1-\varphi^2)^2/2$. When $\varepsilon \to 0$, we have proved in [20] that the interface moves according to the velocity (1.4) in the sense of Brakke with a suitable regularity assumptions on u. To be more precise, we use a regularized version of (ACT) as we present later for the result of [20] to be applicable. The result of [20] was built upon those of many earlier works, most relevant being [14, 15] which analyzed (ACT) with $u = 0$, and also [13, 35, 30, 29].

We mention a number of results related to the two-phase flow problem. In the case without surface tension ($\kappa_1 = \kappa_2 = 0$), Solonnikov [32] proved the time-local existence of classical solution. The time-local existence of weak solution is proved by Solonnikov [33], Beale [5], Abels [1], and others. For time-global existence of weak solution, Beale [6] proved in the case that the initial data is small. Nouri-Poupaud [27] considered the case of multi-phase fluid. Giga-Takahashi [11] considered the problem within the framework of level set method. When $\kappa_1 > 0$, $\kappa_2 = 0$, Plotnikov [28] proved the time-global existence of varifold solution for $d = 2$, $p > 2$, and Abels [2] proved the time-global existence of measure-valued solution for $d = 2, 3$, $p > \frac{2d}{d+2}$. When $\kappa_1 > 0$, $\kappa_2 > 0$, Maekawa [23] proved the time-local existence of classical solution with $p = 2$ and for all dimension. Abels-Röger [3] considered a coupled problem of Navier-Stokes and Mullins-Sekerka (instead of motion by mean curvature in the present paper) and proved the existence of weak solutions. As for related phase field approximations of sharp interface model which we adopt in this paper, Liu and Walkington [22] considered the case of fluids containing visco-hyperelastic particles. Perhaps the most closely related work to the present paper is that of Mugnai and Röger [26] which studied the identical problem with $p = 2$ (linear viscosity case) and $d = 2, 3$. There they introduced the notion of L^2 velocity and showed that (1.4) is satisfied in a weak sense different from that of Brakke for the limiting interface. The additional property which we have with $p > \frac{d+2}{2}$ is the density upper bound obtained in [20]. Kim-Consiglieri-Rodrigues [16] dealt with a coupling of

Cahn-Hilliard and Navier-Stokes equations to describe the flow of non-Newtonian two-phase fluid with phase transitions. Soner [34] dealt with a coupling of Allen-Cahn and heat equations to approximate the Mullins-Sekerka problem with kinetic undercooling.

Finally we should note that we perhaps raised more questions than answers by proving our main results. We expect that the solution u would be more regular than what we proved. So would be the moving interface, which we expect to be smooth a.e. in space-time under some mild density conditions. The case $d = 2$ and $p = 2$ corresponds to the critical exponent case which our result does not cover. This is the linear viscosity case and is naturally the very interesting one. We expect that some smallness assumption on the initial energy should suffice to show the existence of a time-global weak solution in Brakke's sense, but it remains an open question.

The organization of this paper is as follows. In Section 2, we summarize the basic notations and main results. Section 3 describes the result of [20] which establishes the upper density ratio bound for surface energy and which proves (1.4). In Section 4 we construct a sequence of approximating solution for the two-phase flow problem via Galerkin method and phase field method. In the last Section 5 we combine the results from Section 3 and 4 and obtain the desired weak solution for the two-phase flow problem.

2 Preliminaries and Main results

For $A, B \in \mathbb{R}^{d^2}$ we denote $A : B = \sum A_{ij}B_{ij}$ and $|A| := \sqrt{A : A}$. For $a \in \mathbb{R}^d$, we denote by $a \otimes a$ the matrix with the entries $a_i a_j,\ i, j = 1, \dots, d$.

2.1 Function spaces

Set $\Omega = \mathbb{T}^d$ throughout this paper. We set function spaces for $p > \frac{d+2}{2}$ as follows:

$$\begin{aligned}
&\mathcal{V} = \left\{ v \in C^\infty(\Omega)^d;\ \operatorname{div} v = 0 \right\}, \\
&\text{for } s \in \mathbb{Z}^+ \cup \{0\},\ W^{s,p}(\Omega) = \{ v\ :\ \nabla^j v \in L^p(\Omega) \text{ for } 0 \le j \le s \} \\
&V^{s,p} = \text{closure of } \mathcal{V} \text{ in the } W^{s,p}(\Omega)^d\text{-norm},
\end{aligned}$$

We denote the dual space of $V^{s,p}$ by $(V^{s,p})^*$ and similarly for other spaces. The pairing between the dual spaces is tacitly denoted by $(\cdot, \cdot)$ whenever there should be no confusion.

2.2 Varifold notations

We recall some notions from geometric measure theory and refer to [4, 7, 31] for more details. A *general k-varifold* in $\mathbb{R}^d$ is a Radon measure on $\mathbb{R}^d \times G(d,k)$, where $G(d,k)$ is the space of k-dimensional subspaces in $\mathbb{R}^d$. We denote the set of all general k-varifolds by $\mathbf{V}_k(\mathbb{R}^d)$. When S is a k-dimensional subspace, we also use S to denote the orthogonal projection matrix corresponding to $\mathbb{R}^d \to S$. The first variation of V can be written as

$$\delta V(g) = \int_{\mathbb{R}^d \times G(d,k)} \nabla g(x) : S \, dV(x,S) = - \int_{\mathbb{R}^d} g(x) \cdot H(x) \, d\|V\|(x) \quad \text{if } \|\delta V\| \ll \|V\|.$$

Here $V \in \mathbf{V}_k(\mathbb{R}^d)$, $\|V\|$ is the mass measure of V, $g \in C_c^1(\mathbb{R}^d)^d$, $H = H_V$ is the generalized mean curvature vector if it exists and $\|\delta V\| \ll \|V\|$ denotes that $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$.

We call a Radon measure μ *k-integral* if μ is represented as $\mu = \theta \mathcal{H}^k\lfloor_X$, where X is a locally k-rectifiable, $\mathcal{H}^k$-measurable set, and $\theta \in L^1_{\mathrm{loc}}(\mathcal{H}^k\lfloor_X)$ is positive and integer-valued $\mathcal{H}^k$ a.e on X. We denote the set of k-integral Radon measure by $\mathcal{IM}_k$. We say that a k-integral varifold is of *unit density* if θ is $\mathcal{H}^k$ a.e. equal to 1 on X. For each such k-integral measure μ corresponds a unique k-varifold V defined by

$$\int_{\mathbb{R}^d \times G(d,k)} \phi(x, S)\, dV(x, S) = \int_{\mathbb{R}^d} \phi(x, T_x\mu)\, d\mu(x) \quad \text{for } \phi \in C_c(\mathbb{R}^d \times G(d,k)),$$

where $T_x\mu$ is the approximate tangent k-plane. Note that $\mu = \|V\|$. We make such identification in the following. For this reason we define H_μ as H_V (or simply H) if the latter exists. When X is a C^2 submanifold without boundary and θ is constant on X, H corresponds to the usual mean curvature vector for X. In the following we suitably adopt the above notions on $\Omega = \mathbb{T}^d$ such as $\mathbf{V}_k(\Omega)$, which present no essential difficulties.

2.3 Weak formulation of free boundary motion

For sufficiently smooth surface $\Gamma(t)$ moving by the velocity (1.4), the following holds for any $\phi \in C^2(\Omega; \mathbb{R}^+)$ due to the first variation formula (1.7):

$$\frac{d}{dt}\int_{\Gamma(t)} \phi\, d\mathcal{H}^{d-1} \le \int_{\Gamma(t)} (-\phi H + \nabla\phi) \cdot \{\kappa_2 H + (u \cdot n)n\}\, d\mathcal{H}^{d-1}. \tag{2.1}$$

One can check that having this inequality for any $\phi \in C^2(\Omega; \mathbb{R}^+)$ implies (1.4) thus (2.1) is equivalent to (1.4). This is Brakke's approach for the mean curvature flow and we suitably modify it to incorporate the transport term u. To do this we recall

Theorem 2.1. (Meyers-Ziemer inequality) *For a Radon measure μ on $\mathbb{R}^d$with*

$$D = \sup_{r>0,\, x\in\mathbb{R}^d} \frac{\mu(B_r(x))}{\omega_{d-1} r^{d-1}},$$

$$\int_{\mathbb{R}^d} |\phi|\, d\mu \le c_{MZ} D \int_{\mathbb{R}^d} |\nabla\phi|\, dx \tag{2.2}$$

for $\phi \in C_c^1(\mathbb{R}^d)$. Here $c_{MZ} = c_{MZ}(d)$.

See [25] and [36, p.266]. By localizing Theorem 2.1 to $\Omega = \mathbb{T}^d$ we obtain (with r in the definition of D above replaced by $0 < r < 1/2$)

$$\int_\Omega |\phi|^2\, d\mu \le c_{MZ} D \|\phi\|_{L^2(\Omega)} \|\nabla\phi\|_{L^2(\Omega)} \tag{2.3}$$

where the constant c_{MZ} may be different due to the localization but depends only on d. The inequality allows us to define $\int_\Omega |\phi|^2\, d\mu$ for $\phi \in W^{1,2}(\Omega)$ by the standard density argument.

We define for any Radon measure μ, $u \in L^2(\Omega)^d$ and $\phi \in C^2(\Omega : \mathbb{R}^+)$

$$\mathcal{B}(\mu, u, \phi) = \int_\Omega (-\phi H + \nabla\phi) \cdot \{\kappa_2 H + (u \cdot n)n\}\, d\mu \tag{2.4}$$

if $\mu \in \mathcal{IM}_{d-1}(\Omega)$ with generalized mean curvature $H \in L^2(\mu)$ and with $\sup_{\frac{1}{2}>r>0,\, x\in\Omega} \frac{\mu(B_r(x))}{\omega_{d-1} r^{d-1}} <$ ∞ and $u \in W^{1,2}(\Omega)$. It gives a well-defined finite value due to the stated conditions and (2.3). If any one of the conditions is not satisfied, we define $\mathcal{B}(\mu, u, \phi) = -\infty$.

Next we note

Proposition 2.2. *For any* $0 < T < \infty$

$$\{u \in L^q([0,T]; V^{1,q}) \,|\, \frac{\partial u}{\partial t} \in L^{\frac{q}{q-1}}([0,T]; (V^{1,q})^*)\} \hookrightarrow C([0,T],\, V^{0,2})$$

for $q > \frac{2d}{d+2}$.

The Sobolev embedding gives $V^{1,q} \hookrightarrow V^{0,2}$ for such q and we may apply the result [24, p. 35, Lemma 2.45]) to obtain the above embedding. Thus for this class of u we may define $u(\cdot,t) \in V^{0,2}$ for all $t \in [0,T]$ instead of a.e. t and we may tacitly assume that we redefine u in this way for all t.

Finally for $\{\mu_t\}_{t\in[0,\infty)}$, $u \in L^q_{loc}([0,\infty); V^{1,q})$ with $\frac{\partial u}{\partial t} \in L^{\frac{q}{q-1}}_{loc}([0,\infty); (V^{1,q})^*)$ for $q \geq 2$ and $\phi \in C^2(\Omega; \mathbb{R}^+)$, we define $\mathcal{B}(\mu_t,\, u(\cdot,t),\, \phi)$ for all $t \geq 0$.

2.4 The main results

Our main results are the following.

Theorem 2.3. *Let* $d = 2$ *or* 3 *and* $p > \frac{d+2}{2}$. *Let* $\Omega = \mathbb{T}^d$. *Assume that* $\tau^{\pm}$ *satisfy* (1.1). *For any initial data* $u_0 \in V^{0,2}$ *and* $\Omega^+(0) \subset \Omega$ *having* C^1 *boundary* $\partial\Omega^+(0)$, *there exist*

(i) $u \in L^\infty_{loc}([0,\infty); V^{0,2}) \cap L^p_{loc}([0,\infty); V^{1,p})$ *with* $\frac{\partial t}{\partial t} \in L^{\frac{p}{p-1}}_{loc}([0,\infty); (V^{1,p})^*)$,

(ii) a family of Radon measures $\{\mu_t\}_{t\in[0,\infty)}$ *with* $\mu_t \in \mathcal{IM}_{d-1}$ *for a.e.* $t \in [0,\infty)$ *and*

(iii) $\varphi \in BV_{loc}(\Omega \times [0,\infty)) \cap L^\infty_{loc}([0,\infty); BV(\Omega)) \cap C^{\frac{1}{2}}_{loc}([0,\infty); L^1(\Omega))$

such that the following properties hold:

(i) The triplet $(u(\cdot,t),\, \varphi(\cdot,t),\, \mu_t)_{t\in[0,\infty)}$ *is a weak solution of* (1.5). *More precisely, for any* $T > 0$ *we have*

$$\int_0^T \int_\Omega -u \cdot \frac{\partial v}{\partial t} + (u \cdot \nabla u) \cdot v + \tau(\varphi, e(u)) : e(v)\, dxdt = \int_\Omega u_0 \cdot v(0)\, dx + \int_0^T \int_\Omega \kappa_1 H \cdot v\, d\mu_t dt \tag{2.5}$$

for any $v \in C^\infty([0,T]; \mathcal{V})$ *such that* $v(T) = 0$. *Here* $H \in L^2_{loc}([0,\infty); L^2(\mu_t)^d)$ *is the generalized mean curvature vector corresponding to* μ_t.

(ii) For all $0 \leq t_1 < t_2 < \infty$ *and* $\phi \in C^2(\Omega; \mathbb{R}^+)$ *we have*

$$\mu_{t_2}(\phi) - \mu_{t_1}(\phi) \leq \int_{t_1}^{t_2} \mathcal{B}(\mu_t,\, u(\cdot,t),\, \phi)\, dt. \tag{2.6}$$

Moreover $\sup_{0<r<1/2,\, x\in\Omega} \frac{\mu_t(B_r(x))}{\omega_{d-1} r^{d-1}} \in L^\infty_{loc}([0,\infty))$ *and* $\mathcal{B}(\mu_t,\, u(\cdot,t),\, \phi) \in L^1_{loc}([0,\infty))$.

(iii) The function φ *satisfies the following properties.*
(1) $\varphi = \pm 1$ a.e. on Ω *for all* $t \in [0,\infty)$.
(2) $\varphi(x,0) = \chi_{\Omega^+(0)} - \chi_{\Omega\setminus\Omega^+(0)}$ a.e. on Ω.
(3) $\mathrm{spt}|\nabla\chi_{\{\varphi(\cdot,t)=1\}}| \subset \mathrm{spt}\mu_t$ *for all* $t \in [0,\infty)$.

(iv) There exists

$$T_1 = T_1(\|u_0\|_H, \Omega^+(0), p)$$

such that μ_t *has unit density for a.e.* $t \in [0,T_1]$. *In addition* $|\nabla\chi_{\{\varphi=1\}}| = \mu_t$ *for a.e.* $t \in [0,T_1]$.

Remark 2.4. *Somewhat different from $u = 0$ case we do not expect that*

$$\limsup_{\Delta t \to 0} \frac{\mu_{t+\Delta t}(\phi) - \mu_t(\phi)}{\Delta t} \le \mathcal{B}(\mu_t, u(\cdot, t), \phi)$$

holds for all $t \in [0, T]$ and $\phi \in C^2(\Omega; \mathbb{R}^+)$ in general. While we know that the right-hand side is $< \infty$ (by definition) for all t, we do not know in general if the left-hand side is finite. One may even expect that at a time when $\int_\Omega |\nabla u(\cdot, t)|^2 \, dx = \infty$, it is infinite. Thus we should be content with the integral form (2.6) for the definition of Brakke's flow, which in its original form is infinitesimally defined.

Remark 2.5. *The difficulty of multiplicities have been often encountered in the measure-theoretic setting like ours. Varifold solutions constructed by Brakke [7] have the same properties in this regard. On the other hand, (iv) says that there is no 'folding', where $\theta_t \ge 2$, for some time.*

Remark 2.6. *In the following we set $\kappa_1 = \kappa_2 = 1$ without loss of generality.*

2.5 Theorems to be used

We use the following

Theorem 2.7. (Korn's inequality) *Let $1 < p < \infty$. Then there exists a constant $c_K = c(p, d)$ such that*

$$\|v\|_{W^{1,p}(\Omega)} \le c_K(\|e(v)\|_{L^p(\Omega)} + \|v\|_{L^1(\Omega)})$$

holds for all $v \in W^{1,p}(\Omega)^d$.

See [24, p.196] and the reference therein.

3 Results from [20]

In this section we summarize the results from [20] which are the essential ingredients to obtain the velocity law (1.4). First we state the upper density bound of the diffused surface energy. Since the estimate is of independent interest and is true for all dimensions, we state the assumptions in the form independent of the present aim. Also we warn that u in Theorem 3.1 will not be the same u, but will be a regularized u.

Theorem 3.1. *Suppose $d \ge 2$, $\Omega = \mathbb{T}^d$, $p > \frac{d+2}{2}$, $\frac{1}{2} > \gamma \ge 0$, $1 \ge \varepsilon > 0$ and φ satisfies*

$$\frac{\partial \varphi}{\partial t} + u \cdot \nabla \varphi = \Delta \varphi - \frac{W'(\varphi)}{\varepsilon^2} \qquad \text{on } \Omega \times [0, T], \tag{3.1}$$

$$\varphi(x, 0) = \varphi_0(x) \qquad \text{on } \Omega, \tag{3.2}$$

where $\nabla^i u, \nabla^j \varphi, \nabla^k \varphi_t \in C(\Omega \times [0, T])$ for $0 \le i, k \le 1$ and $0 \le j \le 3$. Let μ_t be the Radon measure on Ω defined by

$$\int_\Omega \phi(x) \, d\mu_t(x) = \frac{1}{\sigma} \int_\Omega \phi(x) \left(\frac{\varepsilon |\nabla \varphi(x, t)|^2}{2} + \frac{W(\varphi(x, t))}{\varepsilon} \right) dx$$

for $\phi \in C(\Omega)$, where $\sigma = \int_{-1}^{1} \sqrt{2W(s)}\,ds$. We assume also that

$$\sup_{\Omega} |\varphi_0| \le 1 \text{ and } \sup_{\Omega} \varepsilon^i |\nabla^i \varphi_0| \le c_1 \text{ for } 1 \le i \le 3, \tag{3.3}$$

$$\sup_{\Omega} \left(\frac{\varepsilon |\nabla \varphi_0|^2}{2} - \frac{W(\varphi_0)}{\varepsilon} \right) \le \varepsilon^{-\gamma}, \tag{3.4}$$

$$\sup_{\Omega\times[0,T]} \left\{ \varepsilon^{\gamma}|u|,\, \varepsilon^{1+\gamma}|\nabla u| \right\} \le c_2, \tag{3.5}$$

$$\int_0^T \|u(\cdot,t)\|^p_{W^{1,p}(\Omega)}\,dt \le c_3. \tag{3.6}$$

Define for $t \in [0,T]$

$$D(t) = \max\left\{ \sup_{x\in\Omega,\, 0<r\le\frac{1}{2}} \frac{1}{\omega_{d-1} r^{d-1}} \mu_t(B_r(x)), 1 \right\}, \qquad D(0) \le D_0. \tag{3.7}$$

Then there exist $\epsilon_1 > 0$ which depends only on d, p, W, c_1, c_2, c_3, D_0, γ and T, and c_4 which depends only on c_3, d, p, D_0 and T such that for all $0 < \varepsilon \le \epsilon_1$ and $t \in [0,T]$,

$$D(t) \le c_4. \tag{3.8}$$

Once above is established, the following two theorems can be obtained with some minor modification of the argument in [20].

Theorem 3.2. *Suppose that sequences φ^{ε_i} and u^{ε_i} with $\lim_{i\to\infty} \varepsilon_i = 0$ satisfy all the assumptions in Theorem 3.1 where ε, φ_0 and μ_t there are replaced by ε_i, $\varphi_0^{\varepsilon_i}$ and $\mu_t^{\varepsilon_i}$, respectively. We assume that c_1, c_2, c_3, D_0, γ and T are independent of i. In addition we assume that $d = 2$ or 3 and that*

$$u^{\varepsilon_i} \rightharpoonup u \text{ weakly in } L^p([0,T]; W^{1,p}(\Omega)^d), \qquad u^{\varepsilon_i} \to u \text{ strongly in } L^2([0,T]; L^2(\Omega)^d). \tag{3.9}$$

Then there exists a subsequence (denoted by the same index) and and a family of measures $\{\mu_t\}_{0\le t\le T}$ such that

(a) $\lim_{i\to\infty} \mu_t^{\varepsilon_i}(\phi) = \mu_t(\phi)$ for all $t \in [0,T]$ and $\phi \in C(\Omega)$,

(b) $\mu_t \in \mathcal{IM}_{d-1}$ for a.e. $t \in [0,T]$,

(c) $H \in L^2(0,T; L^2(\mu_t)^d)$ where $H(\cdot,t)$ is the generalized mean curvature of μ_t,

(d) for any $0 \le t_1 < t_2 \le T$,

$$\lim_{i\to\infty} \frac{1}{\sigma} \int_{t_1}^{t_2} \int_{\Omega} \varepsilon_i u^{\varepsilon_i} \cdot \nabla \varphi^{\varepsilon_i} \left(-\Delta \varphi^{\varepsilon_i} + \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} \right) dxdt = \int_{t_1}^{t_2} \int_{\Omega} H \cdot u\, d\mu_t dt, \tag{3.10}$$

(e) for any $\phi \in C^2(\Omega; \mathbb{R}^+)$ and $0 \le t_1 < t_2 \le T$,

$$\mu_{t_2}(\phi) - \mu_{t_1}(\phi) \le \int_{t_1}^{t_2} \mathcal{B}(\mu_t,\, u(\cdot,t),\, \phi)\,dt. \tag{3.11}$$

Theorem 3.3. *Under the same assumptions as in Theorem 3.2 we have a subsequence $\{\varphi^{\varepsilon_i}\}$ and a function $\varphi \in BV(\Omega\times[0,T]) \cap L^\infty([0,T]; BV(\Omega)) \cap C^{\frac{1}{2}}([0,T]; L^2(\Omega))$ such that*

(i) $\lim_{i\to\infty} \|\varphi^{\varepsilon_i} - \varphi\|_{L^\alpha(\Omega\times[0,T])} = 0$ for $1 \le \alpha < \infty$ and pointwise a.e. on $\Omega \times [0,T]$,

(ii) $\varphi = \pm 1$ a.e. on $\Omega \times [0,T]$.

(iii) Define $\Gamma(t)$ by $\mu_t = \theta_t \mathcal{H}^{d-1}\lfloor_{\Gamma(t)}$. Then $\mathcal{H}^{d-1}(\partial^*\{\varphi(\cdot,t) = 1 \setminus \Gamma(t)) = 0$ for a.e. $t \in [0,T]$.

4 Existence of approximate solution

In this section we construct the weak solution of approximate solution to (1.2)-(1.4) by the Galerkin method. The proof is a suitable modification of [18] for the non-Newtonian setting but we include the proof for the completeness.

First we prepare a few definitions. We fix a sequence $\{\varepsilon_i\}$ with $\lim_{i\to\infty}\varepsilon_i = 0$ and fix a radially symmetric function $\zeta \in C_c^\infty(\mathbb{R}^d)$ with $\operatorname{spt}\zeta \subset B_1(0)$ and $\int \zeta\, dx = 1$. For a fixed $0 < \gamma < \frac{1}{2}$ we define

$$\zeta^{\varepsilon_i}(x) = \frac{1}{\varepsilon_i^{\gamma}}\zeta\left(\frac{x}{\varepsilon_i^{\gamma/d}}\right). \tag{4.1}$$

We defined ζ^{ε_i} so that $\int \zeta^{\varepsilon_i}\, dx = 1$, $|\zeta^{\varepsilon_i}| \le c(d)\varepsilon_i^{-\gamma}$ and $|\nabla\zeta^{\varepsilon_i}| \le c(d)\varepsilon_i^{-1-\gamma}$.

For a given initial data $\Omega^+(0) \subset \Omega$ with C^1 boundary $\partial\Omega^+(0)$, we can approximate $\Omega^+(0)$ by a sequence of domains with C^3 boundaries. Thus we may assume that $\partial\Omega^+(0)$ is C^3 in the following. Let $d(x)$ be the signed distance function to $\partial\Omega^+(0)$ so that $d(x) > 0$ on $\Omega^+(0)$ and $d(x) < 0$ on $\Omega^-(0)$. Choose $b > 0$ so that d is C^2 function on the b-neighborhood of $\partial\Omega^+(0)$. Let $h \in C^\infty(\mathbb{R})$ be a function such that h is monotone increasing, $h(s) = s$ for $0 \le s \le b/4$ and $h(s) = b/2$ for $b/2 < s$, and define $h(-s) = -h(s)$ for $s < 0$. Then define $\tilde{d}(x) = h(d(x))$ and

$$\varphi_0^{\varepsilon_i}(x) = \tanh(\tilde{d}(x)/\varepsilon_i). \tag{4.2}$$

For all sufficiently small ε_i, $\varphi_0^{\varepsilon_i} \in C^3(\Omega)$ and

$$\lim_{i\to\infty}\varphi_0^{\varepsilon_i} = \chi_{\Omega^+(0)} - \chi_{\Omega^-(0)}, \quad \frac{1}{\sigma}\int_\Omega\left(\frac{\varepsilon_i|\nabla\varphi_0^{\varepsilon_i}|^2}{2} + \frac{W(\varphi_0^{\varepsilon_i})}{\varepsilon_i}\right)dx \le \mathcal{H}^{d-1}(\partial\Omega^+(0)) + 1. \tag{4.3}$$

For $V^{s,2}$ with $s > \frac{d}{2} + 1$ let $\{\omega^i\}_{i=1}^\infty$ be a set of complete orthogonal basis of $V^{s,2}$ such that it is orthonormal in $V^{0,2}$. The choice of s is made so that the Sobolev embedding theorem implies $W^{s-1,2}(\Omega) \hookrightarrow L^\infty(\Omega)$ thus $\nabla\omega^i \in L^\infty(\Omega)^{d^2}$.

Let $P_i : V^{0,2} \to V_i^{0,2} = \operatorname{span}\{\omega_1, \omega_2, \cdots, \omega_i\}$ be the orthogonal projection. We then project the problem (1.2)-(1.4) to $V_i^{0,2}$ by using the orthogonality in $V^{0,2}$. Note that just as in [18], we approximate the mean curvature term in (1.5) by the appropriate phase field approximation. For any $0 < T < \infty$ we consider the following problem:

$$\frac{\partial u^{\varepsilon_i}}{\partial t} = P_i\left(\operatorname{div}\tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) - u^{\varepsilon_i}\cdot\nabla u^{\varepsilon_i} - \frac{\varepsilon_i}{\sigma}\operatorname{div}\left((\nabla\varphi^{\varepsilon_i}\otimes\nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i}\right)\right) \quad \text{in } \Omega\times[0,T], \tag{4.4}$$

$$u^{\varepsilon_i}(\cdot,t) \in V_i^{0,2} \quad \text{in } \Omega\times[0,T], \tag{4.5}$$

$$\frac{\partial\varphi^{\varepsilon_i}}{\partial t} + (u^{\varepsilon_i} * \zeta^{\varepsilon_i})\cdot\nabla\varphi^{\varepsilon_i} = \Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} \quad \text{in } \Omega\times[0,T], \tag{4.6}$$

$$u^{\varepsilon_i}(x,0) = P_i u_0(x), \quad \varphi^{\varepsilon_i}(x,0) = \varphi_0^{\varepsilon_i}(x) \quad \text{in } \Omega. \tag{4.7}$$

Here $*$ is the usual convolution. We first prove the following theorem.

Theorem 4.1. *For any $i \in \mathbb{N}$, $T \in (0,\infty)$, $u_0 \in V^{0,2}$ and $\varphi_0^{\varepsilon_i}$, there exists a weak solution $(u^{\varepsilon_i}, \varphi^{\varepsilon_i})$ of (4.4)-(4.7) on $\Omega\times[0,T]$ such that $u^{\varepsilon_i} \in L^\infty([0,T];V^{0,2}) \cap L^p([0,T];V^{1,p})$, $|\varphi^{\varepsilon_i}| \le 1$, $\varphi^{\varepsilon_i} \in L^\infty([0,T];C^3(\Omega))$ and $\frac{\partial\varphi^{\varepsilon_i}}{\partial t} \in L^\infty([0,T];C^1(\Omega))$.*

We write the above system in terms of $u^{\varepsilon_i} = \sum_{k=1}^{i} c_k^{\varepsilon_i}(t)\omega_k(x)$ first. Since

$$\begin{aligned}\left(\frac{d}{dt}u^{\varepsilon_i},\, \omega_j\right) &= \left(\frac{d}{dt}\sum_{k=1}^{i} c_k^{\varepsilon_i}(t)\,\omega_k,\, \omega_j\right) = \frac{d}{dt}c_j^{\varepsilon_i}(t),\\ (u^{\varepsilon_i}\cdot\nabla u^{\varepsilon_i},\, \omega_j) &= \sum_{k,l=1}^{i} c_k^{\varepsilon_i}(t)c_l^{\varepsilon_i}(t)(\omega_k\cdot\nabla\omega_l,\, \omega_j),\\ \varepsilon_i(\operatorname{div}((\nabla\varphi^{\varepsilon_i}\otimes\nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i}),\, \omega_j) &= -\varepsilon_i\int_\Omega(\nabla\varphi^{\varepsilon_i}\otimes\nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i} : \nabla\omega_j\, dx,\\ (\operatorname{div}\tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})),\, \omega_j) &= -\int_\Omega \tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) : e(\omega_j)\, dx\end{aligned}$$

for $j = 1, \cdots, i$, (4.4) is equivalent to

$$\begin{aligned}\frac{d}{dt}c_j^{\varepsilon_i}(t) = &-\int_\Omega \tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) : e(\omega_j)\, dx - \sum_{k,l=1}^{i} c_k^{\varepsilon_i}(t)c_l^{\varepsilon_i}(t)(\omega_k\cdot\nabla\omega_l,\, \omega_j)\\ &+\frac{\varepsilon_i}{\sigma}\int_\Omega(\nabla\varphi^{\varepsilon_i}\otimes\nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i} : \nabla\omega_j\, dx = A_j^{\varepsilon_i}(t) + B_{klj}c_k^{\varepsilon_i}(t)c_l^{\varepsilon_i}(t) + D_j^{\varepsilon_i}(t).\end{aligned} \tag{4.8}$$

Moreover, the initial condition of $c_j^{\varepsilon_i}$ is

$$c_j^{\varepsilon_i}(0) = (u_0,\, \omega_j) \quad \text{for } j = 1, 2, \ldots, i.$$

We also set

$$E_0 = \mathcal{H}^{d-1}(\partial\Omega^+(0)) + 1 + \frac{1}{2}\int_\Omega |u_0|^2\, dx$$

and note that

$$\frac{1}{\sigma}\int_\Omega\left(\frac{\varepsilon_i|\nabla\varphi_0^{\varepsilon_i}|^2}{2} + \frac{W(\varphi_0^{\varepsilon_i})}{\varepsilon_i}\right)dx + \frac{1}{2}\sum_{j=1}^{i}(c_j^{\varepsilon_i}(0))^2 \le E_0 \tag{4.9}$$

for all i by (4.3).

We use the following lemma to prove Theorem 4.1.

Lemma 4.2. *There exists a constant $T_0 = T_0(E_0, i) > 0$ such that (4.4)-(4.7) has a weak solution $(u^{\varepsilon_i}, \varphi^{\varepsilon_i})$ in $\Omega\times[0, T_0]$ such that $u^{\varepsilon_i} \in L^\infty([0,T_0]; V^{0,2}) \cap L^p([0,T_0]; V^{1,p})$, $|\varphi^{\varepsilon_i}| \le 1$, $\varphi^{\varepsilon_i} \in L^\infty([0,T_0]; C^3(\Omega))$ and $\frac{\partial\varphi^{\varepsilon_i}}{\partial t} \in L^\infty([0,T_0]; C^1(\Omega))$.*

Proof. Assume that we are given a function $u(x,t) = \sum_{j=1}^{i} c_j^{\varepsilon_i}(t)\omega_j(x) \in C([0,T]; V^{s,2})$ with

$$c_j^{\varepsilon_i}(0) = (u_0,\, \omega_j), \qquad \max_{t\in[0,T]}\frac{1}{2}\sum_{j=1}^{i}|c_j^{\varepsilon_i}(t)|^2 \le 2E_0. \tag{4.10}$$

We let $\varphi(x,t)$ be the solution of the following parabolic equation:

$$\begin{aligned}\frac{\partial}{\partial t}\varphi + (u * \zeta^{\varepsilon_i})\cdot\nabla\varphi &= \Delta\varphi - \frac{W'(\varphi)}{\varepsilon_i^2},\\ \varphi(x,0) &= \varphi_0^{\varepsilon_i}(x).\end{aligned} \tag{4.11}$$

The existence of such φ with $|\varphi| \le 1$ is guaranteed by the standard theory of parabolic equations ([17]). By (4.11) and Cauchy-Schwarz inequality, we can estimate

$$\frac{d}{dt}\int_\Omega \left(\frac{\varepsilon_i|\nabla\varphi|^2}{2} + \frac{W(\varphi)}{\varepsilon_i^2}\right) dx \le -\frac{\varepsilon_i}{2}\int_\Omega \left(\Delta\varphi - \frac{W'(\varphi)}{\varepsilon_i^2}\right)^2 dx + \frac{\varepsilon_i}{2}\int_\Omega \{(u * \zeta^{\varepsilon_i})\cdot\nabla\varphi\}^2\, dx.$$

Since for any $t \in [0,T]$

$$\|u * \zeta^{\varepsilon_i}\|^2_{L^\infty(\Omega)} \le \varepsilon_i^{-2\gamma}\|u\|^2_{L^\infty(\Omega)} \le i\varepsilon_i^{-2\gamma}\max_{1\le j\le i}\|\omega_j(x)\|^2_{L^\infty(\Omega)}\sum_{j=1}^{i}|c_j^{\varepsilon_i}(t)|^2 \le c(i)E_0,$$

$$\frac{d}{dt}\int_\Omega \left(\frac{\varepsilon_i|\nabla\varphi|^2}{2} + \frac{W(\varphi)}{\varepsilon_i}\right) dx \le c(i)E_0\int_\Omega \frac{\varepsilon_i|\nabla\varphi|^2}{2}\, dx.$$

This gives

$$\sup_{0\le t\le T}\frac{1}{\sigma}\int_\Omega \left(\frac{\varepsilon_i|\nabla\varphi|^2}{2} + \frac{W(\varphi)}{\varepsilon_i}\right) dx \le e^{c(i)E_0T}E_0. \tag{4.12}$$

Hence as long as $T \le 1$,

$$|D_j^{\varepsilon_i}(t)| \le c\|\nabla\omega_j\|_{L^\infty(\Omega)}\frac{1}{\sigma}\int_\Omega\int_\Omega \varepsilon_i|\nabla\varphi(y)|^2\zeta^{\varepsilon_i}(x-y)\, dydx \le c(i)e^{c(i)E_0}E_0$$

by $\nabla\omega_j \in L^\infty(\Omega)^{d^2}$ and (4.12).

Next we substitute the above solution φ into the place of φ^{ε_i}, and solve (4.8) with the initial condition $c_j^{\varepsilon_i}(0) = (u_0, \omega_j)$. Since τ is locally Lipschitz with respect to $e(u)$, there is at least some short time T_1 such that (4.8) has a unique solution $\tilde{c}_j^{\varepsilon_i}(t)$ on $[0,T_1]$ with the initial condition $\tilde{c}_j^{\varepsilon_i}(0) = (u_0, \omega_j)$ for $1 \le i \le i$. We show that the solution exists up to $T_0 = T_0(i, E_0)$ satisfying (4.10). Let $\tilde{c}(t) = \frac{1}{2}\sum_{j=1}^m |\tilde{c}_j^{\varepsilon_i}(t)|^2$. Then,

$$\frac{d}{dt}\tilde{c}(t) = A_j^{\varepsilon_i}\tilde{c}_j^{\varepsilon_i} + B^i_{klj}\tilde{c}_k^{\varepsilon_i}\tilde{c}_l^{\varepsilon_i}\tilde{c}_j^{\varepsilon_i} + D_j^{\varepsilon_i}\tilde{c}_j^{\varepsilon_i}.$$

By (1.1) $A_j^{\varepsilon_i}\tilde{c}^{\varepsilon_i} \le 0$ hence

$$\frac{d}{dt}\tilde{c}(t) \le c(i, E_0)(\tilde{c}^{3/2} + \tilde{c}^{1/2}).$$

Therefore,

$$\tanh\sqrt{\tilde{c}(t)} \le \tanh\sqrt{E_0} + 2c(i, E_0)t.$$

Then, by choosing T_0 small depending only on i and E_0 we have the existence of solution for $t \in [0, T_0]$ satisfying (4.10). We then prove the existence of a weak solution on $\Omega \times [0, T_0]$ by using Leray-Schauder fixed point theorem (see [17]). We define

$$\tilde{u}(x,t) = \sum_{j=1}^{i}\tilde{c}_j^{\varepsilon_i}(t)\omega_j(x)$$

and we define a map $\mathcal{L} : u \mapsto \tilde{u}$ as in the above procedure. Let

$$V(T_0) := \left\{ u(x,t) = \sum_{j=1}^{i} c_j^{\varepsilon_i}(t)\omega_j(x) ; \right.$$
$$\left. \frac{1}{2}\sum_{j=1}^{i} |\tilde{c}_j^{\varepsilon_i}(t)|^2 \le 2E_0 \text{ for } t \in [0,T_0],\ c_j^{\varepsilon_i}(0) = (u_0, \omega_j),\ c_j^{\varepsilon_i} \in C([0,T_0]) \right\}.$$

Then $V(T_0)$ is a closed, convex subset of $C([0,T_0]; V_i^{0,2})$ equipped with the norm

$$\|u\|_{V(T_0)} = \sup_{0\le t\le T_0} \left(\sum_{j=1}^{i} |c_j^{\varepsilon_i}(t)|^2 \right)^{\frac{1}{2}}$$

and by the above argument $\mathcal{L} : V(T_0) \to V(T_0)$. Moreover by the Ascoli-Arzelà compactness theorem $\mathcal{L}$ is a compact operator. Therefore by using the Leray-Schauder fixed point theorem, $\mathcal{L}$ has a fixed point $u^{\varepsilon_i} \in V(T_0)$. We denote by φ^{ε_i} the solution of (4.6) and (4.7). Then $(u^{\varepsilon_i}, \varphi^{\varepsilon_i})$ is a weak solution of (4.4)-(4.7) in $\Omega \times [0,T_0]$. □

Theorem 4.3. *Let $(u^{\varepsilon_i}, \varphi^{\varepsilon_i})$ be the weak solution of (4.4)-(4.7) in $\Omega\times[0,T]$. Then the following energy estimate holds:*

$$\sup_{0\le t\le T} \int_\Omega \frac{1}{\sigma}\left(\frac{\varepsilon|\nabla\varphi^{\varepsilon_i}|^2}{2} + \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i} \right) + \frac{|u^{\varepsilon_i}|^2}{2}\, dx$$
$$+ \int_0^T \int_\Omega \frac{\varepsilon_i}{\sigma}\left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} \right)^2 + \nu_0 |e(u^{\varepsilon_i})|^p \, dxdt \le E_0. \tag{4.13}$$

Moreover

$$\int_0^T \|u^{\varepsilon_i}(\cdot,t)\|^p_{W^{1,p}(\Omega)}\, dt \le c_K \nu_0^{-1}(E_0 + TE_0^{\frac{p}{2}}). \tag{4.14}$$

Proof. Since $(u^{\varepsilon_i}, \varphi^{\varepsilon_i})$ is the weak solution of (4.4)-(4.7), we derive

$$\begin{aligned} &\frac{d}{dt}\int_\Omega \frac{1}{\sigma}\left(\frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i}|^2}{2} + \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i} \right) + \frac{|u^{\varepsilon_i}|^2}{2}\, dx \\ &= \int_\Omega -\frac{\varepsilon_i}{\sigma}\frac{\partial\varphi^{\varepsilon_i}}{\partial t}\left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} \right) + \frac{\partial u^{\varepsilon_i}}{\partial t}\cdot u^{\varepsilon_i}\, dx \\ &= \int_\Omega -\frac{\varepsilon_i}{\sigma}\left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2} - (u^{\varepsilon_i} * \zeta^{\varepsilon_i})\cdot\nabla\varphi^{\varepsilon_i} \right)\left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon^2} \right) dx \\ &+ \int_\Omega \left\{ \operatorname{div}\tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) - u^{\varepsilon_i}\cdot\nabla u^{\varepsilon_i} - \frac{\varepsilon}{\sigma}\operatorname{div}\left((\nabla\varphi^{\varepsilon_i}\otimes\nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i} \right) \right\}\cdot u^{\varepsilon_i}\, dx = I_1 + I_2. \end{aligned} \tag{4.15}$$

Since $\operatorname{div}(u^{\varepsilon_i} * \zeta^{\varepsilon_i}) = (\operatorname{div} u^{\varepsilon_i}) * \zeta^{\varepsilon_i} = 0$,

$$\sigma I_1 = -\int_\Omega \varepsilon_i \left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi)}{\varepsilon_i^2} \right)^2 dx + \varepsilon_i \int_\Omega (u^{\varepsilon_i} * \zeta^{\varepsilon_i})\cdot\nabla\varphi^{\varepsilon_i}\Delta\varphi^{\varepsilon_i}\, dx.$$

For I_2, with (1.1)

$$\int_\Omega \operatorname{div}\tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i}))\cdot u^{\varepsilon_i}\, dx = -\int_\Omega \tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) : e(u^{\varepsilon_i})\, dx \le -\nu_0 \int_\Omega |e(u^{\varepsilon_i})|^p\, dx.$$

Moreover the second term of I_2 vanishes by $\mathrm{div}\, u^{\varepsilon_i} = 0$ and

$$-\int_\Omega \varepsilon_i \mathrm{div}\, (\nabla\varphi^{\varepsilon_i} \otimes \nabla\varphi^{\varepsilon_i} * \zeta^{\varepsilon_i}) \cdot u^{\varepsilon_i}\, dx = -\int_\Omega \varepsilon_i \left(\nabla \frac{|\nabla\varphi^{\varepsilon_i}|^2}{2} + \nabla\varphi^{\varepsilon_i}\Delta\varphi^{\varepsilon_i}\right) * \zeta^{\varepsilon} \cdot u^{\varepsilon_i}\, dx$$
$$= -\varepsilon_i \int_\Omega (u^{\varepsilon_i} * \zeta^{\varepsilon_i}) \cdot \nabla\varphi^{\varepsilon_i}\Delta\varphi^{\varepsilon_i}\, dx.$$

Hence (4.15) becomes

$$\frac{d}{dt}\int_\Omega \frac{1}{\sigma}\left(\frac{\varepsilon_i|\nabla\varphi^{\varepsilon_i}|^2}{2} + \frac{W(\varphi^{\varepsilon_i})}{\varepsilon_i}\right) + \frac{|u^{\varepsilon_i}|^2}{2}\, dx \le -\int_\Omega \frac{\varepsilon_i}{\sigma}\left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2}\right)^2 + \nu_0|e(u^{\varepsilon_i})|^p\, dx$$

Integrating with respect to t and taking supremum over all $t \in [0, T]$, we obtain (4.13). The proof of (4.14) follows from (4.13) and Theorem 2.7. □

Proof of Theorem 4.1. For each fixed i we have a short time existence for $[0, T_0]$ where T_0 depends only on i and E_0 at $t = 0$. By Lemma 4.3 the energy at $t = T_0$ is again bounded by E_0. By repeatedly using Lemma 4.2 Theorem 4.1 follows. □

5 Existence of weak solution

Finally in this section, we take the limit $i \to \infty$ and establish the main result. The necessary steps for the proof of the convergence of the phase boundary are all resolved in Section 3 and 4. The proof of the convergence of the velocity field can be handled by the standard method (see [19, p.207]) combined with the observation on the varifold convergence ([28]). Here we only sketch the outline of the proof with reference to [19]. First using the equation (4.4) and energy inequalities (4.3) one can show

$$\int_0^T \left\|\frac{\partial u^{\varepsilon_i}}{\partial t}\right\|_{(V^{s,2})^*}^{\frac{p}{p-1}} dt \le c$$

where c depends only on E_0, c_K and ν_0 and is independent of i. The application of Aubin-Lions compactness Theorem [19, p.57] with $B_0 = V^{s,2}$, $B = V^{0,2}$, $B_1 = (V^{s,2})^*$, $p_0 = p$ and $p_1 = \frac{p}{p-1}$ there shows the existence of a subsequence still denoted by $\{u^{\varepsilon_i}\}_{i=1}^\infty$ such that

$$u^{\varepsilon_i} \to u \quad \text{in } L^p([0,T]; V^{0,2}). \tag{5.1}$$

Since $p > 2$ and $L^\infty([0,T]; L^2(\Omega)^d)$ bound, we also have the strong convergence in $L^2([0,T]; L^2(\Omega)^d)$. As for the convergence of $\{\mu^{\varepsilon_i}\}_{i=1}^\infty$ we have all the assumptions on φ^{ε_i} and $u^{\varepsilon_i} * \zeta^{\varepsilon_i}$ satisfied to apply Theorem 3.1. Thus we have the upper density ratio bound, and then we can apply Theorem 3.2 and Theorem 3.3 since $u^{\varepsilon_i} * \zeta^{\varepsilon_i}$ also converges in the sense of (3.9). We may extract a further subsequence so that

$$\begin{gathered}\frac{\partial u^{\varepsilon_i}}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ weakly in } L^{\frac{p}{p-1}}([0,T]; (V^{s,2})^*),\\ \tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) \rightharpoonup \hat{\tau} \text{ weakly in } L^{\frac{p}{p-1}}([0,T]; L^{\frac{p}{p-1}}(\Omega)^{d^2}).\end{gathered} \tag{5.2}$$

For $\omega_j \in V^{s,2}$ $(j = 1, \cdots)$ and $h \in C_c^\infty((0,T))$ we have

$$\int_\Omega \mathrm{div}((\nabla\varphi^{\varepsilon_i} \otimes \nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i}) \cdot h\omega_j\, dx = \int_\Omega \left(\Delta\varphi^{\varepsilon_i} - \frac{W'(\varphi^{\varepsilon_i})}{\varepsilon_i^2}\right)\nabla\varphi^{\varepsilon_i} \cdot h\omega_j * \zeta^{\varepsilon_i}\, dx$$

by $\operatorname{div}\omega_j = 0$. Thus the argument in [19, p.212] and the similar convergence argument in Section 4

$$\int_0^T \left\{ \left(\frac{\partial u}{\partial t}, h\omega_j \right) + \int_\Omega (u \cdot \nabla u) \cdot h\omega_j + h\hat{\tau} : e(\omega_j)\, dx \right\} dt = \int_0^T \int_\Omega H \cdot h\omega_j \, d\mu_t dt. \tag{5.3}$$

Again by the similar argument using the density ratio bound and Theorem 2.1 one show by the density argument and (5.3) that $\frac{\partial u}{\partial t} \in L^{\frac{p}{p-1}}([0,T];(V^{1,p})^*)$ and

$$\int_0^T \left\{ \left(\frac{\partial u}{\partial t}, v \right) + \int_\Omega (u \cdot \nabla u) \cdot v + \hat{\tau} : e(v)\, dx \right\} dt = \int_0^T \int_\Omega H \cdot v \, d\mu_t dt. \tag{5.4}$$

for all $v \in L^p([0,T];V^{1,p})$. The only thing to be left now is to prove that

$$\int_0^T \int_\Omega \hat{\tau} : e(v)\, dxdt = \int_0^T \int_\Omega \tau(\varphi, e(u)) : e(v)\, dxdt \tag{5.5}$$

for all $v \in C_c^\infty((0,T);\mathcal{V})$. As in [19, p.213 (5.43)], we may deduce that

$$\frac{1}{2}\|u(t_1)\|^2_{L^2(\Omega)} + \int_0^{t_1} \int_\Omega \hat{\tau} : e(u)\, dxdt \geq \int_0^{t_1} \int_\Omega H \cdot u \, d\mu_t dt + \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} \tag{5.6}$$

for a.e. $t_1 \in [0,T]$. We set for any $v \in V^{1,p}$

$$A_i^{t_1} = \int_0^{t_1} \int_\Omega (\tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) - \tau(\varphi^{\varepsilon_i}, e(v))) : (e(u^{\varepsilon_i}) - e(v))\, dxdt + \frac{1}{2}\|u^{\varepsilon_i}(t_1)\|^2_{L^2(\Omega)}. \tag{5.7}$$

The monotonicity property of $e(\cdot)$ (1.1) shows that the first term of (5.7) is non-negative. We may further assume that $u^{\varepsilon_i}(t_1)$ converges weakly to $u(t_1)$ in $L^2(\Omega)^d$ thus we have

$$\liminf_{i\to\infty} A_i^{t_1} \geq \frac{1}{2}\|u(t_1)\|^2_{L^2(\Omega)}. \tag{5.8}$$

By (4.4) we have

$$\begin{aligned} A_i^{t_1} =& \frac{1}{2}\|u^{\varepsilon_i}(0)\|^2_{L^2(\Omega)} - \frac{\varepsilon_i}{\sigma} \int_0^{t_1} \int_\Omega \operatorname{div}((\nabla\varphi^{\varepsilon_i} \otimes \nabla\varphi^{\varepsilon_i}) * \zeta^{\varepsilon_i}) \cdot u^{\varepsilon_i} \\ &- \int_0^{t_1} \int_\Omega \tau(\varphi^{\varepsilon_i}, e(u^{\varepsilon_i})) : e(v) + \tau(\varphi^{\varepsilon_i}, e(v)) : (e(u^{\varepsilon_i}) - e(v))\, dxdt \end{aligned}$$

which converges to

$$A^{t_1} = \frac{1}{2}\|u(0)\|^2_{L^2(\Omega)} + \int_0^{t_1} \int_\Omega H \cdot u \, d\mu_t dt - \int_0^{t_1} \int_\Omega \hat{\tau} : e(v) + \tau(\varphi, e(v)) : (e(u) - e(v))\, dxdt. \tag{5.9}$$

Here we used that φ^{ε_i} converges to φ a.e. on $\Omega \times [0,T]$. By (5.6), (5.8) and (5.9), we deduce that

$$\int_0^{t_1} \int_\Omega (\hat{\tau} - \tau(\varphi, e(v))) : (e(u) - e(v))\, dxdt \geq 0.$$

By choosing $v = u + \epsilon\tilde{v}$, divide by ϵ and letting $\epsilon \to 0$, we prove (5.5). This concludes the proof of Theorem 2.3 $\square$

References

[1] H. Abels, *The initial value problem for the Navier-Stokes equations with a free surface in L^q-Sobolev spaces*, Adv. Diff. Eqns. **10** (2005), 45–64.

[2] H. Abels, *On generalized solutions of two-phase flows for viscous incompressible fluids*, Interface. Free Bound. **9** (2007), 31–65.

[3] H. Abels, M. Röger, *Existence of weak solutions for a non-classical sharp interface model for a two-phase flow of viscous, incompressible fluids*, preprint.

[4] W. Allard, *On the first variation of a varifold*, Ann. of Math. **95** (1972), 417–491.

[5] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Comm. Pure. Appl. Math. **34** (1981), 359–392.

[6] J. T. Beale, *Large-time regularity of viscous surface waves*, Arch. Ration. Mech. Anal. **84** (1984), 307–352.

[7] K. Brakke, *The motion of a surface by its mean curvature*, Princeton University Press, Princeton, NJ, (1978).

[8] Y. Chen, M. Struwe, *Existence and partial regularity for the solutions to evolution problems for harmonic maps*, Math. Z. **201** (1989), 83–103.

[9] L. C. Evans, *Partial differential equations*, Graduate Studies in Math. AMS, (1998).

[10] L. C. Evans, R. F. Gariepy, *Measure theory and fine properties of functions*, Studies in Advanced Math. CRC Press, (1992).

[11] Y. Giga, S. Takahashi, *On global weak solutions of the nonstationary two phase Stokes flow*, SIAM J. Math. Anal. **25** (1994), 876–893.

[12] G. Huisken, *Asymptotic behavior for singularities of the mean curvature flow*, J. Diff. Geom. **31** (1990), 285–299.

[13] J. E. Hutchinson,Y. Tonegawa, *Convergence of phase interfaces in the van der Waals-Cahn-Hilliard theory*, Calc. Var. PDE **10** (2000), 49–84.

[14] T. Ilmanen, *Convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, J. Diff. Geom. **38** (1993), 417–461.

[15] T. Ilmanen, *Elliptic regularization and partial regularity for motion by mean curvature*, Mem. Amer. Math. Soc. **108** (1994), no. 520.

[16] N. Kim, L. Consiglieri, J. F. Rodrigues, *On non-Newtonian incompressible fluids with phase transitions*, Math. Meth. Appl. Sci. **29** (2006), 1523–1541.

[17] O. A. Ladyzhenskaya, N. A. Solonnikov, N. N. Uraltseva, *Linear and Quasilinear Equations Of Parabolic Type*, Transl. Math. Monographs, Vol. 23, Amer. Math. Soc. (1968).

[18] F. H. Lin, C. Liu, *Nonparabolic dissipative systems modeling the flow of liquid crystals*, Comm. Pure. Appl. Math. **48** (1995), 501–537.

[19] J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Paris.

[20] C. Liu, N. Sato, Y. Tonegawa, *On the existence of mean curvature flow with transport term*, Interface. Free Bound. **12** (2010), 251–277.

[21] C. Liu, N. Sato, Y. Tonegawa, *On generalized solutions of two-phase flow problem coupled with mean curvature flow*, in preparation.

[22] C. Liu, N. J. Walkington, *An Eulerian description of fluids containing visco-hyperelastic particles*, Arch. Ration. Mech. Anal. **159** (2001), 229–252.

[23] Y. Maekawa, *On a free boundary problem for viscous incompressible flows*, Interface. Free Bound. **9** (2007), 549–589.

[24] J. Málek, J. Nečas, M. Rokyta, M. Růžička, *Weak and measure-valued solutions to evolutionary PDEs*, Appl. Math. Math. Comput. 13, Chapman & Hall, London (1996).

[25] N. G. Meyers, W. P. Ziemer, *Integral inequalities of Poincaré and Wirtinger type for BV functions*, Amer. J. Math. **99** (1977), 1345–1360.

[26] L. Mugnai, M. Röger, *Convergence of perturbed Allen-Cahn equations to forced mean curvature flow*, preprint.

[27] A. Nouri, F. Poupaud, *An existence theorem for the multifluid Navier-Stokes Problem*, J. Diff. Eqns. **122** (1995), 71–88.

[28] P. I. Plotnikov *Generalized solutions to a free boundary problem of motion of a non-Newtonian fluid*, Siberian Math. J. **34** (1993), 704–716.

[29] M. Röger, R. Schätzle, *On a modified conjecture of De Giorgi*, Math. Z. **254** (2006), 675–714.

[30] N. Sato, *A simple proof of convergence of the Allen-Cahn equation to Brakke's motion by mean curvature*, Indiana Univ. Math. J. **57** (2008), 1743–1751.

[31] L. Simon, *Lectures on geometric measure theory*, Proc. Centre Math. Anal. Austral. Nat. Univ. **3** (1983).

[32] V. A. Solonnikov, *Estimates of the solution of a certain initial-boundary value problem for a linear nonstationary system of Navier-Stokes equations*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **59** (1976), 178–254, 257 (in Russian).

[33] V. A. Solonnikov, *On the transient motion of an isolated volume of viscous incompressible fluid*, Math. USSR-Izv. **31** (1988), 381–405.

[34] H. M. Soner, *Convergence of the phase-field equations to the Mullins-Sekerka problem with kinetic undercooling*, Arch. Ration. Mech. Anal. **131** (1995), 139–197.

[35] Y. Tonegawa, *Integrality of varifolds in the singular limit of reaction-diffusion equations*, Hiroshima Math. J. **33**, (2003), 323–341.

[36] W. P. Ziemer, *Weakly differentiable functions*, Springer-Verlag (1989).

Department of Mathematics, Hokkaido University, Sapporo 060-0810 Japan
tonegawa@math.sci.hokudai.ac.jp